6.2 Normal distribution

Mean = $\mu$
Standard Deviation = $\sigma$

Donation: $X \sim N(\mu, \sigma^2)$
Heights of Adult Men and Women

Women:
\[ \mu = 63.6 \]
\[ \sigma = 2.5 \]

Men:
\[ \mu = 69.0 \]
\[ \sigma = 2.8 \]
Standard Normal Distribution:
a normal probability distribution that has a mean of 0 and a standard deviation of 1.

Notation: $Z \sim N(0,1)$
<table>
<thead>
<tr>
<th>$z$</th>
<th>.00</th>
<th>.01</th>
<th>.02</th>
<th>.03</th>
<th>.04</th>
<th>.05</th>
<th>.06</th>
<th>.07</th>
<th>.08</th>
<th>.09</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>.5000</td>
<td>.5040</td>
<td>.5080</td>
<td>.5120</td>
<td>.5160</td>
<td>.5199</td>
<td>.5239</td>
<td>.5279</td>
<td>.5319</td>
<td>.5359</td>
</tr>
<tr>
<td>0.1</td>
<td>.5398</td>
<td>.5438</td>
<td>.5478</td>
<td>.5517</td>
<td>.5557</td>
<td>.5596</td>
<td>.5636</td>
<td>.5675</td>
<td>.5714</td>
<td>.5753</td>
</tr>
<tr>
<td>0.2</td>
<td>.5793</td>
<td>.5832</td>
<td>.5871</td>
<td>.5910</td>
<td>.5948</td>
<td>.5987</td>
<td>.6026</td>
<td>.6064</td>
<td>.6103</td>
<td>.6141</td>
</tr>
<tr>
<td>0.3</td>
<td>.6179</td>
<td>.6217</td>
<td>.6255</td>
<td>.6293</td>
<td>.6331</td>
<td>.6368</td>
<td>.6406</td>
<td>.6443</td>
<td>.6480</td>
<td>.6517</td>
</tr>
<tr>
<td>0.4</td>
<td>.6554</td>
<td>.6591</td>
<td>.6628</td>
<td>.6664</td>
<td>.6700</td>
<td>.6736</td>
<td>.6772</td>
<td>.6808</td>
<td>.6844</td>
<td>.6879</td>
</tr>
<tr>
<td>0.5</td>
<td>.6915</td>
<td>.6950</td>
<td>.6985</td>
<td>.7019</td>
<td>.7054</td>
<td>.7088</td>
<td>.7123</td>
<td>.7157</td>
<td>.7190</td>
<td>.7224</td>
</tr>
<tr>
<td>0.6</td>
<td>.7257</td>
<td>.7291</td>
<td>.7324</td>
<td>.7357</td>
<td>.7389</td>
<td>.7422</td>
<td>.7454</td>
<td>.7486</td>
<td>.7517</td>
<td>.7549</td>
</tr>
<tr>
<td>0.7</td>
<td>.7580</td>
<td>.7611</td>
<td>.7642</td>
<td>.7673</td>
<td>.7704</td>
<td>.7734</td>
<td>.7764</td>
<td>.7794</td>
<td>.7823</td>
<td>.7852</td>
</tr>
<tr>
<td>0.8</td>
<td>.7881</td>
<td>.7910</td>
<td>.7939</td>
<td>.7967</td>
<td>.7995</td>
<td>.8023</td>
<td>.8051</td>
<td>.8078</td>
<td>.8106</td>
<td>.8133</td>
</tr>
<tr>
<td>0.9</td>
<td>.8159</td>
<td>.8186</td>
<td>.8212</td>
<td>.8238</td>
<td>.8264</td>
<td>.8289</td>
<td>.8315</td>
<td>.8340</td>
<td>.8365</td>
<td>.8389</td>
</tr>
<tr>
<td>1.0</td>
<td>.8413</td>
<td>.8438</td>
<td>.8461</td>
<td>.8485</td>
<td>.8508</td>
<td>.8531</td>
<td>.8554</td>
<td>.8577</td>
<td>.8599</td>
<td>.8621</td>
</tr>
