Noisy nets and transient limits

Fred Richman

Florida Atlantic University

29 August 2019

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Paul Halmos $\longrightarrow$ Errett Bishop $\longrightarrow$ Y.K. Chan

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Schizophrenia in contemporary mathematics, August 1973

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Our interest is in long-term collective opinion; in particular, the possible persistence of a minority opinion.

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$$\begin{matrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{matrix}$$

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At the next time the opinions of the first 5 individuals will switch to ■■□□□.

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Alternatively, we could posit that the configuration ■■□□■□ is a circle—the first individual being adjacent to the last. In that case the state would switch to ■■□□□■, which is where it would stay.

Discrete dynamical systems

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For $m = 4$ these are exactly the six strings

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There is also a pair of strings of period two (only for m even):

■□■□ $\longleftrightarrow$ □■□■

The 12 fixed points for the 5-circle are

□■■■□ □■■□□ □□■■□ □□■■■ □□□■■ □□□□□

and their complements.

The 12 fixed points for the 5-circle are

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The 20 fixed points for the 6-circle are

□■■■■□ □■■□□□ □■■■□□ □□■■■□ □□□■■□
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There are 122 fixed points for the 10-circle
and 2206 fixed points for the 16-circle.

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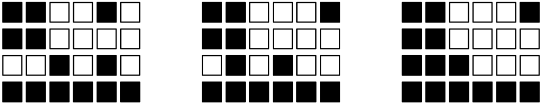
Here's a 4-by-6 example and how it steps to a fixed point:



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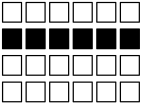

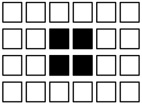

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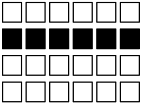

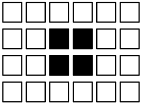

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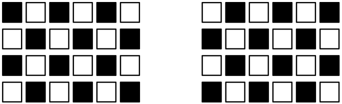

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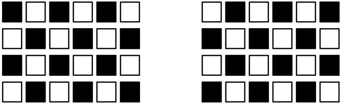

If we change these states of period two by filling in all four squares in the upper right corner, we get another pair of states of period two:

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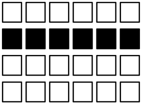
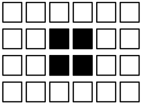

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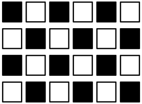
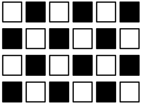

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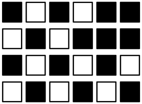
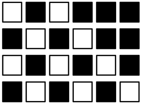

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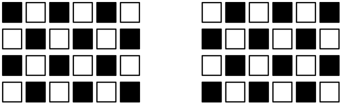

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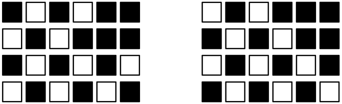

That's essentially the same as filling in the four central squares.

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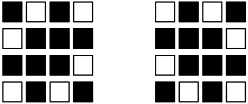

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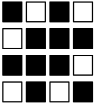
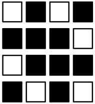
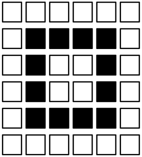
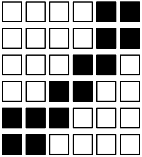




Here are two fixed points of the 6-by-6 torus:





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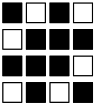

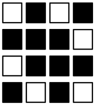

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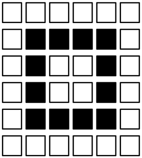
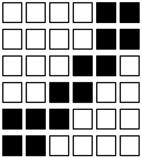
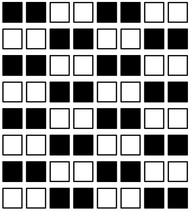




and a point of period two of the 8-by-8 torus:



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Then pass to the parameterized density function

$$\frac{1}{\sigma} f\left(\frac{x}{\sigma}\right)$$

Here are graphs of the Gaussian density function for $\sigma = 1$ and $\sigma = 2$.

