Intuitionistic notions of boundedness in $\mathbb{N}$

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Abstract

We consider notions of boundedness of subsets of the natural numbers $\mathbb{N}$ that occur when doing mathematics in the context of intuitionistic logic. We obtain a new characterization of the notion of a pseudobounded subset and formulate the closely related notion of a detachably finite subset. We establish metric equivalents for a subset of $\mathbb{N}$ to be detachably finite and to satisfy the ascending chain condition. Following Ishihara, we spell out the relationship between detachable finiteness and sequential continuity. Most of the results do not require countable choice.

1 Pseudobounded subsets

A subset A of the natural numbers $\mathbb{N}$ is **bounded** if A is contained in a finite subset of $\mathbb{N}$. In this paper we consider other notions of boundedness of subsets of $\mathbb{N}$ that occur when working within intuitionistic logic. The goal is to understand a notion introduced by Ishihara [3] who called a subset A of $\mathbb{N}$ **pseudobounded** if for any sequence a_n in A, the sequence a_n/n converges to zero. This seems like a peculiar definition, but we know from [3] that the concept has some interesting connections with sequential continuity of functions between metric spaces.

The following theorem provides some equivalent conditions for a subset of $\mathbb{N}$ to be pseudobounded. Examples of the sequence s_n in the theorem are n, n^2, and $\lfloor\sqrt{n}\rfloor$.

Theorem 1 *Let s_n be a fixed weakly increasing sequence in $\mathbb{N}$ with $\lim s_n = \infty$. A subset A of $\mathbb{N}$ is pseudobounded if and only if each sequence in A is eventually bounded by s_n.*

Proof. Suppose A is pseudobounded and let a_i be a sequence in A. We want to show that a_i is eventually bounded by s_i. We may assume that a_i is weakly increasing. Let $t_n = \min\{i : n \leq s_i\}$. Note that t_n is weakly increasing and that $t_n \leq i$ if and only if $n \leq s_i$. As A is pseudobounded, there exists k in $\mathbb{N}$ such that $a_{t_{n+1}} \leq n$ for all $n \geq k$. So, if $n \geq k$ and $t_n \leq i \leq t_{n+1}$, then $a_i \leq n \leq s_i$. Thus $a_i \leq s_i$ for each $i \geq t_k$.

Conversely, suppose each sequence in A is eventually bounded by s_n, and let a_i be a sequence in A. We want to show that a_i/i converges to zero. We may assume that a_i is weakly increasing. Given m in $\mathbb{N}$, there is k in $\mathbb{N}$ so that $a_{ms_{n+1}} \leq s_n$ for each $n \geq k$. So, if $n \geq k$ and $ms_n \leq i \leq ms_{n+1}$, then $a_i \leq s_n \leq i/m$. Thus $a_i \leq i/m$ for each $i \geq ms_k$, whence a_i/i converges to zero. ■

The author is indebted to the referee for pointing out that the special case of Theorem 1, for $s_n = n - 1$, is proved as Lemma 3 in [4].

The following characterization of pseudobounded can be applied to subsets of arbitrary discrete sets, not just subsets of $\mathbb{N}$. Although it is not clear that the general concept is useful, the fact that the restricted concept can be phrased without reference to the order on $\mathbb{N}$ is worth noting.

Theorem 2 *A subset A of $\mathbb{N}$ is pseudobounded if and only if every nonempty subset of A that is detachable from $\mathbb{N}$ is finite.*

Proof. Suppose A is pseudobounded and D is a nonempty detachable subset of $\mathbb{N}$ that is contained in A. Let $b \in D$ and define $a_n = b$ if $n \notin D$ and $a_n = n$ if $n \in D$. By Theorem 1, eventually $a_n \leq n - 1$ so D is finite.

Conversely, suppose every nonempty detachable subset of $\mathbb{N}$ that is contained in A is finite, and let a_n be sequence in A. By Theorem 1, it suffices to show that $a_n \leq n$ eventually. The nonempty set $D = \{a_n : a_n > n\} \cup \{a_0\}$ is detachable from $\mathbb{N}$ because $i \in D$ if and only if $i = a_0$ or there exists $n < i$ such that $i = a_n$. So D is finite whence eventually $a_n \leq n$. ■

Omitting the requirement that the detachable subset be nonempty results in a slightly stronger notion: A subset A of $\mathbb{N}$ is **detachably finite** if every detachable subset of $\mathbb{N}$ contained in A is finite. So if A is detachably finite, then A is pseudobounded. Conversely, if A is nonempty and pseudobounded, then A is detachably finite because if D is a detachable subset of $\mathbb{N}$ contained in A, then $D \cup \{a\}$ is detachable for any element $a \in A$. An example of a pseudobounded subset of $\mathbb{N}$ that is not detachably finite is obtained by taking a binary sequence b_n with at most one 1 and letting $A = \{n : b_n = 1\}$. The subset A itself is detachable but not necessarily finite. However A is pseudobounded because any sequence of elements of A is constant.

