IKP and Friends

Robert S. Lubarsky*
1755 NE 18th St.
Ft. Lauderdale, FL 33305
USA
Robert.Lubarsky@alum.mit.edu
Robert.Lubarsky@att.net

December 2001

1 Introduction

There has been increasing interest in intuitionistic methods over the years. Still, there has been relatively little work on intuitionistic set theory, and most of that has been on intuitionistic ZF. This investigation is about intuitionistic admissibility and theories of similar strength.

There are several more particular goals for this paper. One is just to get some more Kripke models of various set theories out there. Those papers that have dealt with IZF usually were more proof-theoretic in nature, and did not provide models. Furthermore, the inspirations for many of the constructions here are classical forcing arguments. Although the correspondence between the forcing and the Kripke constructions are not made tight, the relationship between these two methods is of interest (see [6] for instance) and some examples, even if only suggestive, should help us better understand the relationship between forcing and Kripke constructions. Along different lines, the subject of least and greatest fixed points of inductive definitions, while of interest to computer scientists, has yet to be studied constructively, and probably holds some surprises. Admissibility is of course the proper set-theoretic context for this study. Finally, while most of the classical material referred to here has long been standard, some of it has not been well codified and may even be unknown, so along the way we'll even fill in a gap in the classical literature.

*I would like to thank Anil Nerode for bringing the subject of intuitionistic set theory to my attention, Michael Rathjen for his encouragement in bringing this work to completion, and the referee for the careful reading and insistence on including the details, even when I didn't think they were necessary.

The next section develops the basics of IKP, including some remarks on fixed points of inductive definitions. After that some classical theories related to KP are presented, and the question of which imply which others is completely characterized. While they are not equivalent in general, when restricted to initial segments of (classical) L they are. However, the section after that shows that in intuitionistic L this equivalence breaks down. We close with some questions.

2 IKP

The axioms of classical KP are: Empty Set, Pairing, Union, Extensionality, Foundation (as a schema for all definable classes), Δ_0 Comprehension (also known as Δ_0 Separation), and Δ_0 Bounding (also known as Δ_0 Collection). Often Infinity is adjoined; we will also use the axiomatisation with Infinity in this paper. There is not much trouble adapting these to an intuitionistic setting. The concept of a Δ_0 formula needs no change. As usual, Foundation must be replaced by $\in$-Induction: $\forall x(\forall y \in x\ \phi(y) \rightarrow \phi(x)) \rightarrow \forall x\ \phi(x)$, where ϕ is a formula in the language of set theory. Also, the standard version of Infinity, as in [9] for instance ($\exists x\ [(\exists y\ y \in x) \wedge (\forall y \in x\ \exists z \in x\ y \in z)]$), won't do, because you need Power Set and Π_1 Comprehension to get ω; instead, take the version of Infinity from [7] ($\exists x\ [0 \in x\ \wedge\ \forall y \in x\ \exists z \in x(z = y \cup \{y\})\ \wedge\ \forall y \in x(y = 0 \vee \exists z \in x\ y = z \cup \{z\})]$), which axiomatizes ω straightforwardly. We take IKP to be KP with these few adaptations.

This choice can be further justified. Two other axiomatizations of KP are to replace Δ_0 Bounding with either Σ Bounding or Σ Reflection. Taking the Σ formulas to be the closure of the Δ_0 formulas under $\wedge, \vee, \exists$, and bounded quantification, these equivalences hold intuitionistically, for the same reasons they do classically. Rather than repeating these completely standard arguments here, the reader is referred to [1], theorems I.4.3 and I.4.4, the proofs of which go through unchanged in IKP. In addition, under IKP every Σ formula is equivalent to a Σ_1 formula, proven inductively on formulas.

One could also consider many of the standard consequences of KP. The ones of interest to us are:

1. Σ Replacement,
2. Strong Σ Replacement,
3. Δ Comprehension,
4. Σ Recursion,
5. the Second Recursion Theorem, and
6. Σ Inductive Definitions.

Σ Replacement and Strong Σ Replacement ($\forall x \in A\ \exists y\ \phi(x,y) \rightarrow \exists f\ \forall x \in A\ f(x) \neq \emptyset \wedge \forall y \in f(x)\ \phi(x,y)$) hold intuitionistically, by the same proofs as in the classical case; again, the reader is referred to [1], theorems I.4.6 and I.4.7. Δ Comprehension requires considerably more care. Usually a Δ property is defined as being given by a pair ϕ, ψ of Σ and Π formulas respectively, but when it's used the first thing you do is deal with $\neg\psi$ as a Σ formula. This won't work intuitionistically. The next obvious guess at what a Δ property should be is a pair ϕ, ψ of Σ formulas such that $\forall x\ [(\phi(x) \vee \psi(x))\ \wedge\ \neg(\phi(x) \wedge \psi(x))]$. The problem here is that too much is being demanded: such a definition would be almost impossible to meet. Consider a very simple property, such as "x = $\emptyset$". What would be a Δ definition of this, according to the above notion of Δ? If $\phi(x)$ is "x = $\emptyset$" and $\psi(x)$ is "x $\neq \emptyset$" then we certainly do not have $\forall x\ (\phi(x) \vee \psi(x))$. To address these (and other) problems, we will be content with defining Δ_1, noting that Σ resp. Π formulas are under IKP provably equivalent to Σ_1 resp. Π_1 formulas. A Δ_1 property of a variable x is given by a pair of Δ_0 formulas $\phi(x,y), \psi(x,y)$ with free variables x,y such that

$$\forall x[\exists y \neg\neg(\phi(x,y) \vee \psi(x,y))\ \wedge \forall y \neg(\phi(x,y) \wedge \psi(x,y))].$$

With this notion of Δ_1, Δ_1 Comprehension is provable in IKP; see [1] theorem I.4.5.

That a function defined by Σ recursion is Σ_1 definable holds in both KP and IKP; again, the proof from [1], theorem I.6.4 holds. The Second Recursion Theorem, as stated and proved in [1] V.2, pretty much stands as is, except that when syntax and semantics are introduced in ch. III all of the standard Boolean connectives must be included as primitives (since, for instance, in our context $\rightarrow$ cannot be defined in terms of $\neg$ and $\vee$). The fact that the least fixed point of a positive inductive Σ definition is Σ_1 definable actually requires a bit of care in one point, so the development of this theory will be summarized here, with the reader referred to [1] for the details.

In IZF, suppose $\Gamma : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is a monotone inductive operator on X. Consider $\{Y | \Gamma(Y) \subseteq Y\}$. This set is non-empty, since it contains X itself as a member. Let $\Gamma_{fix} = \bigcap\{Y | \Gamma(Y) \subseteq Y\}$. By the monotonicity of $\Gamma, \Gamma(\Gamma_{fix}) \subseteq \Gamma_{fix}$. Applying Γ again we get $\Gamma\Gamma(\Gamma_{fix}) \subseteq \Gamma\Gamma_{fix}$, so $\Gamma(\Gamma_{fix}) \in \{Y | \Gamma(Y) \subseteq Y\}$, and $\Gamma_{fix} \subseteq \Gamma(\Gamma_{fix})$. Hence $\Gamma_{fix} = \Gamma(\Gamma_{fix})$, and Γ_{fix} is a fixed point. Moreover, by its definition, Γ_{fix} is the least fixed point. Γ_{fix} can also be defined from the bottom up. Inductively on ordinals β, let $\Gamma_{<\beta} = \bigcup_{\alpha<\beta} \Gamma_\alpha, \Gamma_\beta = \Gamma(\Gamma_{<\beta})$. (Note the standard way to induct on ordinals intuitionistically, which avoids the successor-or-limit case split.) Let $\Gamma_\infty = \bigcup_{\beta \in ORD} \Gamma_\beta$, which can be shown to be a set using Bounding. (Note that Replacement will not do! Let $\psi(Y, \beta)$ be "Y = Γ_β"; since such a β cannot be uniquely chosen, we need (Comprehension and) Bounding in order to get a range for ψ, a set A of ordinals such that $\forall Y \subseteq X$ if $\exists\beta$ Y = Γ_β then $\exists\beta \in$ A Y = Γ_β. Letting γ = TC(A), $\Gamma_\infty = \Gamma_\gamma$. This is where the current argument differs from its classical version, and recurs when discussing admissible sets proper.) It's easy to see that $\Gamma_\beta \subseteq \Gamma_{fix}$, inductively on β, and that Γ_∞ is a fixed point, making Γ_∞ equal to Γ_{fix}.

Now particularize to the case where Γ is given by an X-positive Σ formula ψ: $\Gamma(Y) = \{ x \in X \mid \psi(x, X) \}$. (What follows is adapted from [1], VI.2.6.) If $\mathbf{M}$ is admissible (i.e. $\mathbf{M} \models$ IKP), then, letting $\alpha = \mathrm{ORD}(\mathbf{M})$, $\Gamma_\infty = \Gamma_{<\alpha}$, as follows. In $\mathbf{M}$, the relation "$x \in \Gamma_\beta$" is a Σ_1 relation, using the Second Recursion Theorem. So $\phi(x, \Gamma_{<\alpha})$ is a Σ relation, where "$y \in \Gamma_{<\alpha}$" is interpreted as "$\exists\beta$ $y \in \Gamma_\beta$". If $\phi(x, \Gamma_{<\alpha})$ holds, then, by Σ Bounding, there is a set of ordinals $A \in \mathbf{M}$ such that $\phi(x, \bigcup_{\beta \in A} \Gamma_\beta)$ holds. Letting $\gamma = \mathrm{TC}(A)$, $\phi(x, \Gamma_{<\gamma})$, and $x \in \Gamma_\gamma$. So $\Gamma_\alpha = \Gamma_{<\alpha}$ is a fixed point. Since $\Gamma_{<\alpha} \subseteq \Gamma_\infty$ too, $\Gamma_{<\alpha} = \Gamma_\infty$. It has already been observed that $\Gamma_{<\alpha}$ is Σ_1 definable over $\mathbf{M}$.