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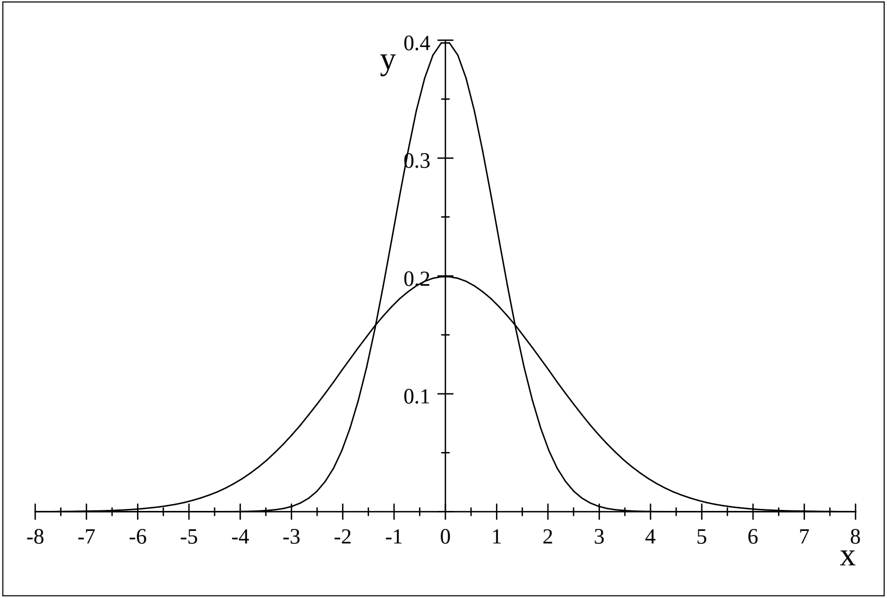

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For the Gaussian distribution, $[4](1) = 0.00003167$ and $[4](2) = 0.02275$.

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	000	001	010	011	100	101	110	111
000	$\bar{2}\bar{3}\bar{2}$	$\bar{2}\bar{3}2$	$\bar{2}3\bar{2}$	$\bar{2}32$	$2\bar{3}\bar{2}$	$2\bar{3}2$	$23\bar{2}$	232
001	$\bar{2}\bar{1}0$	$\bar{2}\bar{1}0$	$\bar{2}10$	$\bar{2}10$	$2\bar{1}0$	$2\bar{1}0$	210	210
010	$0\bar{2}0$	$0\bar{2}0$	020	020	$0\bar{2}0$	$0\bar{2}0$	020	020
011	012	$01\bar{2}$	$0\bar{1}2$	$0\bar{1}\bar{2}$	012	$01\bar{2}$	$0\bar{1}2$	$0\bar{1}\bar{2}$
100	$0\bar{1}\bar{2}$	$0\bar{1}2$	$01\bar{2}$	012	$0\bar{1}\bar{2}$	$0\bar{1}2$	$01\bar{2}$	012
101	020	020	$0\bar{2}0$	$0\bar{2}0$	020	020	$0\bar{2}0$	$0\bar{2}0$
110	210	210	$2\bar{1}0$	$2\bar{1}0$	$\bar{2}10$	$\bar{2}10$	$\bar{2}\bar{1}0$	$\bar{2}\bar{1}0$
111	232	$23\bar{2}$	$2\bar{3}2$	$2\bar{3}\bar{2}$	$\bar{2}32$	$\bar{2}3\bar{2}$	$\bar{2}\bar{3}2$	$\bar{2}\bar{3}\bar{2}$

Here is the matrix $A(0)$ for the three-node line.

	000	001	010	011	100	101	110	111
000	1	0	0	0	0	0	0	0
001	1/2	1/2	0	0	0	0	0	0
010	1/4	1/4	0	0	1/4	1/4	0	0
011	0	0	0	1/2	0	0	0	1/2
100	1/2	0	0	0	1/2	0	0	0
101	0	0	1/4	1/4	0	0	1/4	1/4
110	0	0	0	0	0	0	1/2	1/2
111	0	0	0	0	0	0	0	1

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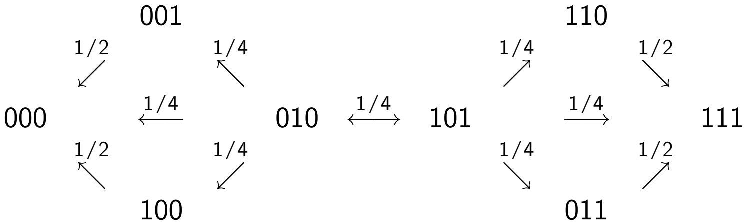
	000	001	010	011	100	101	110	111
000	1	0	0	0	0	0	0	0
001	1/2	1/2	0	0	0	0	0	0
010	1/4	1/4	0	0	1/4	1/4	0	0
011	0	0	0	1/2	0	0	0	1/2
100	1/2	0	0	0	1/2	0	0	0
101	0	0	1/4	1/4	0	0	1/4	1/4
110	0	0	0	0	0	0	1/2	1/2
111	0	0	0	0	0	0	0	1