In addition to its uses later on, the notion of detachably finite enjoys a simple property that the notion of pseudobounded lacks: if A is detachably finite, and F is finite, then $A \cup F$ is detachably finite. One could argue that any decent notion of boundedness for subsets of $\mathbb{N}$ should have this property. However, if A is the pseudobounded set of the last paragraph, then $A \cup \{0\}$ need not be pseudobounded: let $a_n = nb_n$.

We leave it to the reader to verify that A is detachably finite if and only if every choice function for the family $S_i = \{0\} \cup \{1 : i \in A\}$ is eventually zero.

The principle **BD**, that nonempty pseudobounded sets are bounded, was introduced in [3] along with the principle **BD-**$\mathbb{N}$ which says that countable nonempty pseudobounded sets are bounded. We get equivalent principles by replacing "nonempty pseudobounded" by the weaker "detachably finite" because if A is detachably finite, then $A \cup \{0\}$ is nonempty pseudobounded. This is convenient, for example in Theorem 13, because we don't have to worry about constructing a *nonempty* set A.

In the context of an arbitrary discrete set, these notions are relative: whether or not A is detachably finite (or pseudobounded) depends on the discrete containing set S. The bigger S is, the easier it is for A to be detachably finite. The hardest thing is for A to be detachably finite in itself: if it is, then as A is detachable from itself, it must be finite. On the other hand, there is a certain amount of invariance: if $A \subset S \subset S'$, and S is detachable from S', then A is detachably finite in S if and only if it is detachably finite in S'.

2 The ascending chain condition

One natural notion of a bounded set goes back to Brouwer [5],[2, 3.4.4]. A set A is **n-bounded** if whenever $a_0, \ldots, a_n$ are in A, then $a_i = a_j$ for some $i < j$. A set is **bounded in number** if it is n-bounded for some natural number

n. The original, more negative, versions of this are weaker for arbitrary sets but equivalent for discrete sets.

A natural boundedness condition for a subset A of $\mathbb{N}$ is the **ascending chain condition** or **ACC**: If $a_0 \leq a_1 \leq a_2 \leq \cdots$ is a sequence in A, then $a_n = a_{n+1}$ for some n. For arbitrary sets the generalization is that A be **ω-bounded**: for any sequence $a_0, a_1, a_2, \ldots$ in A, there are $i < j$ such that $a_i = a_j$.

Theorem 3 *Let A be a subset of $\mathbb{N}$. Then A satisfies the ascending chain condition if and only if A is ω-bounded.*

Proof. One direction is trivial. For the other, suppose that A satisfies the ascending chain condition and that $a_0, a_1, a_2, \ldots$ is a sequence in A. Define a sequence b in A by $b_0 = a_0$ and $b_{n+1} = \max\{a_i : 0 \leq i \leq b_n + 1\}$. Note that $b_1 \geq b_0$ so, by induction, $b_{n+1} \geq b_n$. Moreover, if $b_{n+1} = b_n$, then two of the elements $a_0, a_1, \ldots a_{b_n+1}$, which are all at most $b_{n+1} = b_n$, must be equal. ■

As in the case of pseudobounded, we this theorem gives us an equivalent formulation for the ascending chain condition that does not refer to order.

Theorem 4 *Every pseudobounded subset of $\mathbb{N}$ is ω-bounded.*

Proof. Let A be a pseudobounded subset of $\mathbb{N}$. By the preceding theorem, it suffices to prove that A satisfies the ACC. So suppose a_n is a weakly increasing sequence in A. Because A is a pseudobounded subset of $\mathbb{N}$, eventually $a_n \leq n - 2$. But if $a_n \leq n - 2$, then there exist $i < j \leq n$ such that $a_i = a_j$. ■

It doesn't seem likely that this result generalizes to arbitrary discrete sets. Also, there seems no reason to believe that a pseudobounded subset of $\mathbb{N}$ must be bounded in number.

For the purpose of verifying the ACC, we can start with an ascending chain in **normal form**, that is, if $a_{n+1} = a_n$, then $a_{n+2} = a_{n+1}$. Indeed, let $b_n = a_n$ if $a_1 < a_2 < \cdots < a_n$ and $b_n = b_{n-1}$ otherwise.