So where are the differences between KP and IKP? Typically properties that a classical set theorist identifies automatically become inequivalent in an intuitionistic setting. No one would believe that every property around admissibility has the same status in IKP.

If one of the pleasures of intuitionism is to surprise our intuitions, then you're in for a treat. You would have expected to find some difference between KP and IKP among the most common properties, those already cited. So there's no use having the differences there, where you're already looking for them. Rather, the differences show up where you'd never think to check, right under your nose. Consider the basic axiom, Δ_0 Bounding:

$$\forall x \in A\ \exists y\ \phi(x, y) \rightarrow \exists B\ \forall x \in A\ \exists y \in B\ \phi(x, y),\ \phi\ \in\ \Delta_0.$$

Consider the contrapositive with the negations pushed through (the "classical contrapositive", with ϕ absorbing the negation):

$$\forall B\ \exists x \in A\ \forall y \in B\ \phi(x, y) \rightarrow \exists x \in A\ \forall y\ \phi(x, y),\ \phi\ \in\ \Delta_0.$$

We call this latter property Δ_0 Uniformity. Of course classically Δ_0 Uniformity and Δ_0 Bounding are equivalent as contrapositives, but intuitionistically they're not, as we'll see later.

An alternative to Δ_0 Bounding classically is Σ Reflection:

$$\phi \rightarrow \exists A\ \phi^{(A)},\ \phi \in \Sigma,$$

where the superscript means bound all as yet unbound quantifiers by the superscript. The classical contrapositive is:

$$\forall A\ \phi^{(A)} \rightarrow \phi,\ \phi \in \Pi.$$

The latter property is called Π Persistence. Finally, regarding the last alternative axiomatization of KP, Σ Bounding, its classical contrapositive is Π Uniformity.

Proposition 2.0.1 *Over IKP - Δ_0 Bounding, the following are equivalent:*

1. Π *Persistence*

2. Π *Uniformity*

3. Δ_0 *Uniformity.*

proof: In what follows, we will rely heavily on the fact that, for ϕ a Π formula, if B $\subseteq$ A then $\phi^{(A)} \rightarrow \phi^{(B)}$. This is proved by induction on Π formulas. The same also holds for A = V: if B is a set, then $\phi \rightarrow \phi^{(B)}$.

(1) $\rightarrow$ (2) Suppose that $\forall B\ \exists x \in A\ \forall y \in B\ \phi(x, y), \phi$ a Π formula. By the above mentioned fact, $\forall B\ \exists x \in A\ \forall y \in B\ \phi(x, y)^{(B)}$, i.e. $\forall B\ (\exists x \in A\, \forall y\ \phi(x, y))^{(B)}$. By Π Persistence, $\exists x \in A\, \forall y\ \phi(x, y)$, as was to be shown.

(2) $\rightarrow$ (3) trivial

(3) $\rightarrow$ (1) We show that Π Persistence holds for all Π formulas ϕ by induction on ϕ.

The base case, ϕ Δ_0, is trivial, since $\phi^{(A)} = \phi$.

Suppose $\phi = \psi_0 \vee \psi_1$. Assume $\forall A(\psi_0 \vee \psi_1)^{(A)}$, which equals $\forall A(\psi_0^{(A)} \vee \psi_1^{(A)})$. We claim that $\forall B\ \exists x \in \{0, 1\}\ \forall y \in B\ ((x = 0 \wedge \psi_0^{(y)}) \vee (x = 1 \wedge \psi_1^{(y)}))$. To see this, by assumption $\forall A(\psi_0^{(A)} \vee \psi_1^{(A)})$; letting be A be $\bigcup$ B, we have $\psi_0^{(\bigcup B)} \vee \psi_1^{(\bigcup B)}$. If $\psi_0^{(\bigcup B)}$ let x be 0, and note that $\psi_0^{(\bigcup B)} \rightarrow \psi_0^{(y)}$ since y $\subseteq \bigcup$ B. Similarly if $\psi_1^{(\bigcup B)}$. Applying Δ_0 Uniformity to the claim, we get $\exists x \in \{0, 1\}\ \forall y\ ((x = 0 \wedge \psi_0^{(y)}) \vee (x = 1 \wedge \psi_1^{(y)}))$. If the value of x which witnesses this sentence is 0, then $\forall y\ \psi_0^{(y)}$, and by induction ψ_0. Similarly, if the value of x which witnesses this sentence is 1, we get ψ_1. Since either 0 or 1 witnesses this sentence, $\psi_0 \vee \psi_1$, as was to be shown.

Suppose $\phi = \exists u \in v\ \psi$. Assume $\forall A(\exists u \in v\ \psi)^{(A)}$, which equals $\forall A\ \exists u \in v\ \psi^{(A)}$. We claim $\forall B\ \exists u \in v\ \forall y \in B\ \psi^{(y)}$. To see this, again use the assumption with A = $\bigcup$B. By Δ_0 Uniformity, $\exists u \in v\ \forall y\ \psi^{(y)}$, and by induction $\exists u \in v\ \psi$.

The other cases are easier. If $\phi = \psi_0 \wedge \psi_1$, $\forall A(\psi_0 \wedge \psi_1)^{(A)} = \forall A(\psi_0^{(A)} \wedge \psi_1^{(A)}) \rightarrow \forall A\psi_0^{(A)} \wedge \forall A\psi_1^{(A)} \rightarrow$ (by induction) $\psi_0 \wedge \psi_1$. If $\phi = \forall u \in v\ \psi$, $\forall A\ (\forall u \in v\ \psi)^{(A)} = \forall A\ \forall u \in v\ \psi^{(A)} \rightarrow \forall u \in v\ \forall A\ \psi^{(A)} \rightarrow$ (by induction) $\forall u \in v\ \psi$. Finally, if $\phi = \forall u\, \psi$, suppose $\forall A\, (\forall u\, \psi)^{(A)}$ which equals $\forall A\, (\forall u \in A\, \psi^{(A)})$. It follows that $\forall u\, \forall B\, \psi^{(B)}$: for arbitrary values v and C for u and B respectively, let A = C $\cup$ {v} in the assumption above, yielding $\forall u \in C \cup \{v\}\ \psi^{(C \cup \{v\})}(u)$. In particular, for u = v, $\psi^{(C \cup \{v\})}(v)$, and, shrinking the bound, $\psi^{(C)}(v)$. So $\forall u\, \forall B\, \psi^{(B)}(u)$, and, by induction, $\forall u\, \psi(u)$. ▮

In what follows, Π Persistence will refer to the theory IKP - Δ_0 Bounding + Π Persistence (or Π Uniformity or Δ_0 Uniformity, from the proposition), as well as to the axiom scheme, except when such an ambiguity might cause confusion.

Although KP and Π Persistence are equivalent classically, intuitionistically they are (implicationally) incomparable, as follows. To see that IKP does not imply Π Persistence, consider the partial order that has a bottom element ⊥ and ω-many incomparable nodes n ($n \geq 1$) larger than it. Let the Kripke structure **M** have $L_{\omega_1^{CK}}$ at ⊥, $L_{\omega_n^{CK}}$ (ω_n^{CK} being the nth admissible ordinal) at n, and Id for transition functions. $\bot \models$ IKP, as follows. $\mathbf{M}_\bot$ already contains ∅ and ω, and the universe at each node is closed under pairing and union. These universes are also transitive sets, and the ∈-relation of **M** is merely a restriction of ∈ (of **V**), so Extensionality and Foundation hold. Both Δ_0 Comprehension and Δ_0 Bounding are based on the fact that sets in **M** don't grow, or, to put it more formally, $\mathbf{M}_n$ is an end-extension of $\mathbf{M}_\bot$. As a consequence, Excluded Middle holds in **M** for Δ_0 formulas. So for Δ_0 Comprehension, suppose $\phi(x)$ is a Δ_0 formula with parameters from $\mathbf{M}_\bot = L_{\omega_1^{CK}}$, and X ∈ $\mathbf{M}_\bot$. Let A = { x ∈ X | $\phi(x)$}. A ∈ $L_{\omega_1^{CK}} = \mathbf{M}_\bot$, and $\bot \models$ " A = { x ∈ X | $\phi(x)$}". Regarding Δ_0 Bounding, if $\bot \models$ "f is total" then $1 \models$ "f is total", and $L_{\omega_1^{CK}} \models$ "f is total". By the admissibility of $L_{\omega_1^{CK}}$, A = rng(f) ∈ $L_{\omega_1^{CK}}$. By Excluded Middle and Absoluteness for Δ_0 formulas, $\bot \models$ "A = rng(f)". But Δ_0 Uniformity fails: $\bot \models \forall a$ ($\exists n \in \omega\ \forall x \in a$ if x is a sequence of admissible ordinals then length(x) < n), but $\bot \not\models \exists n \in \omega\ \forall x$ (if x is a sequence of admissible ordinals then length(x) < n).

In the other direction, consider ω as a partial order, and let α_n be an ω-sequence cofinal in ω_1^{CK}. Let the Kripke model **M** have L_{α_n} at node n (with Id as the transition functions). It is easy to see that no node forces IKP: given n, just pick a witness that L_{α_n} is not admissible, such a witness being a Δ_0 function unbounded in L_{α_n}. But Π Persistence holds: Suppose for a node n, $n \models$ "$\forall A\ \phi^{(A)}$" (and hence the same for all nodes $m, m \geq n$). For $A \in L_{\alpha_n}$, $\phi^{(A)}$ is Δ_0, and $\mathbf{M}_m$ is an end-extension of $\mathbf{M}_n$ for $m \geq n$; this means that the truth of $\phi^{(A)}$ can be determined locally, i.e. $L_{\alpha_n} \models \phi^{(A)}$. "A" ranges over all sets in L_{α_m} for all $m \geq n$, hence over $L_{\omega_1^{CK}}$, so we have $L_{\omega_1^{CK}} \models \forall A\ \phi^{(A)}$. By Π Persistence classically, $L_{\omega_1^{CK}} \models \phi$. In showing that $n \models \phi$, when unraveling ϕ choices will have to be made, with ∃ and ∨. Use the truth of ϕ in $L_{\omega_1^{CK}}$ as a guide.

