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A distance between orbits that controls commutator estimates and invertibility of operators. (English. English summary)

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The most common methods of interpolation of Banach spaces, the real method and the complex method, each produce a parametrized family of Banach spaces. That is, given a pair of Banach spaces, X_i, $i = 0, 1$, these methods produce families of Banach spaces, $\{X_\gamma\}$, with the property that any linear operator T bounded on both X_i will also be bounded on all the X_γ. More refined results show that there is some continuity with respect to the parameter γ. In particular, there is a result of I. Ya. Shneiberg [Dokl. Akad. Nauk SSSR 212 (1973), 57–59; MR0636330 (58 #30443)] that if T is invertible on some X_γ then it is also invertible on $X_{\gamma'}$ for γ' near γ. This result and variations and extensions have been used frequently in potential theory, although not always with cognizance of Shneiberg's contribution. Recent developments and further references are in [N. J. Kalton and M. Mitrea, Trans. Amer. Math. Soc. 350 (1998), no. 10, 3903–3922; MR1443193 (98m:46094)]. Other results can also be obtained by, in effect, differentiating the boundedness estimates with respect to the parameter. For instance, in [R. Rochberg and G. Weiss, Ann. of Math. (2) 118 (1983), no. 2, 315–347; MR0717826 (86a:46099)] it is shown that if T is bounded on both X_i then on each of the spaces X_γ obtained by complex interpolation the commutator $[T, \Omega_\gamma]$ is also bounded. Here Ω_γ is an operator associated to the interpolation construction which is generally unbounded and may be nonlinear. Such results have proven useful in nonlinear analysis; see, for instance, Chapter 13 of [T. Iwaniec and G. J. Martin, Geometric function theory and non-linear analysis, Oxford Univ. Press, New York, 2001; MR1859913 (2003c:30001)].

Although less used in practice, the orbital method of interpolation of N. Aronszajn and E. Gagliardo [Ann. Mat. Pura Appl. (4) 68 (1965), 51–117; MR0226361 (37 #1951)] is conceptually more fundamental. However, that method produces single spaces, not parametrized families. Thus it was not clear if, or how, the parametric results extend to this situation.

Here the authors develop versions of the result of Shneiberg and of the commutator estimates for spaces obtained by the method of orbits generated by a single element. They begin by introducing a metric to quantify the idea of two interpolation spaces being close to each other. They then prove a finite version of the commutator theorem concerning spaces which are close to each other and an infinitesimal version concerning maps of a space to itself. The finite version is used later to start an iterative procedure used in proving the surjectivity half of their version of Shneiberg's theorem.

When formulating their infinitesimal commutator theorem the authors need, in effect, a notion of infinitesimal deformation of the Banach space structure. They construct a Banach space of such deformations which they name the "Benson space". As sometimes happens in highly abstract treatments, the main theorem follows quite easily from well-selected definitions, but much further work is required to instantiate the abstract definitions in cases of interest. Thus the authors spend substantial effort obtaining concrete descriptions of various Benson spaces. They also show that the metric they have introduced is compatible with the parameter metric of real or complex interpolation, and, more generally, the parametric structure of the methods in [M. Cwikel et al., Adv. Math. 169 (2002), no. 2, 241–312; MR1926224 (2003k:46104)]. With these tools in hand they show that their results both generalize and provide alternative developments of the earlier results.

These are substantial results and they are quite satisfying aesthetically. Furthermore, beyond the particulars of the theorems and proofs, there is a great deal going on here conceptually. The ideas and techniques in this paper represent a fundamental expansion of the theory of interpolation of linear operators. *Richard Rochberg* (1-WASN)

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