The metric space $2^{-A} \cup \{0\}$, for A a subset of $\mathbb{N}$, plays an important role in the study of pseudoboundedness in [3]. We will characterize the ACC for A in terms of the sequential closure of the space 2^{-A}. As we do not assume countable choice, we will pay some attention to the difference between a point being in the closure of a set, a point being the limit of a sequence of elements in a set, and a point being the modulated limit of a sequence of elements in a set. To say that x is the **modulated limit** of a sequence c_n means that there is a function $\omega : \mathbb{N} \to \mathbb{N}$ such that $|c_n - x| \leq 1/m$ for all $n \geq \omega(m)$. The function ω is a modulus of convergence.

We will have occasion to use the following simple-minded lemma about when an element is a modulated limit.

Lemma 5 *Let $b_m \leq 1/m$ be a sequence of positive real numbers. Let S be a subset of a metric space, and x an element of that metric space such that $d(x, s) = b_m$ or $d(x, s) \neq b_m$ for every m and every $s \in S$. Then*

1. *Any sequence in S that converges to x is modulated.*
2. *If S is nonempty and countable, then x is the limit of a sequence in S.*

Proof. Let s_n be a sequence in S that converges to x. For each m, let

$$\omega(m) = \min\{n : d(x, s_k) \leq b_m \text{ for all } k \geq n\}.$$

The indicated minimum exists because the set $\{n : d(x, s_n) > b_m\}$ is detachable and bounded, and $\omega(m) = 1 + \max\{n : d(x, s_n) > b_m\}$.

Now let s_n be a sequence that enumerates S and let i_m be the smallest index such that $d(x, s_{i_m}) \leq b_m$. Then the sequence s_{i_m} converges to x. ■

In particular, if a sequence of rational numbers converges to a rational number, then the convergence is modulated (take $b_m = 1/m$). So, for example, the limit in the original definition of "pseudobounded" is automatically modulated.

In the proof of Lemma 5, and elsewhere, the following principle of unique choice is used: if for each x in X there exists a unique y in Y such that $R(x, y)$, then there is a function $f : X \to Y$ such that $R(x, f(x))$ for all x in X."

Lemma 6 *Let A be a subset of $\mathbb{N}$. If x is in the closure of 2^{-A}, then for each $a \in A$, either $x = 2^{-a}$ or $x \neq 2^{-a}$. If $x \neq 0$, then $x \in 2^{-A}$. If A is countable, then x is a modulated limit of a sequence in 2^{-A}.*

Proof. For the first part, suppose $a \in A$. There exists $a' \in A$ such that $\left|x - 2^{-a'}\right| < 2^{-a-1}$. If $a' \leq a$, then $\left|x - 2^{-a'}\right| < 2^{-a'-1}$, so $x = 2^{-a'}$ and we are done because 2^{-A} is discrete. If $a' > a$, then

$$\left|x - 2^{-a}\right| \geq 2^{-a} - 2^{-a'} - \left|x - 2^{-a'}\right| > 2^{-a} - 2^{-a'} - 2^{-a-1} \geq 0.$$

For the second part, suppose $x \neq 0$, whence $x > 0$ so $x > 2^{-m}$ for some m. Then $x = 2^{-a}$ for some $a = 1, \ldots, m-1$ in A.

The third part follows from Lemma 5 if we show that $|2^{-a} - x|$ is equal to or different from 3^{-m} for each $a \in A$. If $x > 0$ this follows from the second part, so we may assume that $x < \min(|2^{-a} - 3^{-m}|, 2^{-a})$. If $3^{-m} > 2^{-a}$, then $2^{-a} - x < 3^{-m}$ because $x \geq 0$. If $3^{-m} < 2^{-a}$, then $2^{-a} - x > 3^{-m}$ because $x < 3^{-m} - 2^{-a}$. ■

We pause to ask the question whether the closure of 2^{-A} is the same as its sequential closure. Zero is in the closure of 2^{-A} if and only if A is unbounded (in the strong positive sense). Zero is in the sequential closure of 2^{-A} if and only if A contains a countable unbounded subset. So a weaker question is whether every unbounded subset of $\mathbb{N}$ contains a countable unbounded subset. No doubt the answer to that is no (without choice).

Here is the promised characterization of the ascending chain condition.