So we have two different theories, IKP and Π Persistence. Is there a difference between the mathematics you can do in them? Yes, again by duality. We have already seen that in IKP the least fixed point of a positive inductive Σ definition is Σ_1 definable. Classically it would follow that the greatest fixed point of a positive inductive Π definition is Π_1 definable. However, intuitionistically, this latter property seems to require Π Persistence as well as IKP.

In a bit more detail, if $\Gamma : \mathcal{P}(X) \to \mathcal{P}(X)$ is a monotone inductive operator on X, let A be $\{Y | Y \subseteq \Gamma(Y)\}$ and $\Gamma_{fix} = \bigcup A$. By the monotonicity of $\Gamma, \Gamma_{fix} \subseteq \Gamma(\Gamma_{fix})$. Applying Γ again, $\Gamma(\Gamma_{fix}) \subseteq \Gamma(\Gamma(\Gamma_{fix}))$, so $\Gamma(\Gamma_{fix}) \in$ A and $\Gamma(\Gamma_{fix}) \subseteq \Gamma_{fix}$. Hence $\Gamma_{fix} = \Gamma(\Gamma_{fix})$, and Γ_{fix} is a fixed point. Moreover, by its definition, Γ_{fix} is the greatest fixed point. From the bottom up, let $\Gamma_{<\beta} = \bigcap_{\alpha<\beta} \Gamma_\alpha$, $\Gamma_\beta = \Gamma(\Gamma_{<\beta})$, and $\Gamma_\infty = \bigcap_{\beta \in ORD} \Gamma_\beta$. $\Gamma_{fix} \subseteq \Gamma_\beta$, by induction on β, and Γ_∞ is a fixed point, so $\Gamma_{fix} = \Gamma_\infty$. If Γ is given by a Π formula

ϕ, then, over IKP + Π Persistence, "x $\in \Gamma_\beta$" is a Π relation, by the duals of the arguments for the Σ case. (This can be seen by adapting [1], III.1 and V.1 and 2. The satisfaction relation $\mathbf{M} \models \phi$[s] is Δ_1 over IKP. That the universal Π predicate Π-Sat(ϕ, s) can be defined as "$\forall A \phi^{(A)}$[s]" uses Π Persistence. This fact is then plugged into the dual to the proof of the Second Recursion Theorem to get that for all R-positive Π formulas ϕ(x, y, R) there is a Π formula ψ(x, y) such that ψ(x, y) iff ψ(x, y, λx.ψ(x, y)). This suffices to get "x $\in \Gamma_\beta$" to be a Π relation of x and β.) Let $\mathbf{M} \models$ IKP + Π Persistence, and α = ORD($\mathbf{M}$). The claim is that $\Gamma_{<\alpha} = \Gamma_\alpha$. To see this, suppose x $\in \Gamma_{<\alpha}$, i.e. $\mathbf{M} \models \forall\beta\phi(x, \Gamma_{<\beta})$, a Π relation. Letting Γ_A be $\bigcap_{\alpha \in A \cap ORD} \Gamma_\alpha$, $\mathbf{M} \models \forall A \phi(x, \Gamma_A)$ (because β can be chosen to be TC(A) ∩ ORD, and then $\Gamma_{<\beta} \subseteq \Gamma_A$), and $\mathbf{M} \models \forall A \ \phi^{(A)}(x, \Gamma_A)$. By Π Persistence, $\mathbf{M} \models \phi(x, \Gamma_{<\alpha})$, so x $\in \Gamma_\alpha$ and $\Gamma_{<\alpha} \subseteq \Gamma_\alpha$. On general principles $\Gamma_\alpha \subseteq \Gamma_{<\alpha}$, so $\Gamma_\alpha = \Gamma_{<\alpha}$, and $\Gamma_{<\alpha}$ is a fixed point. Since $\Gamma_{<\alpha} \supseteq \Gamma_\infty$, $\Gamma_{<\alpha} = \Gamma_\infty$.

So, while IKP suffices to get least fixed points to be Σ_1 definable, it seems as though Π Persistence is necessary to get greatest fixed points to be Π_1 definable (although, to be fair, such necessity has yet to be proven). Furthermore, the second of the models above, intended to show that Π Persistence does not imply IKP, also shows that Π Persistence doesn't prove that lfp's are Σ_1 definable (Kleene's O has an appropriate inductive definition but is not Σ_1 definable over **M**.). So these two constructions, least and greatest fixed point, so near to each other classically, are apparently more easily splittable intuitionistically. This matter will be pursued in the questions at the end of this paper.

It bears observation that Π Persistence, as a theory, is quite weak, in that one cannot easily construct sets in it. For instance, it does not even prove the totality of the function $\alpha \mapsto L_\alpha$, as the following model shows. Let $\langle \alpha_n \mid n \in \omega \rangle$ be a strictly increasing sequence of limit ordinals cofinal in ω_1^{CK}. For X a set, let Pair(X) be the set of all pairs from X (including singletons, as degenerate pairs), Union(X) the set of all unions from X (i.e. $\{\bigcup x \mid x \in$ X$\}$), and Δ_0(X) the set of all Δ_0-definable subsets of members of X (assume here that X is transitive, just for simplicity). Let Close(X) be X ∪ Pair(X) ∪ Union(X) ∪ Δ_0(X). The Kripke model under construction has for its partial order an increasing ω-sequence. For node n, start with $X_0 = L_{\alpha_n} \cup \alpha_{n+1}$, and let X_{m+1} = Close(X_m). The nth node is $\bigcup_m X_m$ (with the inclusion function as the transition functions). IKP - Δ_0 Bounding is easily seen to hold, as is Π Persistence, by the same argument as in the second model above. But α_n is an ordinal at every node n and L_{α_n} isn't a set until node n+1. (To see that L_{α_n} is not a set at node n, show inductively on m that for all finite Y $\subseteq X_m$ $L_{\alpha_n} \not\subseteq$ TC(Y).)

It is an interesting question whether IKP + Π Persistence - Δ_0 Bounding proves the Π_1 definability of greatest fixed points of positive inductive Π definitions, or whether the full power of IKP + Π Persistence is needed. The status of Π Persistence will be further pursued in the questions at the end.

3 Classical Friends

In the following section we will turn to the intuitionistic versions of various theories related to KP. Since these classical theories are not all well known, though, a brief discussion of them for their own sakes is in order.

The axioms of interest to us are:

1. Σ_1 Dependent Choice: If $\phi(x, y)$ is Σ_1, and $\forall x \; \exists y \; \phi(x, y)$, then there is a function f with domain ω such that $\forall n \; \phi$(f(n), f(n+1)).

2. Resolvability: There is a Δ_1 definable function f on the ordinals such that V = $\bigcup$range(f).

3. Π_2 Reflection: If $\forall x \; \exists y \; \phi(x, y)$, $\phi \in \Delta_0$, then for every A there is a B $\supseteq$ A such that $\forall x \in B \; \exists y \in B \; \phi(x, y)$.

4. Δ_0 Bounding: from KP.

The implicational relations among these theories are as follows:

$$\begin{array}{ccc} \Sigma_1 \text{ DC + KP} & & \\ & \searrow & \\ & & \Pi_2 \text{ Reflection + KP} \rightarrow \text{KP} \\ & \nearrow & \\ \text{Resolvability + KP} & & \end{array}$$

Over the base theory KP - Δ_0 Bounding, Resolvability alone implies nothing of interest; after all, L_λ for any arbitrary limit λ is resolvable. Similarly, Σ_1 DC alone is rather weak, as $L_{\omega_1 \cdot 2}$ models Σ_1 DC (because cf$(\omega_1 \cdot 2) > \omega$). However, over this base theory Π_2 Reflection itself picks up Δ_0 Bounding.

All of these implications are easy to prove; for details see [1]. Furthermore, in L they are all equivalent; this is also easy, and can be found in [1]. However, if an implication does not follow from the preceding then it is not in general true. Some of these non-implications are easy to see, and must certainly already be known if not written down; others are actually a bit tricky. In any event these proofs are not so easy to find in the literature, and they also provide the models for the proofs in the next section, so it is well to include them here. (For background on forcing, as well as arguments similar to those that follow, including the use of symmetric submodels to falsify the Axiom of Choice, the reader is referred to [4] and [5].)

1. Resolvability + KP + $\neg\Sigma_1$ DC: Adjoin to $L_{\omega_1^{CK}}$ (or, for that matter, to L $\models$ ZF, for the stronger result Resolvability + ZF + $\neg\Sigma_1$ DC) ω-many Cohen reals (that is, reals generic for the forcing partial order $2^{<\omega}$), as well as the set (not sequence!) G consisting of all those reals. This is a symmetric submodel of the model obtained by the set forcing to add an ω-sequence of Cohen reals, hence is admissible. It is already a standard argument (see [5])

that not even all of ZF is enough to build an ω-sequence of distinct elements of G. This violation of the Axiom of Choice is actually a violation of Σ_1 DC: let $\phi(x,y)$ be "if x is a sequence of distinct elements of G then y is one also and is a proper end-extension of x". The function which shows Resolvability is $f(\alpha) = L_\alpha[G]$.