<tr>
<td>1.1</td>
<td>.8643</td>
<td>.8665</td>
<td>.8686</td>
<td>.8708</td>
<td>.8729</td>
<td>.8749</td>
<td>.8770</td>
<td>.8790</td>
<td>.8810</td>
<td>.8830</td>
</tr>
<tr>
<td>1.2</td>
<td>.8849</td>
<td>.8869</td>
<td>.8888</td>
<td>.8907</td>
<td>.8925</td>
<td>.8944</td>
<td>.8962</td>
<td>.8980</td>
<td>.8997</td>
<td>.9015</td>
</tr>
<tr>
<td>1.3</td>
<td>.9032</td>
<td>.9049</td>
<td>.9066</td>
<td>.9082</td>
<td>.9099</td>
<td>.9115</td>
<td>.9131</td>
<td>.9147</td>
<td>.9162</td>
<td>.9177</td>
</tr>
<tr>
<td>1.4</td>
<td>.9192</td>
<td>.9207</td>
<td>.9222</td>
<td>.9236</td>
<td>.9251</td>
<td>.9265</td>
<td>.9279</td>
<td>.9292</td>
<td>.9306</td>
<td>.9319</td>
</tr>
<tr>
<td>1.5</td>
<td>.9332</td>
<td>.9345</td>
<td>.9357</td>
<td>.9370</td>
<td>.9382</td>
<td>.9394</td>
<td>.9406</td>
<td>.9418</td>
<td>.9429</td>
<td>.9441</td>
</tr>
<tr>
<td>1.6</td>
<td>.9452</td>
<td>.9463</td>
<td>.9474</td>
<td>.9484</td>
<td>.9495</td>
<td>*9505</td>
<td>.9515</td>
<td>.9525</td>
<td>.9535</td>
<td>.9545</td>
</tr>
<tr>
<td>1.7</td>
<td>.9554</td>
<td>.9564</td>
<td>.9573</td>
<td>.9582</td>
<td>.9591</td>
<td>.9599</td>
<td>.9608</td>
<td>.9616</td>
<td>.9625</td>
<td>.9633</td>
</tr>
<tr>
<td>1.8</td>
<td>.9641</td>
<td>.9649</td>
<td>.9656</td>
<td>.9664</td>
<td>.9671</td>
<td>.9678</td>
<td>.9686</td>
<td>.9693</td>
<td>.9699</td>
<td>.9706</td>
</tr>
<tr>
<td>1.9</td>
<td>.9713</td>
<td>.9719</td>
<td>.9726</td>
<td>.9732</td>
<td>.9738</td>
<td>.9744</td>
<td>.9750</td>
<td>.9756</td>
<td>.9761</td>
<td>.9767</td>
</tr>
<tr>
<td>2.0</td>
<td>.9772</td>
<td>.9778</td>
<td>.9783</td>
<td>.9788</td>
<td>.9793</td>
<td>*9798</td>
<td>.9803</td>
<td>.9808</td>
<td>.9812</td>
<td>.9817</td>
</tr>
<tr>
<td>2.1</td>
<td>.9821</td>
<td>.9826</td>
<td>.9830</td>
<td>.9834</td>
<td>.9838</td>
<td>.9842</td>
<td>.9846</td>
<td>.9850</td>
<td>.9854</td>
<td>.9857</td>
</tr>
<tr>
<td>2.2</td>
<td>.9861</td>
<td>.9864</td>
<td>.9868</td>
<td>.9871</td>
<td>.9875</td>
<td>.9878</td>
<td>.9881</td>
<td>.9884</td>
<td>.9887</td>
<td>.9890</td>
</tr>
</tbody>
</table>
Example: If thermometers have an average (mean) reading of 0 degrees and a standard deviation of 1 degree for freezing water and if one thermometer is randomly selected, find the probability that, at the freezing point of water, the reading is less than 1.58 degrees. (Assuming the reading at freezing water is normally distributed.)

\[ P(Z < 1.58) = 0.9429 \]

