The simplex algorithm

Fred Richman

Florida Atlantic University

7 November 2012
Maximization problem

\[ \begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\end{array} \]

Minimization problem

\[ \begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\end{array} \]

Solution to maximum problem:

\[ s_1 = s_2 = 0, \quad t_1 = 3, \quad t_2 = 6, \quad v = 9. \]

Solution to minimum problem:

\[ t_1 = t_2 = 0, \quad s_1 = 7, \quad s_2 = 8, \quad v = 9. \]
Maximization problem

\[ \begin{align*}
s_1 + 2s_2 & \leq 3 \\
4s_1 + 5s_2 & \leq 6 \\
\text{maximize} & \quad -7s_1 - 8s_2 + 9
\end{align*} \]

Subject to

\[ \begin{array}{ccc}
st_1 & s_2 & t_1 \\
t_2 & \phantom{t_2} & \phantom{t_2} \\
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{array} \]

Solution to maximum problem:

\[ \begin{align*}
s_1 & = s_2 = 0, \\
t_1 & = 3, \\
t_2 & = 6, \\
v & = 9. \\
\end{align*} \]

Minimization problem

\[ \begin{align*}
t_1 + 4t_2 & \leq 2 \\
t_2 + 5t_2 & \leq 8 \\
\text{minimize} & \quad 3t_1 + 6t_2 + 9 \\
\end{align*} \]

Subject to

\[ \begin{array}{ccc}
st_1 & s_2 & t_1 \\
t_2 & \phantom{t_2} & \phantom{t_2} \\
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{array} \]

Solution to minimum problem:

\[ \begin{align*}
s_1 & = s_2 = 7, \\
t_1 & = t_2 = 0, \\
v & = 9. \\
\end{align*} \]
Maximization problem

\[ \begin{align*}
 s_1 + 2s_2 & \leq 3 \\
 4s_1 + 5s_2 & \leq 6 \\
 \text{maximize} & \quad -7s_1 - 8s_2 + 9
\end{align*} \]

\[ t_1 = -s_1 - 2s_2 + 3 \]
\[ t_2 = -4s_1 - 5s_2 + 6 \]

payoff = \[ -7s_1 - 8s_2 + 9 \]

Minimization problem

\[ \begin{align*}
 t_1 + 4t_2 & \geq -7 \\
 2t_1 + 5t_2 & \geq -8 \\
 \text{minimize} & \quad 3t_1 + 6t_2 + 9
\end{align*} \]

\[ s_1 = t_1 + 4t_2 + 7 \]
\[ s_2 = 2t_1 + 5t_2 + 8 \]

cost = \[ 3t_1 + 6t_2 + 9 \]
Maximization problem

\[ s_1 + 2s_2 \leq 3 \]
\[ 4s_1 + 5s_2 \leq 6 \]
maximize \[ -7s_1 - 8s_2 + 9 \]

\[ t_1 = -s_1 - 2s_2 + 3 \]
\[ t_2 = -4s_1 - 5s_2 + 6 \]

payoff = \[ -7s_1 - 8s_2 + 9 \]

Minimization problem

\[ t_1 + 4t_2 \geq -7 \]
\[ 2t_1 + 5t_2 \geq -8 \]
minimize \[ 3t_1 + 6t_2 + 9 \]

\[ s_1 = t_1 + 4t_2 + 7 \]
\[ s_2 = 2t_1 + 5t_2 + 8 \]

cost = \[ 3t_1 + 6t_2 + 9 \]

Solution to maximum problem: \[ s_1 = s_2 = 0, \ t_1 = 3, \ t_2 = 6, \ \nu = 9. \]
Maximization problem

\[
\begin{align*}
    s_1 + 2s_2 &\leq 3 \\
    4s_1 + 5s_2 &\leq 6 \\
    \text{maximize} & \quad -7s_1 - 8s_2 + 9
\end{align*}
\]

\[
\begin{align*}
    t_1 &= -s_1 - 2s_2 + 3 \\
    t_2 &= -4s_1 - 5s_2 + 6
\end{align*}
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payoff = \( -7s_1 - 8s_2 + 9 \)

Minimization problem

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    t_1 + 4t_2 &\geq -7 \\
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\begin{align*}
    s_1 &= t_1 + 4t_2 + 7 \\
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cost = \( 3t_1 + 6t_2 + 9 \)

Solution to maximum problem: \( s_1 = s_2 = 0, \ t_1 = 3, \ t_2 = 6, \ \nu = 9 \).
Solution to minimum problem: \( t_1 = t_2 = 0, \ s_1 = 7, \ s_2 = 8, \ \nu = 9 \).
Maximum problem is unfeasible: 

\[ s_1 + 2s_2 < 3. \]

Minimum problem is unbounded: cost is 

\[ 3t_1 + 6t_2 + 9. \]

Pivot at the 4.