2. Σ_1 DC + KP + $\neg$Resolvability: Adjoin to $L_{\omega_1^{CK}}$ any finite subset G_{fin} of a countable set G of Cohen reals, and take the union of all such adjunctions: $\mathbf{M} = \bigcup_{G_{fin} \subseteq G,\, G_{fin} finite} L_{\omega_1^{CK}}[G_{fin}]$. Σ_1 DC and Δ_0 Bounding hold, as follows. It suffices to consider formulas of the form $\exists y \phi(x,y)$, with $\phi(x,y)$ a Δ_0 formula. If $\exists y \phi(x,y)$ holds for a particular set x, then let $G_{fin} \subseteq G$ be the finite set of those reals used in the construction of x and of ϕ's parameters (i.e. $\exists y \phi(x,y)$ is a formula over $L_{\omega_1^{CK}}[G_{fin}]$). We claim there is a witness y not merely in $\mathbf{M}$, as hypothesized, but already in $L_{\omega_1^{CK}}[G_{fin}]$. To see this, let y witness $\exists y \phi(x,y)$ in $\mathbf{M}$. Let $G_y \supseteq G_{fin}$ be a finite set such that $y \in L_{\omega_1^{CK}}[G_y]$. By the absoluteness of Δ_0 formulas, $L_{\omega_1^{CK}}[G_y] \models \phi(x,y)$; again by absoluteness, $L_\alpha[G_y] \models \phi(x,y)$ for some $\alpha < \omega_1^{CK}$. $G_y \setminus G_{fin}$ is generic over $L_{\omega_1^{CK}}[G_{fin}]$ (for the forcing p.o. which is the product of finitely many copies of $2^{<\omega}$), hence the last assertion is forced over $L_{\omega_1^{CK}}[G_{fin}]$ by some condition p: p $\Vdash$ "$L_\alpha[G_y] \models \phi(x, \mathring{y})$", where $\mathring{y}$ is a name for y. In $L_{\omega_1^{CK}}[G_{fin}]$ a generic over $L_\alpha[G_{fin}]$ through p can be built (by the countability of $L_\alpha[G_{fin}]$ in $L_{\omega_1^{CK}}[G_{fin}]$); let G_α be such a generic. Then $L_\alpha[G_{fin}, G_\alpha] \models \phi(x, \mathring{y}(G_\alpha))$, where "$\mathring{y}(G_\alpha)$" refers to the interpretation of $\mathring{y}$ with the canonical name for the generic interpreted as G_α. By the absoluteness of Δ_0 formulas $L_{\omega_1^{CK}}[G_{fin}] \models \phi(x, \mathring{y}(G_\alpha))$, and $L_{\omega_1^{CK}}[G_{fin}] \models \exists y\, \phi(x,y)$. Now it's easy to see that Σ_1 DC holds in $\mathbf{M}$. If $\mathbf{M} \models$ "$\forall x \exists y \phi(x,y)$", $\phi \in \Delta_0$, then $L_{\omega_1^{CK}}[G_{fin}]$ models the same. $L_{\omega_1^{CK}}[G_{fin}]$ also satisfies Σ_1 DC, so contains an appropriate ω-sequence, which by absoluteness also works in $\mathbf{M}$. Δ_0 Bounding is similar.

Resolvability fails, as follows. Let f be a function over $\mathbf{M}$ (given via a Δ_1 definition) with domain $\mathrm{ORD}^{\mathbf{M}}$. Let G_{fin} be a finite subset of G such that all of f's parameters are in $L_{\omega_1^{CK}}[G_{fin}]$. By the same argument as in the last paragraph, since f is total over $\mathbf{M}$, the same Δ_1 definition produces the same total function over $L_{\omega_1^{CK}}[G_{fin}]$. So $\bigcup$ range(f) $\subseteq L_{\omega_1^{CK}}[G_{fin}]$, which itself is a proper subset of $\mathbf{M}$, so $\bigcup$ range(f) $\neq \mathbf{M}$.

3. Π_2-Reflection + $\neg\Sigma_1$DC + $\neg$Resolvability: This is a combination of the previous two arguments. To construct $\mathbf{M}$, first adjoin an infinite set G of Cohen reals, then all possible finite subsets of another such infinite set H. Σ_1 DC and Resolvability fail as before. Π_2 Reflection holds for much the same reason that Σ_1 DC did in the previous example. If $\mathbf{M} \models \exists y \phi(x,y)$ then all you need to build such a y is G and the finite subset H_{fin} of H used for x and ϕ. So if $\mathbf{M} \models$ "$\forall x \exists y\, \phi(x,y)$", $\phi \in \Delta_0$, and if $A \in \mathbf{M}$, then $L_{\omega_1^{CK}}[G, H_{fin}] \models$ "$\forall x \exists y\, \phi(x,y)$" ($H_{fin}$ merely being large enough to contain all of the reals in H needed to construct ϕ's parameters and A). Since $L_{\omega_1^{CK}}[G, H_{fin}]$ satisfies

Π_2 Reflection, it contains an appropriate B $\supseteq$ A, as desired.

4. KP + $\neg\Pi_2$-Reflection: The model from [8] suffices, and will be described briefly here. A Steel generic tree over $L_{\omega_1^{CK}}$ is not well-founded but has no infinite descending sequence in the generic extension. This forcing can be modified slightly to include a distinguished path through the tree which does not destroy admissibility. Force an ω-sequence of such trees $< T_n >$ and paths $< B_n >$. Let W be a recursive linear ordering of ω which is not well-founded but has no hyperarithmetic infinite descending sequence. Think of the trees as ordered by W. Let $\mathbf{M}_i$ be $L_{\omega_1^{CK}}[< T_n \mid n \in \omega >, < B_n \mid n \geq_W i >]$. $\mathbf{M} = \bigcup_{i \in I} \mathbf{M}_i$ for I some final segment of W with no least element, chosen exactly so that $\mathbf{M}$ is admissible. In [8] it was observed that $\mathbf{M}$ falsifies Σ_1 DC; in fact it's even Π_2 Reflection that fails: $\mathbf{M} \models$ "$\forall j$ if there is a branch through T_j then $\exists i <_W j$ and a branch through T_i", but any set in $\mathbf{M}$ is a set in some $\mathbf{M}_i$, in which W is still well-founded.

4 Intuitionistic Models

As remarked in the introduction, even though the constructions just given show that none of these theories are equivalent to each other in general, in L they all are. However, intuitionistically the ordinals do funny things. With Kripke models there is an extra degree of freedom, the choice of the partial order. If the underlying partial order is chosen to be some forcing partial order, then by coding forcing conditions as small ordinals one could reasonably hope to get a generic G, although still not in V, actually in $L^{V[G]}$. This is basically the approach followed in the upcoming arguments. Notice, though, that these intuitions are not made precise. In particular, of the four constructions from the preceding section, only three are adapted here, because we couldn't see how to do the fourth. The omitted one is Σ_1 DC + KP + V=L + $\neg$Resolvability, the problem being Σ_1 DC. There's no trouble picking some least set in classical L, but IL does not apparently come equipped with a well-founded linear order.

4.1 General Considerations

There will be some common constructions in these models. It would be well to isolate them, as well as to give some of the standard background on IORD and IL.

4.1.1 Intuitionistic Ordinals

Intuitionistically, an ordinal is a transitive set of transitive sets. The standard weak counter-example to show that the ordinals are not linearly ordered goes as follows. Let $\alpha = \{0 \mid \phi\}$. α is an ordinal. If the ordinals are linearly ordered, then $\alpha \in 0, \alpha = 0$, or $0 \in \alpha$. The first case is not possible, since 0 is the empty set. In the second case, $\neg\phi$. In the last case, ϕ. So $\phi \vee \neg\phi$. Hence if Excluded Middle is to fail, then the ordinals are not linearly ordered. It is not hard to give examples of non-classical ordinals in Kripke models. Let P be any partial order. A trivial Kripke structure on P has some universe of sets M (with M's $\in$-relation) at each node and the identity function as the transition functions. (Think of M as a model of ZF, or a weaker set theory, or even the universe V for a class-sized Kripke structure. A class-sized Kripke structure is like a Kripke structure, but rather than requiring that the relations and functions in question (e.g. "is a set at node σ", or the transition functions between nodes) be sets, they must merely be definable relations on a definable subset of V.) Consider a larger or ambient Kripke structure, that is, a structure with the same partial order P but more sets at each node. Any set in this ambient structure which at each node contains only ordinals from the trivial Kripke structure is an intuitionistic ordinal which cannot be construed as classical.

4.1.2 Intuitionistic Definability

Regarding IL, definability can be developed within IZF much as it is classically, so def(X), the set of definable subsets of a set X, makes perfect sense. As usual, the definition of L_α cannot depend on whether α is a limit or a successor, since intuitionistically not every ordinal is a limit or a successor. So L_α is $\bigcup_{\beta\in\alpha}$ def(L_β). For details see [7]. In what follows, however, we will typically not be dealing with L as an inner model of an already given model of IZF, which is merely the semantic version of L's axiomatic development in IZF, hence implicit in [7]. Rather, L will be handled *sui generis*: given a Kripke structure $\mathbf{M}$ for the language of set theory, we will have occasion to consider def($\mathbf{M}$). While not difficult, this could stand some exposition.