The probability that the chosen thermometer will measure freezing water less than 1.58 degrees is 0.9429.
<table>
<thead>
<tr>
<th>$z$</th>
<th>.00</th>
<th>.01</th>
<th>.02</th>
<th>.03</th>
<th>.04</th>
<th>.05</th>
<th>.06</th>
<th>.07</th>
<th>.08</th>
<th>.09</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-3.50$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-3.4$</td>
<td>.0001</td>
<td>.0003</td>
<td>.0003</td>
<td>.0003</td>
<td>.0003</td>
<td>.0003</td>
<td>.0003</td>
<td>.0003</td>
<td>.0003</td>
<td>.0002</td>
</tr>
<tr>
<td>$-3.3$</td>
<td>.0005</td>
<td>.0005</td>
<td>.0005</td>
<td>.0004</td>
<td>.0004</td>
<td>.0004</td>
<td>.0004</td>
<td>.0004</td>
<td>.0004</td>
<td>.0003</td>
</tr>
<tr>
<td>$-3.2$</td>
<td>.0007</td>
<td>.0007</td>
<td>.0006</td>
<td>.0006</td>
<td>.0006</td>
<td>.0006</td>
<td>.0006</td>
<td>.0006</td>
<td>.0005</td>
<td>.0005</td>
</tr>
<tr>
<td>$-3.1$</td>
<td>.0010</td>
<td>.0009</td>
<td>.0009</td>
<td>.0009</td>
<td>.0008</td>
<td>.0008</td>
<td>.0008</td>
<td>.0008</td>
<td>.0007</td>
<td>.0007</td>
</tr>
<tr>
<td>$-3.0$</td>
<td>.0013</td>
<td>.0013</td>
<td>.0013</td>
<td>.0012</td>
<td>.0012</td>
<td>.0011</td>
<td>.0011</td>
<td>.0011</td>
<td>.0010</td>
<td>.0010</td>
</tr>
<tr>
<td>$-2.9$</td>
<td>.0019</td>
<td>.0018</td>
<td>.0018</td>
<td>.0017</td>
<td>.0016</td>
<td>.0015</td>
<td>.0015</td>
<td>.0014</td>
<td>.0014</td>
<td>.0014</td>
</tr>
<tr>
<td>$-2.8$</td>
<td>.0026</td>
<td>.0025</td>
<td>.0024</td>
<td>.0023</td>
<td>.0023</td>
<td>.0022</td>
<td>.0021</td>
<td>.0021</td>
<td>.0020</td>
<td>.0019</td>
</tr>
<tr>
<td>$-2.7$</td>
<td>.0035</td>
<td>.0034</td>
<td>.0033</td>
<td>.0032</td>
<td>.0031</td>
<td>.0030</td>
<td>.0029</td>
<td>.0028</td>
<td>.0027</td>
<td>.0026</td>
</tr>
<tr>
<td>$-2.6$</td>
<td>.0047</td>
<td>.0045</td>
<td>.0044</td>
<td>.0043</td>
<td>.0041</td>
<td>.0040</td>
<td>.0039</td>
<td>.0038</td>
<td>.0037</td>
<td>.0036</td>
</tr>
<tr>
<td>$-2.5$</td>
<td>.0062</td>
<td>.0060</td>
<td>.0059</td>
<td>.0057</td>
<td>.0055</td>
<td>.0054</td>
<td>.0052</td>
<td>.0051</td>
<td>.0049</td>
<td>.0048</td>
</tr>
<tr>
<td>$-2.4$</td>
<td>.0082</td>
<td>.0080</td>
<td>.0078</td>
<td>.0075</td>
<td>.0073</td>
<td>.0071</td>
<td>.0069</td>
<td>.0068</td>
<td>.0066</td>
<td>.0064</td>
</tr>
<tr>
<td>$-2.3$</td>
<td>.0107</td>
<td>.0104</td>
<td>.0102</td>
<td>.0099</td>
<td>.0096</td>
<td>.0094</td>
<td>.0091</td>
<td>.0089</td>
<td>.0087</td>
<td>.0084</td>
</tr>
<tr>
<td>$-2.2$</td>
<td>.0139</td>
<td>.0136</td>
<td>.0132</td>
<td>.0129</td>
<td>.0125</td>
<td>.0122</td>
<td>.0119</td>
<td>.0116</td>
<td>.0113</td>
<td>.0110</td>
</tr>
<tr>
<td>$-2.1$</td>
<td>.0179</td>
<td>.0174</td>
<td>.0170</td>
<td>.0166</td>
<td>.0162</td>
<td>.0158</td>
<td>.0154</td>
<td>.0150</td>
<td>.0146</td>
<td>.0143</td>
</tr>
<tr>
<td>$-2.0$</td>
<td><strong>.0228</strong></td>
<td>.0222</td>
<td>.0217</td>
<td>.0212</td>
<td>.0207</td>
<td>.0202</td>
<td>.0197</td>
<td>.0192</td>
<td>.0188</td>
<td>.0183</td>
</tr>
<tr>
<td>$-1.9$</td>
<td>.0287</td>
<td>.0281</td>
<td>.0274</td>
<td>.0268</td>
<td>.0262</td>
<td>.0256</td>
<td>.0250</td>
<td>.0244</td>
<td>.0239</td>
<td>.0233</td>
</tr>
<tr>
<td>$-1.8$</td>
<td>.0359</td>
<td>.0351</td>
<td>.0344</td>
<td>.0336</td>
<td>.0329</td>
<td>.0322</td>
<td>.0314</td>
<td>.0307</td>
<td>.0301</td>
<td>.0294</td>
</tr>
<tr>
<td>$-1.7$</td>
<td>.0446</td>
<td>.0436</td>
<td>.0427</td>
<td>.0418</td>
<td>.0409</td>
<td>.0401</td>
<td>.0392</td>
<td>.0384</td>
<td>.0375</td>
<td>.0367</td>
</tr>
<tr>
<td>$-1.6$</td>
<td>.0548</td>
<td>.0537</td>
<td>.0526</td>
<td>.0516</td>
<td>.0505</td>
<td>.0495</td>
<td>.0485</td>
<td>.0475</td>
<td>.0465</td>
<td>.0455</td>
</tr>
<tr>
<td>$-1.5$</td>
<td>.0668</td>
<td>.0655</td>
<td>.0643</td>
<td>.0630</td>
<td>.0618</td>
<td>.0606</td>
<td>.0594</td>
<td>.0582</td>
<td>.0571</td>
<td>.0559</td>
</tr>
</tbody>
</table>