Solution to max problem:

\[ t_2 = s_2 = 0, \quad t_1 = \frac{3}{2}, \quad s_1 = \frac{3}{2}, \quad v = \frac{39}{2}. \]

Solution to min problem:

\[ t_1 = s_1 = 0, \quad t_2 = \frac{7}{4}, \quad s_2 = \frac{67}{4}, \quad v = \frac{39}{2}. \]
Maximum problem is unfeasible: $s_1 + 2s_2 \leq -3$. 

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Solution to max problem: 
$t_2 = s_2 = 0, t_1 = 3/2, s_1 = 3/2, v = 39/2$. 

Solution to min problem: 
$t_1 = s_1 = 0, t_2 = 7/4, s_2 = 67/4, v = 39/2$. 

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Maximum problem is unfeasible: $s_1 + 2s_2 \leq -3$.
Minimum problem is unbounded: cost is $-3t_1 + 6t_2 + 9$.

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Maximum problem is unfeasible: \( s_1 + 2s_2 \leq -3 \).
Minimum problem is unbounded: cost is \(-3t_1 + 6t_2 + 9\).

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The simplex algorithm
Maximum problem is unfeasible: $s_1 + 2s_2 \leq -3$.

Minimum problem is unbounded: cost is $-3t_1 + 6t_2 + 9$.

$$
\begin{array}{ccc}
  & s_1 & s_2 \\
t_1 & 1 & 2 & -3 \\
t_2 & 4 & 5 & 6 \\
  & 7 & 8 & 9 \\
\end{array}
$$

Pivot at the 4.

$$
\begin{array}{ccc}
  & s_1 & s_2 \\
t_1 & 1 & 2 & 3 \\
t_2 & 4 & 5 & 6 \\
  & -7 & 8 & 9 \\
\end{array}
$$
Maximum problem is unfeasible: \( s_1 + 2s_2 \leq -3 \).
Minimum problem is unbounded: cost is \(-3t_1 + 6t_2 + 9\).

Pivot at the 4.

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<tr>
<td></td>
<td>-1/4</td>
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Solution to max problem: \( t_2 = s_2 = 0 \), \( t_1 = 3/2 \), \( s_1 = 3/2 \), \( v = 39/2 \).

Solution to min problem: \( t_1 = s_1 = 0 \), \( t_2 = 7/4 \), \( s_2 = 67/4 \), \( v = 39/2 \).
Maximum problem is unfeasible: \( s_1 + 2s_2 \leq -3 \).
Minimum problem is unbounded: cost is \(-3t_1 + 6t_2 + 9\).

\[
\begin{array}{ccc|c}
 & s_1 & s_2 \\
\hline 
t_1 & 1 & 2 & -3 \\
t_2 & 4 & 5 & 6 \\
7 & 8 & 9 \\
\end{array}
\]

Pivot at the 4.

\[
\begin{array}{ccc|c}
 & s_1 & s_2 \\
\hline 
t_1 & 1 & 2 & 3 \\
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Solution to max problem: \( t_2 = s_2 = 0, \ t_1 = 3/2, \ s_1 = 3/2, \ v = 39/2 \).
Maximum problem is unfeasible: \( s_1 + 2s_2 \leq -3 \).
Minimum problem is unbounded: cost is \( -3t_1 + 6t_2 + 9 \).

Pivot at the 4.

Solution to max problem: \( t_2 = s_2 = 0, t_1 = \frac{3}{2}, s_1 = \frac{3}{2}, v = \frac{39}{2} \).
Solution to min problem: \( t_1 = s_1 = 0, t_2 = \frac{7}{4}, s_2 = \frac{67}{4}, v = \frac{39}{2} \).
Exercises 3 and 4.

First column: max prob 3 is unbounded.
Second row: max prob 4 is unfeasible.

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Maximum problem is unfeasible: $2s_2 \leq -3$. 
Maximum problem is unfeasible: $2s_2 \leq -3$.
Minimum problem is unfeasible: $-4t_2 \geq 7$. 

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Minimum problem is unfeasible: \(-4t_2 \geq 7\).

Exercises 3 and 4.
Maximum problem is unfeasible: $2s_2 \leq -3$.
Minimum problem is unfeasible: $-4t_2 \geq 7$.

Exercises 3 and 4.
Maximum problem is unfeasible: \(2s_2 \leq -3\).
Minimum problem is unfeasible: \(-4t_2 \geq 7\).

**Exercises 3 and 4.**

```
-2  3  -1  2
0  -1  2  3
-1  0  1  1
-1  1  2  2
```

```
7  5  3  2
6  1  2 -4
-3 -1 -1  1
1  -1  0  0
```
Maximum problem is unfeasible: $2s_2 \leq -3$.
Minimum problem is unfeasible: $-4t_2 \geq 7$.

**Exercises 3 and 4.**

First column: max prob 3 is unbounded.
Maximum problem is unfeasible: $2s_2 \leq -3$.
Minimum problem is unfeasible: $-4t_2 \geq 7$.

Exercises 3 and 4.

First column: max prob 3 is unbounded.
Second row: max prob 4 is unfeasible.
Yay!