So suppose $\mathbf{M}$ is an extensional ($x = y$ iff $\forall z\ z \in x \leftrightarrow z \in y$) Kripke structure (with relations $\in$ and $=$), with underlying partial order P, universe $\mathbf{M}_\sigma$ at node σ, and transition functions $f_{\sigma\tau} : \mathbf{M}_\sigma \to \mathbf{M}_\tau$. In def($\mathbf{M}$), a set at node σ is named by a formula ϕ(x) with free variable x and parameters from $\mathbf{M}$. A set is actually an equivalence class of such formulas, in which $\phi \sim_\sigma \psi$ if $\sigma \models_\mathbf{M} \forall x\ [\phi(x) \leftrightarrow \psi(x)]$. $f_{\sigma\tau}$: def($\mathbf{M}_\sigma$) $\to$ def($\mathbf{M}_\tau$) is given by the equation

$$f_{\sigma\tau}([\phi(x, y_1, ..., y_n)]_\sigma) = [\phi(x, f_{\sigma\tau}(y_1), ..., f_{\sigma\tau}(y_n))]_\tau,$$

where $y_1, ..., y_n$ are the parameters, and the notation $f_{\sigma\tau}$ is used ambiguously for transition functions in both $\mathbf{M}$ and def($\mathbf{M}$). $f_{\sigma\tau}$ is well-defined, because if $\phi \sim_\sigma \psi$, that is, if $\sigma \models_\mathbf{M} \forall x\ [\phi(x, y_1, ..., y_n) \leftrightarrow \psi(x, z_1, ..., z_m)]$, then, by general lemmas on Kripke models, $\tau \models_\mathbf{M} \forall x\ [\phi(x, f_{\sigma\tau}(y_1), ..., f_{\sigma\tau}(y_n)) \leftrightarrow \psi(x, f_{\sigma\tau}(z_1), ..., f_{\sigma\tau}(z_m))]$ ($\sigma \leq \tau$), or $\phi \sim_\tau \psi$. $\mathbf{M}_\sigma$ can be embedded in def($\mathbf{M}_\sigma$), with y being identified with the formula "x $\in$ y". This much being said, $\in$ can now be interpreted: $[\phi]_\sigma \in [\psi]_\sigma$ iff $[\phi]_\sigma = [y]_\sigma$ for some y $\in \mathbf{M}_\sigma$ and $\sigma \models_\mathbf{M} \psi(y)$. (Notice that this definition is independent of the choice

of representative for $[\psi]_\sigma$, by the definition of the equivalence relation; furthermore, there is at most one y such that y $\sim_\sigma \phi$, by the extensionality of $\mathbf{M}$.)
Extensionality holds in def($\mathbf{M}$):
$\sigma \models_{def(\mathbf{M})} \forall x\ (x \in [\phi]_\sigma \leftrightarrow x \in [\psi]_\sigma)$ iff
$\forall \tau \geq \sigma\ \forall x \in \mathbf{M}_\tau \tau \models_{def(\mathbf{M})} [x] \in [\phi]_\tau \leftrightarrow [x] \in [\psi]_\tau$ iff
$\forall \tau \geq \sigma\ \forall x \in \mathbf{M}_\tau \tau \models_{\mathbf{M}} \phi(x) \leftrightarrow \psi(x)$ iff
$\sigma \models_{\mathbf{M}} \forall x\ [\phi(x) \leftrightarrow \psi(x)]$ iff
$[\phi]_\sigma = [\psi]_\sigma$.

To summarize, if $\mathbf{M}$ is an extensional Kripke structure for the language of set theory, so is def($\mathbf{M}$), and $\mathbf{M}$ can be canonically embedded in def($\mathbf{M}$). Furthermore, def($\mathbf{M}$) can be construed as an end-extension of $\mathbf{M}$, since if $\sigma \models_{def(\mathbf{M})} [\phi]_\sigma \in [y]_\sigma$ then $[\phi]_\sigma = [z]_\sigma$ for some z $\in \mathbf{M}_\sigma$ and $\sigma \models_{\mathbf{M}} z \in y$. Also, this process can be iterated through the transfinite: starting with $\mathbf{M}_0 = \mathbf{M}$, let $\mathbf{M}_{\alpha+1} = \text{def}(\mathbf{M}_\alpha)$ and $\mathbf{M}_\lambda = \overset{lim}{\rightarrow} \{\mathbf{M}_\alpha \mid \alpha < \lambda\}$ (or, more colloquially, $\bigcup_{\alpha<\lambda} \mathbf{M}_\alpha$). $\mathbf{M}_\lambda$ is also an extensional Kripke model end-extending each $\mathbf{M}_\alpha$.

Indeed, this process can be iterated not only along a well-founded linear order, but also along a well-founded partial order. Suppose that X is a Kripke structure (for the language of set theory) with underlying partial order P, and that the partial order defined by

$$(\sigma, x) \geq (\tau, y) \leftrightarrow \sigma \leq_P \tau \wedge \tau \models y \in f_{\sigma\tau}(x)$$

$(\sigma, \tau \in P, x \in X_\sigma, y \in X_\tau)$ is well-founded. Then "def" can be iterated along this p.o., as follows. Suppose inductively that for all $(\tau, y) \leq (\sigma, x)$ def$(\mathbf{M}_{(\tau,y)})$ is well-defined as a Kripke structure on the partial order $P_{\geq\tau}$. Then $\mathbf{M}_{(\sigma,x)}$ can be defined as follows:

$$(\mathbf{M}_{(\sigma,x)})_\tau = \bigcup_{\{y \mid \tau \models y \in f_{\sigma\tau}(x)\}} (def(\mathbf{M}_{(\tau,y)}))_\tau.$$

Regarding the transition functions, consider $[\phi(z, y_1, ..., y_n)]_\tau \in (\mathbf{M}_{(\sigma,x)})_\tau$. Then

$$[\phi(z, y_1, ..., y_n)]_\tau \in (def((\mathbf{M}_{(\tau,y)}))_\tau$$

where $\tau \models y \in f_{\sigma\tau}(x)$. Then let

$$f_{\tau\rho}([\phi(z, y_1, ..., y_n)]_\tau) := [\phi(z, f_{\tau\rho}(y_1), ..., f_{\tau\rho}(y_n))]_\rho,$$

since

$$[\phi(z)]_\rho \in (def(\mathbf{M}_{(\rho, f_{\tau\rho}(y))}))_\rho \subseteq (\mathbf{M}_{(\sigma,x)})_\rho,$$

the latter inclusion because $\rho \models f_{\tau\rho}(y) \in f_{\tau\rho}(f_{\sigma\tau}(x)) = f_{\sigma\rho}(x)$.

With $\mathbf{M}_{(\sigma,x)}$ now defined, def$(\mathbf{M}_{(\sigma,x)})$ follows, using the "def" operator developed earlier in this section. The final structure $\mathbf{M}_X$ is then given by

$$(\mathbf{M}_X)_\sigma = \bigcup_{x \in X_\sigma} (def(\mathbf{M}_{(\sigma,x)}))_\sigma.$$

At this point a few words about absoluteness are in order. For starters, the external concept "def" developed above is equivalent to the internal concept "def". In more detail, suppose that $\mathbf{V}$ is some classical meta-universe, and $\mathbf{M}\in\mathbf{V}$ is an extensional Kripke structure for the language of set theory, and $\mathbf{N}$ = def($\mathbf{M}$), again in $\mathbf{V}$. Suppose that $\mathbf{K}$ is a Kripke structure with the same underlying partial order as $\mathbf{M}$ and $\mathbf{N}$ such that $\mathbf{M}$, $\mathbf{N} \in \mathbf{K}$ (so $\mathbf{M}$ and $\mathbf{N}$ are sets in the sense of $\mathbf{K}$). Then $\mathbf{K} \models \mathbf{N}$ = def($\mathbf{M}$), where of course "def" in this case is the traditional, internal concept of definability. We will not prove this here; hopefully the external development of "def" was transparent enough to make this assertion clear. This allows us in the following to be ambiguous as to which version of "def" is intended.

Furthermore, the same holds for the iterations of "def", both into the transfinite as well as along a well-founded partial order. In the first case, suppose α is an ordinal and $\mathbf{K}$ a Kripke structure for the language of set theory (both in our classical meta-universe $\mathbf{V}$, of course). Suppose that there is an $\alpha_{\mathbf{K}} \in \mathbf{K}_\sigma, \sigma \in P$, such that $(\alpha, \in_{\mathbf{V}})$ and $(\alpha_{\mathbf{K}}, \in_{\mathbf{K}_\sigma})$ are isomorphic, and that, for all $\tau \geq \sigma$, $\mathrm{f}_{\sigma\tau}$: $\alpha_{\mathbf{K}} \rightarrow \mathrm{f}_{\sigma\tau}(\alpha_{\mathbf{K}})$ is an isomorphism. Then $\sigma \models$ "$\alpha_{\mathbf{K}}$ is an ordinal", and $\alpha_{\mathbf{K}}$ can be identified with α. Notice that this is nothing other than the extension of the already common practice of using the notation "0" to denote the empty set both in $\mathbf{V}$ and in a Kripke model, "1" as $\{0\}$ again in $\mathbf{V}$ or in a Kripke model, etc., and even "ω" for the ω of a Kripke model, so long as it contains only the internalizations of the standard natural numbers. In addition, for $\mathbf{M}$, $\mathbf{N} \in \mathbf{K}_\sigma$, if $\mathbf{N} = \mathbf{M}_\alpha$ (the α-fold iteration of def, as defined above, of course interpreted in $\mathbf{V}$), then $\sigma \models \mathbf{N} = \mathbf{M}_{\alpha_{\mathbf{K}}}$. (This applies, of course, only when $\mathbf{K}$ has enough set-theoretic power to express α-fold iteration of definability, e.g. in models of IKP (see [7]).) In such a case we will feel free to use the notation α for both the external and internal ordinal.

Regarding iterations along well-founded partial orders, let X be as above (that is, a Kripke structure over P well-founded in the manner previously described). Let $\mathbf{N} = \mathbf{M}_X$, of course in $\mathbf{V}$, and let $\mathbf{K}$ be a Kripke structure containing X, $\mathbf{M}$, and $\mathbf{N}$ as members. If $\mathbf{K}$ satisfies a sufficiently large fragment of IZF then the X-fold iteration of def (notation: X-def) would be definable and provably total. The inductive definition of "X-def" is none other than the already well-known notion of the iteration of definability along an ordinal (i.e. $\mathbf{L}_\alpha$), the point being that α's ordinalhood is unnecessary for the intelligibility of the notion: X-def($\mathbf{M}$) = $\bigcup_{Y\in X}$ def(Y-def($\mathbf{M}$)). The result, again stated without proof, is that $\mathbf{K} \models \mathbf{N}$ = X-def($\mathbf{M}$).

