Algebraic functions, calculus style

Fred Richman

Florida Atlantic University

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Las Cruces
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Implicit ones satisfy polynomials in $X$ with coefficients in $\mathbb{R}[x]$.

Explicit ones are constructed from $\mathbb{R}$ and $x$ by the arithmetic operations, and extracting roots. These are the algebraic functions of the calculus books.

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So $|x|$ is an explicit algebraic function because it is equal to $\sqrt{x^2}$. 
Theorem

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- R. W. Hamming, *Monthly* 1970. “We now indicate the proof that this second definition is contained in the first.”
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- \( \text{dom} \sqrt[m]{f} = \begin{cases} \text{dom} f & \text{if } m \text{ is odd} \\ \{ x \in \text{dom} f : f(x) \geq 0 \} & \text{if } m \text{ is even} \end{cases}. \)
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The signum function defined by

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\text{sgn}(x) = \begin{cases} 
1 & \text{if } x > 0 \\
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**Theorem**

Every explicit algebraic function is continuous on its domain.
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If $f$ and $g$ are algebraic functions, then so are $f + g$, $f \cdot g$, $1/f$, and $\sqrt[m]{f}$. 
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Let \( h \) be one of \( f + g \) or \( f \cdot g \). Let \( D = \text{dom} \ h \).
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We may assume that $D = \text{dom } f = \text{dom } g$ and that $D$ is nonempty.
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The ring $T$ of all functions on $D$ is an $\mathbb{R}[x]$-module.

Let $S$ be the nonzero elements of $\mathbb{R}[x]$ and $A = T_S$. $A$ is an algebra over the field $\mathbb{R}(x)$. 
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\( A \) is an algebra over the field \( \mathbb{R} (x) \).

In \( A \), both \( f \) and \( g \) are integral over \( \mathbb{R} (x) \), hence so is \( h \).
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*If* $f$ *and* $g$ *are algebraic functions, then so are* $f + g$, $f \cdot g$, $1/f$, *and* $\sqrt[m]{f}$.

Let $h$ be one of $f + g$ or $f \cdot g$. Let $D = \text{dom } h$.

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The ring $T$ of all functions on $D$ is an $R[x]$-module.

Let $S$ be the nonzero elements of $R[x]$ and $A = Ts$.

$A$ is an algebra over the field $R(x)$.

In $A$, both $f$ and $g$ are integral over $R(x)$, hence so is $h$.

There is a nonzero polynomial $P$ with coefficients in $R[x]$ such that $cP(h) = 0$ for some $c \in S$. 
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**Theorem**

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**Corollary**

An analytic algebraic function on an open interval is either identically zero or has only a finite number of zeros.
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**Corollary**

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**Theorem**

A continuous algebraic function on an open interval is piecewise analytic with a finite number of pieces.
Theorem

If $f$ is an explicit algebraic function, then there exist points $t_1 < t_2 < \cdots < t_n$ cutting up $\mathbb{R}$ into $n + 1$ open intervals $I_0, I_1, \ldots, I_n$ such that on each $I_i$

- $f$ is undefined or
- $f$ is analytic and never 0 (hence positive or negative) or
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Theorem (splicing)

Let $t_1 < t_2 < \cdots < t_n$ be real numbers. These points break up $\mathbb{R}$ into $n + 1$ closed intervals $I_0, I_1, \ldots, I_n$. If $f_i$ is an explicit algebraic function for $i = 0, \ldots, n$, and $f_{i-1}(t_i) = f_i(t_i)$ (possibly both undefined) for $i = 1, \ldots, n$, then there is an explicit algebraic function that is equal to $f_i$ on each $I_i$. 
We want a solution of $X^3 - 3X + x = 0$. Here is a plot of $x = 3X - X^3$. This is a functional instance of casus irreducibilis, an irreducible cubic with three real roots. You can see the three functions, $f_1(x)$, $f_2(x)$, $f_3(x)$, that are the continuous roots of $X^3 - 3X + x$ for $x \in (-2, 2)$. Analytic.
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At the end of his book, van der Waerden goes into the theory of formally real fields: fields where if a sum of squares is zero, then each square is zero; but in his treatment of *casus irreducibilis*, which comes several chapters before, he seems to be dealing with a subfield of the real numbers.
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Hardy gives the polynomial $X^5 - X - x$ for an example of an algebraic function which is not an explicit algebraic function. Hardy says, the “proof is difficult and cannot be attempted here.”
Real valued partial functions
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Commutative semiring (commutative rig?)
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**Theorem**

*If $e$ is an additive idempotent, then so is $ae$. The additive idempotents are exactly the multiples of 0. For each $a$ there is a unique additive idempotent $e = a0$ such that $a + e = a$ and $ae = e$. If $e$ is an additive idempotent, then $\{a : a + e = a$ and $ae = e\}$ is a ring with additive identity $e$ and multiplicative identity $1 + e$.***
Yay!