**...**

**-1.2** | **.1093**
**Example:** If thermometers have an average (mean) reading of 0 degrees and a standard deviation of 1 degree for freezing water, and if one thermometer is randomly selected, find the probability that it reads (at the freezing point of water) above $-1.23$ degrees.

The probability that the chosen thermometer with a reading above $-1.23$ degrees is $0.8907$. 

$$P(z > -1.23) = 0.8907$$
Example: A thermometer is randomly selected. Find the probability that it reads (at the freezing point of water) between –2.00 and 1.50 degrees.

(1) \( P (Z < -2.00) = 0.0228 \)
(2) \( P (Z < 1.50) = 0.9332 \)
(3) \( P (-2.00 < Z < 1.50) = 0.9332 - 0.0228 = 0.9104 \)

The probability that the chosen thermometer has a reading between –2.00 and 1.50 degrees is 0.9104. If many thermometers are selected and tested at the freezing point of water, then 91.04% of them will read between –2.00 and 1.50 degrees.
Inverse problem: Finding $z$ Scores when Given Probabilities

Finding the 95th Percentile $= 1.645$

(z score will be positive)
Finding $z$ Scores when Given Probabilities

(One $z$ score will be negative and the other positive)

Finding the Bottom 2.5% and Upper 2.5%
Finding $z$ Scores when Given Probabilities

(One $z$ score will be negative and the other positive)

Finding the Bottom 2.5% and Upper 2.5%
Thus, the bottom 2.5% values are less than $-1.96$ and the upper 2.5% values are bigger than 1.96.

(One $z$ score will be negative and the other positive)
Example: What IQ Do You Need to Get Into Mensa?

- Mensa is a society of high-IQ people whose members have a score on an IQ test at the 98th percentile or higher.
Example: What IQ Do You Need to Get Into Mensa?