A case of particular interest is when $\mathbf{K} \models$ "X is an ordinal". Under this circumstance we will use the more common notation α instead of X. For $\mathbf{M}$ take the structure $\mathbf{0} = \emptyset^{\mathbf{K}}$ (equivalently, let $\mathbf{0}$ be the unique function from P to $\{\emptyset\}$). Then there is already standard notation for α-def($\mathbf{0}$), namely $\mathbf{L}_\alpha$. This being the case, we will use the notation "$\mathbf{L}_\alpha$" for $\mathbf{0}_\alpha$ for those α's as above which are ordinals in some Kripke structure, even if such Kripke structure has not yet been introduced in the exposition. Then "$\mathbf{L}_\alpha$" can be interpreted either externally or internally, as the occasion demands. (To keep the reader from being distracted by concerns that any given α will indeed turn out to be an ordinal in some ambient Kripke structure introduced in some unspecified future section, please observe that ordinality is a Δ_0 property for transitive sets, the only kind we will be considering, and hence can be determined locally. We will even go so far as to say that a transitive Kripke structure α *is* an ordinal (or a Kripke ordinal) when it satisfies the intuitionistic definition of such, a transitive set of transitive sets, even in the absence of an ambient Kripke universe, knowing that such an α will be an ordinal in any ambient Kripke structure.)

4.1.3 General Constructions

In the following sections we will be constructing several Kripke models. Here we bring together several steps common to them, in order not to have to do them more than once. In every case we will begin by giving the underlying partial order P; the steps given here are generic and applicable to all P's, assuming only that P has a bottom element $\perp$. (After corollary 4.1.3, we will impose more restrictions on P. Of course for the constructions themselves in future sections, we will specify P explicitly.) Then we will define certain Kripke ordinals with underlying partial order P. These ordinals will be based on the generic reals from the classical forcing constructions of the last section; they will be a kind of internalization of these reals, and will be very low in the power set hierarchy (being collections of subsets of $1 = \{0\}$). It should be noted that the constructions here and in the coming sections take place of necessity in **V**: the only Kripke structures in sight are those being built, and so of no use in their own definitions.

Given a node σ in P, let 1_σ be the set (Kripke set, if you will, as 1_σ will end up being a set in a Kripke structure) that looks like 0 unless you're past or incompatible with σ, where it's 1:

$$[\tau \models x \in 1_\sigma] \leftrightarrow [\sigma < \tau \vee \sigma \perp \tau] \wedge [\tau \models x = \emptyset],$$

or, if you prefer,

$$(1_\sigma)_\tau = \emptyset \leftrightarrow \tau \leq \sigma$$

$$(1_\sigma)_\tau = \{\emptyset\} \leftrightarrow [\sigma < \tau \vee \sigma \perp \tau],$$

with the only possible transition functions (cf. the remark after definition 4.1.5). 1_σ is an ordinal: $\perp \models$ "if $x \in 1_\sigma$ then $x = \emptyset$"; since nothing is in $\emptyset$, 1_σ is transitive, and $\emptyset$ is transitive. (Recall that the intuitionistic definition of an ordinal as a transitive set of transitive sets is Δ_0 for transitive sets, and so can be determined locally, without an ambient Kripke structure.) So L_{1_σ} is well-defined. What is L_{1_σ}? If $\tau \models x \in L_{1_\sigma}$, then, for some $\beta, \tau \models x \in def(L_\beta) \wedge \beta \in 1_\sigma$. If $\tau \leq \sigma$ then $\tau \not\models \beta \in 1_\sigma$, so $\tau \not\models x \in L_{1_\sigma}$ for any x, and L_{1_σ} looks empty. Otherwise $\tau \models \beta \in 1_\sigma$ iff $\tau \models \beta = \emptyset$. $\perp \models L_0 = \emptyset$, and $\perp \models def(L_0) = \{\emptyset\}$. So if $\tau \models x \in L_{1_\sigma}$ then $\tau \models x = \emptyset$. In short, $L_{1_\sigma} = 1_\sigma$.

Furthermore, what's definable over 1_σ? Recall that when taking definitions over a set, truth is evaluated in that set: a definable subset of 1_σ is one of the form $\{ x \in 1_\sigma \mid 1_\sigma \models \phi(x) \}$, where ϕ's parameters must also come from 1_σ. So if both $\tau_0, \tau_1 \leq \sigma$ then $\tau_0 \models$ "$1_\sigma \models \phi(x)$" iff $\tau_1 \models$ "$1_\sigma \models \phi(x)$"; i.e. τ_0 and τ_1 force the same atomic facts about definable subsets of 1_σ. Similarly if both τ_0 and τ_1 extend or are incompatible with σ. Since 1_σ's only possible member is $\emptyset$, the only possible member of a subset of 1_σ is $\emptyset$; by the preceding remarks, either $\emptyset$ is in a given subset at all nodes beyond or incompatible with σ or it's not. So $\mathrm{def}(L_{1_\sigma}) = \{0, 1_\sigma\}$.

Let T be $\{1_\sigma \mid \sigma \in P\}$ (in **V**). Let $\hat{T}$ be a Kripke subset of T, that is, $\hat{T}_\sigma \subseteq$ T and the transition functions are all the identity. Let $\hat{T}_0$ be such that $(\hat{T}_0)_\sigma = \hat{T}_\sigma \cup \{0\}$. $\hat{T}_0$ is an ordinal: T is a set of ordinals, so $\hat{T} \subseteq$ T is also, as is $\hat{T}_0$. So $\hat{T}_0$ is a set of transitive sets. It is also transitive itself: if $\tau \models x \in y \in \hat{T}_0$, then either $\tau \models y = 0$, which contradicts $\tau \models x \in y$, or $\tau \models y \in \hat{T}$, i.e. $\tau \models y = 1_\sigma$ for some σ, and $\tau \models x = 0$, and $0 \in \hat{T}_0$.

Lemma 4.1.1 $L_{\hat{T}_0} = \hat{T}_0$

proof: $L_{\hat{T}_0} = \bigcup_{\alpha \in \hat{T}_0} def(L_\alpha)$
$= [\bigcup_{\alpha \in \hat{T}} def(L_\alpha)] \cup def(L_0)$
$= [\bigcup_{1_\sigma \in \hat{T}} def(L_{1_\sigma})] \cup \{0\}$
$= \bigcup_{1_\sigma \in \hat{T}} \{0, 1_\sigma\} \cup \{0\}$
$= \hat{T}_0$ ▮

Corollary 4.1.2 *$\hat{T}$ is definable over $L_{\hat{T}_0}$.*

proof: $\hat{T} = \hat{T}_0 - \{0\} = \{x \mid x \neq 0\}$ as a definition over $\hat{T}_0$, since, for all $\sigma, \perp \models 1_\sigma \neq 0$. ▮

Corollary 4.1.3 *Suppose $\{\hat{T}_i \mid i \in I\}$ is a collection of Kripke subsets of T. Then $\{\hat{T}_{i0} \mid i \in I\} \cup \bigcup \{\hat{T}_{i0} \mid i \in I\}$ is an ordinal, say ξ, and $\forall i\ \hat{T}_i \in L_\xi$.*

Now suppose that P (the underlying partial order of the Kripke structure to be built) is a tree of height ω. Also assume that P is nowhere degenerate, meaning that each node has incompatible extensions. Much of what follows would work in a more general context, but we have no need of such a detailed investigation. We would like some internal notion of a branch through P; that is, a Kripke set that behaves as such. Given an external branch B (that is, a branch through P in the classical sense), the obvious internalization of B would be $\{1_\sigma \mid \sigma \in B\}$. But if τ is perpendicular to any member of B, then at node

τ this looks like $\{1\}$. Even worse, $\neg\neg$ B = $\{1\}$, and it's not possible to get two such branches forced at any node to be unequal. Hence we would like branches that reappear even after you've fallen off of them.

In the following, we explicitly distinguish between those definitions and lemmas that of necessity are to be evaluated in **V** (labeled "external"), because they refer to non-Kripke objects, and those, labeled "internal", that can be evaluated in an appropriate Kripke structure (meaning one containing certain parameters or satisfying certain exioms from IZF).

Definition 4.1.4 *(external) $\hat{P}$ is the full Kripke subset of T; that is, $\hat{P}_\sigma = T$.*

The following Kripke-internal definition is Δ_0, and hence can be evaluated and applied (such as in the definition and lemma thereafter) in any transitive Kripke set containing $\hat{P}$ even without an ambient Kripke universe. (Cf. the comments on ordinality at the end of section 4.1.2.)

Definition 4.1.5 *(internal) $B \subseteq \hat{P}$ is a branch through $\hat{P}$ if*

1. $\alpha \supseteq \beta \in B \rightarrow \alpha \in B$,
2. $\forall \alpha, \beta \in B \ \ \alpha \subseteq \beta \vee \beta \subseteq \alpha$, *and*
3. $\forall \gamma \in \hat{P} \ ((\forall \beta \in B \ \ (\gamma \subseteq \beta \vee \beta \subseteq \gamma)) \rightarrow \gamma \in B)$.

The reader may have wondered why 1_σ was taken as being 1 at nodes incompatible with σ, instead of 0, which might first have come to mind. The answer is clause 1) in the definition above. Otherwise, if we had defined $(1_\sigma)_\tau = \emptyset$ for $\tau \perp \sigma$, then consider what would hold of a branch, i.e. let $\rho \models$ "B is a branch through $\hat{P}$". By clause 3) $\rho \models$ "$B \neq \emptyset$", so let σ be such that $\rho \models$ "$1_\sigma \in B$". In practice it will be easy to extend ρ to $\tau \perp \sigma$; then $\tau \models$ "$1_\sigma \in B$". $\tau \models$ "$1_\sigma = 0$" because P is a tree (i.e. $\forall \tau' \geq \tau \ \tau' \perp \sigma$), so $\tau \models$ "$0 \in B$". By clause 1) then $\tau \models$ "$B = \hat{P}$", which would then run afoul of clause 2). So in order to have branches at all, 1_σ needed to be defined as it was.