It represents the Markov chain



In the limit, this Markov chain is 000 $\overset{3/4}{\longleftarrow}$ 010 $\overset{1/4}{\longleftrightarrow}$ 101 $\overset{3/4}{\longrightarrow}$ 111 so 010 becomes 000 with probability

$$3/4\left(1+1/16+1/16^{2}+\cdots\right)=\frac{3\cdot 16}{4\left(16-1\right)}=\frac{4}{5}$$

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	000	001	010	011	100	101	110	111
000	1	0	0	0	0	0	0	0
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010	4/5	0	0	0	0	0	0	1/5
011	0	0	0	0	0	0	0	1
100	1	0	0	0	0	0	0	0
101	1/5	0	0	0	0	0	0	4/5
110	0	0	0	0	0	0	0	1
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101	1/5	0	0	0	0	0	0	4/5
110	0	0	0	0	0	0	0	1
111	0	0	0	0	0	0	0	1

This is actually a transient limit for powers of $A\left(\sigma\right)$.

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If $A(\sigma)$ is an indexed (Markov) matrix, then L is a **transient limit** of the sequence $A^n(\sigma)$ if there is a window $u \leq v$ such that for $u \leq n \leq v$, the matrices $A^n(\sigma)$ and L are close for all sufficiently small values of σ.

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Note that $\lim_{\sigma\to 0}(1-\sigma)^{1/\sqrt{\sigma}} = 1$ and $\lim_{\sigma\to 0}(1-\sigma)^{1/\sigma^2} = 0$.

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The transition matrix A is

$$\begin{pmatrix} 1-s_1 & s_1 & 0 & \cdots & 0 & 0 \\ 0 & 1-s_2 & s_2 & \cdots & 0 & 0 \\ 0 & 0 & 1-s_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1-s_{m-1} & s_{m-1} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

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In the window defined by $u = 1$ and $v = \sqrt{1/\left[1\right](\sigma)}$, the transient limit is the matrix

	000	001	010	011	100	101	110	111
000	1	0	0	0	0	0	0	0
001	1	0	0	0	0	0	0	0
010	4/5	0	0	0	0	0	0	1/5
011	0	0	0	0	0	0	0	1
100	1	0	0	0	0	0	0	0
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What about ■□■□■□? There are two rows where ■□■□■□ and □■□■□■ each has an entry that is approximately 1/2. In the algorithm of the first paper, we would create the *mixed state* {■□■□■□, □■□■□■}.

Example 4. The 4-by-4 torus. Let $\tau = \frac{1}{[2](\sigma)}$ and consider the window $u = \tau$ and $v = \tau^2$.

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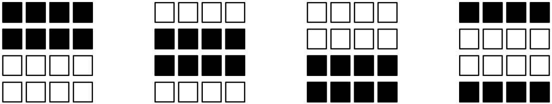
The (four) states that are visible in that window are



together with their transposes, and the two homogeneous states.

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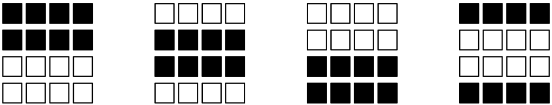

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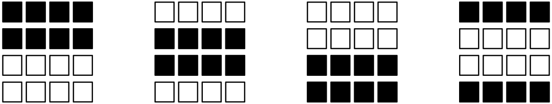

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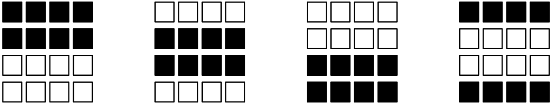

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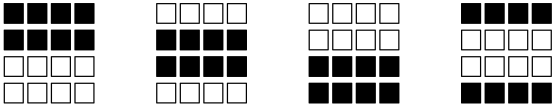
"Visible" in our papers refers to the limiting stationary distribution, which for the 4-by-4 torus is 1/2 on each homogeneous state. A state is **visible in a window** if the Markov matrix has a transient limit in that window with a positive entry in the column corresponding to that state.

Example 4. The 4-by-4 torus. Let $\tau = \frac{1}{[2](\sigma)}$ and consider the window $u = \tau$ and $v = \tau^2$.

The (four) states that are visible in that window are



together with their transposes, and the two homogeneous states.

That is, those states in which each node is next to at least three like-minded nodes.

"Visible" in our papers refers to the limiting stationary distribution, which for the 4-by-4 torus is $1/2$ on each homogeneous state. A state is **visible in a window** if the Markov matrix has a transient limit in that window with a positive entry in the column corresponding to that state. I think :)

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Yay!

Jessica Su