- How many standard deviations above the mean is the 98th percentile?
  - The cumulative probability of 0.980 in the body of Table A corresponds to $z = 2.05$ (inside out method).
  - The 98th percentile is 2.05 standard deviations above $\mu$. 
Example: What IQ Do You Need to Get Into Mensa?

- What is the IQ for that percentile?
  
  - Since $\mu = 100$ and $\sigma = 16$, the 98th percentile of IQ equals:
    
    $$\mu + 2.05\sigma = 100 + 2.05(16) = 133$$
Z-Score for a Value of a Random Variable

• The z-score for a value of a random variable is the number of standard deviations that x falls from the mean µ.

• It is calculated as:

\[ Z = \frac{X - \mu}{\sigma} \]
Example: Finding Your Relative Standing on The SAT

- Scores on the verbal or math portion of the SAT are approximately normally distributed with mean $\mu = 500$ and standard deviation $\sigma = 100$. The scores range from 200 to 800.
Example: Finding Your Relative Standing on The SAT

- If one of your SAT scores was $x = 650$, how many standard deviations from the mean was it?

$$z = \frac{x - \mu}{\sigma} = \frac{650 - 500}{100} = 1.50$$
Example: Finding Your Relative Standing on The SAT

- What percentage of SAT scores was higher than yours?
  - Find the cumulative probability for the z-score of 1.50 from Table A.
  - The cumulative probability is 0.9332.
  - The probability above 650 is 
    \[ 1 - 0.9332 = 0.0668 \]
  - About 6.7% of SAT scores are higher than yours.
Example: What Proportion of Students Get A Grade of B?

On the midterm exam in introductory statistics, an instructor always give a grade of B to students who score between 80 and 90.

One year, the scores on the exam have approximately a normal distribution with mean 83 and standard deviation 5.

About what proportion of students get a B?
Example: What Proportion of Students Get A Grade of B?

- Calculate the z-score for 80 and for 90:

\[
Z = \frac{x - \mu}{\sigma} = \frac{90 - 83}{5} = 1.40
\]

\[
Z = \frac{x - \mu}{\sigma} = \frac{80 - 83}{5} = -0.60
\]
Example: What Proportion of Students Get A Grade of B?

- Look up the cumulative probabilities in Table A.
  - For $z = 1.40$, cum. Prob. = 0.9192
  - For $z = -0.60$, cum. Prob. = 0.2743

- It follows that about $0.9192 - 0.2743 = 0.6449$, or about 64% of the exam scores were in the ‘B’ range.
Using z-scores to Find Normal Probabilities

- If we’re given a value $x$ and need to find a probability, convert $x$ to a z-score using:

$$Z = \frac{x - \mu}{\sigma}$$

- Use a table of normal probabilities to get a cumulative probability.
- Convert it to the probability of interest.
Using z-scores to Find Random Variable x Values

• If we’re given a probability and need to find the value of x, convert the probability to the related cumulative probability.

• Find the z-score using a normal table.

• Evaluate \( x = z\sigma + \mu \).
Example: How Can We Compare Test Scores That Use Different Scales?

• When you applied to college, you scored 650 on an SAT exam, which had mean $\mu = 500$ and standard deviation $\sigma = 100$.
• Your friend took the comparable ACT in 2001, scoring 30. That year, the ACT had $\mu = 21.0$ and $\sigma = 4.7$.
• How can we tell who did better?
What is the z-score for your SAT score of 650?

For the SAT scores: $\mu = 500$ and $\sigma = 100$.

a. 2.15
b. 1.50

c. -1.75
d. -1.25
What percentage of students scored higher than you?

a. 10%
b. 5%
c. 2%
d. 7%
What is the z-score for your friend’s ACT score of 30?

The ACT scores had a mean of 21 and a standard deviation of 4.7.

a. 1.84  

b. -1.56  

c. 1.91  

d. -2.24
What percentage of students scored higher than your friend?

a. 3%

b. 6%

c. 10%

d. 1%
The standard normal distribution is the normal distribution with mean $\mu = 0$ and standard deviation $\sigma = 1$.

It is the distribution of normal z-scores.