Definition 4.1.6 *(external) If $\sigma \models U \subseteq \hat{P}$ then for $\tau \geq \sigma$ $ext_\tau(U) = \{\rho \geq \tau \mid \tau \models 1_\rho \in U\}$: "the externalisation of U at τ".*

Lemma 4.1.7 *(external) $\sigma \models$ "B is a branch through $\hat{P}$" iff for all $\tau \geq \sigma$ $ext_\tau(B)$ is a branch through $P_{\geq \tau}$ and $1_\perp \in B_\sigma$.*

proof: Recall that the notation $\mathrm{Tr}_{\geq \tau}$ for a tree Tr means the subtree of nodes extending τ (including τ itself) and the notation X_σ for a Kripke set X refers to the collection of elements forced at σ into X. Recall also that a branch of a tree classically is a non-empty, linearly ordered subset of a tree which is closed downwards and (if the tree has no terminal leaves, as in our context) has no final element.

$\rightarrow$: Suppose $\tau \geq \sigma$.

a) $\mathrm{ext}_\tau(\mathrm{B})$ is non-empty: For $\tau = \bot$ $\bot \models$ "$\forall\beta \in \hat{P}\ \beta \subseteq 1_\bot$". By clause 3) in the definition of a branch, $\bot \models$ "$1_\bot \in \mathrm{B}$", so $\bot \in \mathrm{ext}_\bot(\mathrm{B})$. For $\tau \neq \bot$ $\tau \models$ "$\forall\beta \in \hat{P}\ \beta = 1 \vee \beta \subseteq 1_\tau$". Again by clause 3), $\tau \models$ "$1_\tau \in \mathrm{B}$", and $\tau \in \mathrm{ext}_\tau(\mathrm{B})$.

b) $\mathrm{ext}_\tau(\mathrm{B})$ is linearly ordered: Suppose $\rho, \xi \in \mathrm{ext}_\tau(\mathrm{B})$. Then $\tau \models$ "$1_\rho, 1_\xi \in B$". By clause 2) in the definition of a branch, $\tau \models$ "$1_\rho \subseteq 1_\xi \ \vee\ 1_\xi \subseteq 1_\rho$". If ρ and ξ were incompatible, then $\rho \not\models$ "$1_\xi \subseteq 1_\rho$" and $\xi \not\models$ "$1_\rho \subseteq 1_\xi$". This, however, contradicts the semantics of "$\vee$" in Kripke structures, by which one of the two disjuncts must hold. Hence ρ and ξ are compatible.

c) $\mathrm{ext}_\tau(\mathrm{B})$ is closed downwards: Suppose $\rho \in \mathrm{ext}_\tau(\mathrm{B})$ and $\tau \leq \xi \leq \rho$. Then $\tau \models$ "$1_\rho \in B$" and $\tau \models$ "$1_\rho \subseteq 1_\xi$". By clause 1) in the definition of a branch, $\tau \models$ "$1_\xi \in B$", and $\xi \in \mathrm{ext}_\tau(\mathrm{B})$.

d) $\mathrm{ext}_\tau(\mathrm{B})$ has no final element: Suppose to the contrary that ρ is the final element of $\mathrm{ext}_\tau(\mathrm{B})$. Consider $\mathrm{ext}_\rho(\mathrm{B})$. From the proof of part a) above, $\rho \in \mathrm{ext}_\rho(\mathrm{B})$. If $\mathrm{ext}_\rho(\mathrm{B}) = \{\rho\}$, then let π be an immediate successor of ρ (which exists since P was assumed to have an ordinal height and no degenerate nodes; see the comments after corollary 4.1.3). Again from part a), we have already seen that $\pi \models$ "$\forall\beta \in \hat{P}\ \beta = 1 \vee \beta \subseteq 1_\pi$"; since branches are, by definition, subsets of $\hat{P}$, we have in particular $\pi \models$ "$\forall\beta \in B\ 1_\pi \subseteq \beta \vee \beta \subseteq 1_\pi$". At an extension ξ of ρ incompatible with π, $\xi \models$ "$1_\pi = 1$", so $\xi \models$ "$\forall\beta \in B\ \beta \subseteq 1_\pi$". Finally, at ρ itself, the only element forced at ρ into B is, by hypothesis, 1_ρ, and $\rho \models$ "$1_\pi \subseteq 1_\rho$". To summarize, $\rho \models$ "$\forall\beta \in B\ 1_\pi \subseteq \beta \vee \beta \subseteq 1_\pi$". By clause 3), $\rho \models$ "$1_\pi \in B$", and $\pi \in \mathrm{ext}_\rho(\mathrm{B})$, contrary to hypothesis. So $\mathrm{ext}_\rho(\mathrm{B}) \neq \{\rho\}$.

Let $\pi \neq \rho$ be some member of $\mathrm{ext}_\rho(\mathrm{B})$. We will show that $\pi \in \mathrm{ext}_\tau(\mathrm{B})$, contradicting the choice of ρ as the final element of $\mathrm{ext}_\tau(\mathrm{B})$. This argument, once again, uses clause 3). Our goal is to show that $\tau \models$ "$\forall\beta \in B\ 1_\pi \subseteq \beta \vee \beta \subseteq 1_\pi$". Once that is accomplished, it follows that $\tau \models$ "$1_\pi \in B$", which means $\pi \in \mathrm{ext}_\tau(\mathrm{B})$.

So let $\xi \geq \tau$. If ξ and ρ are incompatible, then so are ξ and π (π extends ρ). So $\xi \models$ "$1_\pi = 1$", and 1 is a superset of every member of $\hat{P}$. If ξ and ρ are compatible, then either $\rho \leq \xi$ or $\tau \leq \xi < \rho$. In the first case, $\xi \models$ "$1_\pi \in B$", because ρ models the same. By clause 2) in the definition of a branch, $\xi \models$ "$\forall\beta \in B\ 1_\pi \subseteq \beta \vee \beta \subseteq 1_\pi$". In the second case, $\tau \leq \xi < \rho$, we have in any case $\xi \models$ "$1_\rho \in B$", because τ models the same. If $\xi \models$ "$\beta \in B$", then by clause 2) $\xi \models$ "$1_\rho \subseteq \beta \vee \beta \subseteq 1_\rho$". If $\xi \models$ "$1_\rho \subseteq \beta$" then $\xi \models$ "$1_\pi \subseteq \beta$", because $\xi \models$ "$1_\pi \subseteq 1_\rho$". If, on the other hand, $\xi \models$ "$\beta \subseteq 1_\rho$", then $\rho \models$ "$\beta \subseteq 1_\rho$". That implies that $\beta = 1_\mu$ for some $\mu \geq \rho$. Furthermore, by clause 2), $\rho \models$ "$1_\pi \subseteq 1_\mu$" or $\rho \models$ "$1_\mu \subseteq 1_\pi$". In the first case we have $\pi \geq \mu$, in the second $\mu \geq \pi$, and in either case $\xi \models$ "$1_\pi \subseteq 1_\mu$" or $\xi \models$ "$1_\mu \subseteq 1_\pi$".

Finally, for this direction of the proof it remains to show that $1_\bot \in B_\sigma$. $\bot \models$ "$\forall\beta \in \hat{P}\ \beta \subseteq 1_\bot$". By clause 3), $\sigma \models$ "$1_\bot \in B$".

$\leftarrow$:

1. $\sigma \models$ "$\alpha \supseteq \beta \in B \rightarrow \alpha \in B$": Suppose $\sigma \models$ "$1_\tau \supseteq 1_\rho \in B$". If $\tau \bot \sigma$ or

$\tau < \sigma$ then $\sigma \models$ "$1_\tau = 1 = 1_\perp \in B$". Otherwise $\tau \geq \sigma$. Then $\tau \leq \rho$, because otherwise $\tau \models$ "$1_\rho = 1$" and $\tau \not\models$ "$1_\tau = 1$". Since $\rho \in \mathrm{ext}_\sigma(\mathrm{B})$, which is by hypothesis a branch, and hence closed downwards, we have $\tau \in \mathrm{ext}_\sigma(\mathrm{B})$, i.e. $\sigma \models$ "$1_\tau \in B$".

2. $\sigma \models$ "$\forall \alpha, \beta \in B \quad \alpha \subseteq \beta \vee \beta \subseteq \alpha$": Suppose $\sigma \models$ "$1_\tau, 1_\rho \in B$". If $\tau \perp \sigma$ or $\tau < \sigma$ then $\sigma \models$ "$1_\tau = 1 \supseteq 1_\rho$". Similarly if $\rho \perp \sigma$ or $\rho < \sigma$. So $\tau, \rho \geq \sigma$. By hypothesis $\tau, \rho \in \mathrm{ext}_\sigma(\mathrm{B})$, which is a branch, and hence linearly ordered; therefore $\tau \leq \rho$ or $\rho \leq \tau$, implying $\perp \models$ "$\rho \subseteq \tau$" or $\perp \models$ "$\tau \subseteq \rho$" respectively.

3. $\sigma \models$ "$\forall \gamma \in \hat{P}\ (\forall \beta \in B\ \ \gamma \subseteq \beta \vee \beta \subseteq \gamma \rightarrow \gamma \in B)$": Suppose $\sigma \models$ "$\forall \beta \in B\ \ 1_\tau \subseteq \beta \vee \beta \subseteq 1_\tau$". If $\tau \perp \sigma$ or $\tau < \sigma$ then $\sigma \models$ "$1_\tau = 1 = 1_\perp \in B$". Otherwise $\tau \geq \sigma$. If τ were incompatible with some $\rho \in \mathrm{ext}_\sigma(\mathrm{B})$, then $\tau \not\models$ "$1_\rho \subseteq 1_\tau$" and $\rho \not\models$ "$1_\tau \subseteq 1_\rho$", yielding $\sigma \not\models$ "$1_\tau \subseteq 1_\rho \vee 1_\rho \subseteq 1_\tau$", contrary to the hypothesis. Since $\mathrm{ext}_\sigma(\mathrm{B})$ as a branch is a maximal linearly ordered subset of $P_{\geq\sigma}$, $\tau \in \mathrm{ext}_\sigma(\mathrm{B})$, and $\sigma \models$ "$1_\tau \in B$".