Theorem 7 *Let A be a subset of $\mathbb{N}$. If A satisfies the ACC, then 2^{-A} is sequentially closed. If 2^{-A} is modulated sequentially closed, then A satisfies the ACC.*

Proof. Suppose A satisfies the ACC, and let x be the limit of the sequence 2^{-a_n}. For each n in $\mathbb{N}$, define n' as follows. If $x < 2^{-a_n}$, then n' is the first index such that $a_{n'} > a_n$. If $x \geq 2^{-a_n}$, then set $n' = n$. Now define the sequence of indices t_n by $t_0 = 0$ and $t_{n+1} = t'_n$, and consider the sequence $b_n = a_{t_n}$. The sequence b_n is in normal form. The ACC says that it is eventually constant. If $b_{n+1} = b_n$, then $x \geq 2^{-b_m}$ for some $m \leq n$, so $x \in 2^{-A}$.

For the converse, suppose 2^{-A} is modulated sequentially closed. An ascending chain $a_1 \leq a_2 \leq a_3 \leq \cdots$ in normal form gives a modulated Cauchy sequence in 2^{-A}. It must converge to an element of 2^{-A}. That gives a bound on how far out we have to look for a pause in the ascending chain. ■

If A is detachable and satisfies the ACC, then 2^{-A} is closed because any element of the closure is the limit of a sequence. With choice, of course, A satisfies the ACC if and only if 2^{-A} is closed.

3 Detachably finite subsets

We now give a characterization of detachably finite subsets like the one for the ascending chain condition..

Theorem 8 *If A is detachably finite, then $2^{-A} \cup \{0\}$ is closed and 2^{-A} is sequentially closed.*

Proof. For x in the closure of $2^{-A} \cup \{0\}$ let $D = \{n : x = 2^{-n}\}$. This is detachable by Lemma 6, so D is finite. If D is empty, then $x = 0$, otherwise $x = 2^{-n}$ for some $n \in A$. The second part follows from Theorems 4 and 7. ■

We can't weaken the hypothesis of this theorem to read that A is pseudo-bounded because if A is a detachable subset with at most one element, then $2^{-A} \cup \{0\}$ need not be closed: consider the sequence $c_n = 0$ if $m \notin A$ for all $m \leq n$, and $c_n = 2^{-m}$ if $m \in A$ for some $m \leq n$. Similarly, it does not suffice to assume that A is bounded in number.

Here is the converse.

Theorem 9 *If $2^{-A} \cup \{0\}$ and 2^{-A} are sequentially closed, then A is detachably finite.*

Proof. Let D be a detachable subset of $\mathbb{N}$ that is contained in A. Given $a \in A$, or $a = -1$, construct a Cauchy sequence c_m in $2^{-A} \cup \{0\}$ as follows. Let $c_m = 2^{-n}$ where n is the least element of D in the interval $a < n < m+1$, if there is such an element, and $c_m = 0$ otherwise. Then c_m converges to zero or to $2^{-a'}$ where $a' \in A$ and $a' > a$. If c_m converges to zero, then D is bounded by a. In that case set $a' = a$. If c_m converges to zero for $a = -1$, then D is empty, hence finite. Otherwise we get a sequence $a \leq a' \leq a'' \leq \cdots$ in A. As 2^{-A} is sequentially closed, the set A satisfies the ACC, by Theorem 7, so the sequence $a \leq a' \leq a'' \leq \cdots$ pauses, whence D is finite. ■

Theorem 10 *If A is bounded in number, then 2^{-A} is closed.*

Proof. If x is in the closure of 2^{-A}, then there exists an element $a_1 \in A$. Given $a_i \in A$, either $x > 0$, in which case $x \in 2^{-A}$ by Lemma 6, or $x < 2^{-a_i}$. In the latter case there exists $a_{i+1} > a_i$ in A. If A is bounded in number by n, this procedure must stop before a_{n+1} is constructed. Hence $x \in 2^{-A}$. ■

Can "bounded in number" be replaced by "detachably finite"? If x is in the closure of 2^{-A}, then either $x \in 2^{-A}$ or $x = 0$, so it suffices to rule out $x = 0$. But zero is in the closure of 2^{-A} if and only if A is unbounded. Can we rule that out for detachably finite A? One might hope so because detachable finiteness is a boundedness notion, but all we can seem to rule out is that A is sequentially unbounded.

Note that 2^{-A} is discrete, but need not be located, let alone totally bounded.

The following theorem gives another characterization of detachable finiteness and establishes a connection between it and sequential continuity.

Theorem 11 *The characteristic function of* 2^{-A} *in* $2^{-A} \cup \{0\}$ *is sequentially continuous if and only if* A *is detachably finite.*

Proof. Suppose the characteristic function χ of 2^{-A} is sequentially continuous and D is a detachable subset of $\mathbb{N}$ contained in A. Define a sequence c_n in $2^{-A} \cup \{0\}$ by $c_n = 0$ if $n \notin D$, and $c_n = 2^{-n}$ if $n \in D$. So $\chi(c_n) = 0$ if $n \notin D$ and $\chi(c_n) = 1$ if $n \in D$. The sequence c_n clearly converges to zero because $c_n \leq 2^{-n}$. If χ is sequentially continuous, then there exists m such that if $n \geq m$, then $n \notin D$ because $\chi(c_n) = 1$ if $n \in D$. So D is finite.

Conversely, suppose A is detachably finite and the sequence $c_n \in 2^{-A} \cup \{0\}$ converges to 0. The set $D = \{a \in A : c_n = 2^{-a} \text{ for some } n\}$ is detachable from $\mathbb{N}$ because, given $m \in \mathbb{N}$, the sequence c_n is eventually less than 2^{-m}. So D is finite whence $\chi(c_n)$ is eventually zero. ■

On the other hand, the characteristic function of Theorem 11 is continuous if and only if A is bounded. So if sequential continuity always implies continuity, then the principle BD holds. Moreover, if we restrict ourselves to countable A, then the spaces in question are separable, so if sequential continuity always implies continuity for separable spaces, then BD-$\mathbb{N}$ holds. This is half of Ishihara's characterization of when sequential continuity implies continuity. The easy part of the other half does not require choice.

Theorem 12 *Suppose* $f : X \to M$ *is a function between metric spaces, and* $x \in X$*. For each* $\varepsilon > 0$*, let*

$$A_\varepsilon = \{n \in \mathbb{N} : \exists y \in X \text{ such that } d(x, y) < 1/n \text{ and } d(f(x), f(y)) > \varepsilon\}$$

Then f *is continuous at* x *if and only if* A_ε *is bounded for each* ε*.*

Proof. If f is continous at x, then there exists n such that if $d(x, y) < 1/n$, then $d(f(x), f(y)) \leq \varepsilon$. Thus A_ε is bounded. Conversely, if A_ε is bounded, then there exists n such that if $d(x, y) < 1/n$, then $d(f(x), f(y)) < 2\varepsilon$. ■

The sequential part requires choice.

Theorem 13 (Choice) *Suppose $f : X \to M$ is a function between metric spaces, and $x \in X$. For each $\varepsilon > 0$, let*

$$A_\varepsilon = \{n \in \mathbb{N} : \exists y \in X \text{ such that } d(x,y) < 1/n \text{ and } d(f(x), f(y)) > \varepsilon\}$$

Then f is sequentially continuous at x if and only if A_ε is detachably finite for each $\varepsilon > 0$.

Proof. Suppose f is sequentially continuous at x and D is a detachable subset of $\mathbb{N}$ contained in A_ε. Define $c_n \in X$ by $c_n = x$ if $n \notin D$, with $d(x, c_n) < 1/n$ and $d(f(x), f(c_n)) > \varepsilon$ if $n \in D$. This uses choice. Clearly c_n converges to x, so $f(c_n)$ converges to $f(x)$. But that implies that D is finite.

Conversely, suppose A_ε is detachably finite and c_n is a sequence such that $d(x, c_n) < 1/n$ for all n. Define a detachable subset D so that if $n \in D$, then $d(f(x), f(c_n)) > \varepsilon$, and if $n \notin D$, then $d(f(x), f(c_n)) < 2\varepsilon$. This uses choice; indeed it uses the "lambda technique" [1]. Then D is finite, so $d(f(x), f(c_n)) < 2\varepsilon$ eventually. If this happens for all $\varepsilon > 0$, then $f(c_n) \to f(x)$. ■

If X is separable and f is sequentially continuous, then A_ε is countable. If there is y in X such that $d(x,y) < 1/n$ and $d(f(x), f(y)) > \varepsilon$, then there is y' in the countable dense subset so that $d(x,y) < 1/n$ and $d(f(x), f(y')) > \varepsilon$. The set of such y' is countable, so A_ε is countable. Thus, for X separable, BD-$\mathbb{N}$ and sequential continuity implies continuity. Conversely, if sequential continuity implies continuity for X separable, then BD-$\mathbb{N}$ holds because if A is detachably finite, then the characteristic function of 2^{-A} in $2^{-A} \cup \{0\}$ is sequentially continuous, hence continuous which means that A is bounded.

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