▮

4.2 Resolvability + IKP + V=L + $\neg\Sigma_1$ DC

To be perfectly clear, we really should state what Resolvability means in this context. The standard definition, that there is a Δ_1-definable function f : ORD $\rightarrow$ **V** such that $\mathbf{V} = \bigcup \mathrm{rng(f)}$, is perfectly coherent, but could be contested as not being the appropriate correlate to classical resolvability in an intuitionistic setting. After all, maybe the linearity of the domain is what's vital, so a resolution should have domain some linearly ordered set of ordinals, presumably a Δ_1 definable set of ordinals. We will take the latter as the definition of a resolution, because it seems to be a more difficult definition to fit (although it is not clear that a function defined on a Δ_1 set of ordinals could be extended to all the ordinals), and point out that if $\mathbf{V} = \mathbf{L}$ then $\alpha \mapsto L_\alpha$ shows resolvability according to the former notion of a resolution. In fact, the resolution we will be offering will even be extendable to a Δ_1 function on the whole model.

The underlying partial order of the model will be $2^{<\omega}$, with the order being end-extension ($\sigma < \tau$ iff τ is a proper end-extension of σ.). For each real R : $\omega \rightarrow 2$ there is a canonical branch B(R) through T:

$$ext_\tau(B(R)) = \{\rho \geq \tau \mid \forall j\ length(\tau) \leq j < length(\rho) \rightarrow \tau(j) = R(j)\},$$

or, equivalently,

$$B(R)_\tau = \{1_\rho \mid \rho < \tau \vee \rho \perp \tau \vee (\rho \geq \tau \wedge \forall j\ length(\tau) \leq j < length(\rho) \rightarrow \tau(j) = R(j))\}.$$

B(R) is a branch by the preceding lemma. Notice that if R_0 and R_1 differ cofinally then $\perp \models B(R_0) \neq B(R_1)$.

Start with a model **V** of classical ZF. Let $G := \{G_i \mid i \in \omega\}$ be a set of reals mutually Cohen-generic over **V** (that is, reals generic for the forcing partial order $2^{<\omega}$). Actually, the rest of this section save the last lemma applies just as well to any set of (mutually cofinally differing) reals. (This last fact we will use in section 4.4, where many constructions and lemmas similar to those below are used, often with the offered proof being merely a reference to the corresponding proof here, even though the real there is not even generic.) In **V**[G], construct the (Kripke) branches $B(G_i)$, $i \in \omega$. For convenience, call the branch $B(G_i)$ just B_i. Consider $\{B_i \mid i \in \omega\} \cup \{\hat{P}\}$. Let ξ be the (Kripke) ordinal from corollary 4.1.3 applied to this set; in particular, each B_i and $\hat{P}$ is a member of L_ξ.

Lemma 4.2.1 *$\{B_i \mid i \in \omega\}$ is definable over L_ξ, as the set of branches through $\hat{P}$.*

proof: $L_\xi = \bigcup_{\alpha \in \hat{P}_0} \mathrm{def}(L_\alpha) \cup \bigcup_{i \in \omega} \mathrm{def}(L_{B_{i0}}) \cup \mathrm{def}(L_{\hat{P}_0})$. The first union is just $\hat{P}_0$, by lemma 4.1.1, which doesn't contain a branch. Each term in the second union, $\mathrm{def}(L_{B_{i0}})$, is just $\mathrm{def}(B_{i0})$, again by 4.1.1. No branch contains 0, because, if it did, by clause 1) in the definition of a branch, it would then contain all of $\hat{P}$, contradicting clause 2). So any branch which is a subset of B_{i0} is also a subset of B_i. But B_i is itself a branch, and no proper subset of a branch is a branch, because of clause 3), as follows. Suppose B is a branch, $\hat{B} \subseteq B$, and $\hat{B}$ is also a branch. If $\gamma \in B$ and $\beta \in \hat{B}$, $\beta \in B$ too; using that B is a branch, $\gamma \subseteq \beta$ or $\beta \subseteq \gamma$ by clause 2). Since $\hat{B}$ is a branch, $\gamma \in \hat{B}$ by clause 3). So $B \subseteq \hat{B}$, and $B = \hat{B}$. In the final case, $\mathrm{def}(L_{\hat{P}_0})$, or, more simply, again by 4.1.1, $\mathrm{def}(\hat{P}_0)$, consider any definition, say $\phi(x)$, over $\hat{P}_0$. It contains only finitely many parameters 1_σ. Go to a node τ beyond or incompatible with all such σ's. $\tau \models \phi(1_\rho)$ for some extension ρ of τ iff $\tau \models \phi(1_\rho)$ for all extensions ρ of τ of the same length, by symmetry. (There is an automorphism of $\hat{P}_0$ switching any given pair of ρ's.) So ϕ does not define a branch. ∎

The model **M** can now be defined inductively:

$$\mathbf{M}_0 = L_\xi;$$

$$\mathbf{M}_{\alpha+1} = def(\mathbf{M}_\alpha);$$

$$\mathbf{M}_\lambda = \bigcup_{\alpha<\lambda} \mathbf{M}_\alpha; \textit{ and}$$

$$\mathbf{M} = \mathbf{M}_{\omega_1^{CK}}.$$

Lemma 4.2.2 $\mathbf{M} \models$ *IKP.*

proof: Most of the axioms depend on the fact that if $\beta > \alpha$ then $\mathbf{M}_\beta$ is an end-extension of $\mathbf{M}_\alpha$. For instance, as has already been mentioned, $\mathbf{M}_\alpha$ is proven to be extensional inductively on α, using this end-extension property for the limit cases. As another example, consider Pairing. If y, z $\in \mathbf{M}$, let α be such that y, z $\in \mathbf{M}_\alpha$. Now consider ["x = y $\vee$ x = z"] $\in \mathbf{M}_{\alpha+1}$. This set is the pair {y, z} in $\mathbf{M}_{\alpha+1}$; since $\mathbf{M}$ end-extends $\mathbf{M}_{\alpha+1}$, the same fact holds in $\mathbf{M}$. Empty and Union are similar. Δ_0 Comprehension also depends on this, being the reason Δ_0 formulas are absolute between $\mathbf{M}_\alpha$ and $\mathbf{M}$. (So if ϕ(x) is Δ_0, and all of its parameters, as well as X, are in $\mathbf{M}_\alpha$, then [x $\in$ X $\wedge$ ϕ(x)] $\in \mathbf{M}_{\alpha+1}$ is the desired subset in $\mathbf{M}$.) Even Foundation uses end-extensionality. Suppose $\mathbf{M} \not\models$ Foundation. Let ϕ and σ be such that $\sigma \models \forall x((\forall y \in x\ \phi(y)) \rightarrow \phi(x))$ and $\sigma \not\models \forall x\ \phi(x)$. Let $\tau \geq \sigma$ and $x \in (\mathbf{M}_\alpha)_\tau$ be such that $\tau \not\models \phi(x)$; moreover, assume that α is the least such ordinal. By the choice of α, if y $\in (\mathbf{M}_\beta)_\xi$ ($\beta < \alpha$) then $\xi \models \phi(y)$. By end-extensionality, if $\xi \models y \in x$ then y $\in \mathbf{M}_\beta$, some $\beta < \alpha$. So $\tau \models \forall y \in x\ \phi(y)$, hence, by hypothesis, $\tau \models \phi(x)$, a contradiction. So $\mathbf{M} \models$ Foundation.

For Δ_0 Bounding, $\mathbf{M}$ can be considered to be constructed by a Σ_1 induction over the standard $L_{\omega_1^{CK}}[G]$, so any $\Delta_0(\mathbf{M})$ formula ϕ is equivalent to a $\Delta_0(L_{\omega_1^{CK}}[G])$ formula ϕ^*. (This would be proven inductively on formulas, with, for instance, $\phi \rightarrow \psi$ going to $\forall \sigma \in 2^{<\omega} \sigma \models \phi \rightarrow \sigma \models \psi$. The important observation is that only bounded quantifiers are added.) So if $\mathbf{M} \models$ "$\forall x \in A\ \exists y\ \phi(x,y)$" then in $L_{\omega_1^{CK}}[G]$ $\forall x \in_{\mathbf{M}} A\ \exists \alpha\ \exists y \in \mathbf{M}_\alpha\ \phi^*(x,y)$, and by the admissibility of $L_{\omega_1^{CK}}[G]$ the α's needed can be bounded.

$\mathbf{M}$ also satisfies Infinity. To see this, define inductively on n $\in \omega$ the set $\overline{n} \in \mathbf{M}$, the internal version of n, as follows: $\overline{0} = \emptyset^{\mathbf{M}}$, $\overline{n+1} = (n \cup \{n\})^{\mathbf{M}}$. Inductively, $\overline{n} \in \mathbf{M}_n$; in fact, $\overline{n+2} \in \mathbf{M}_n$. Definably over $\mathbf{M}_\omega$, let $\overline{\omega}$ be $\{\alpha \mid \alpha$ is an ordinal, and either $\alpha = 0$ or α is a successor, and $\forall \beta \in \alpha\ \beta$ is 0 or a successor$\}$. $\overline{\omega}$ witnesses that $\mathbf{M} \models$ Infinity; moreover, one can show inductively that $\overline{\omega} \cap \mathbf{M}_n = \{\overline{m} \mid m \leq n+2\}$, so $\overline{\omega} = \{\overline{n} \mid n \in \omega\}$. ▮

Lemma 4.2.3 $\mathbf{M}$ *is resolvable.*

proof: The resolution of $\mathbf{M}$ will be along $\overline{\omega_1^{CK}}$, which is meant as the internalization of ω_1^{CK}, as $\overline{\omega}$ is for ω. $\overline{\omega_1^{CK}}$ is defined (in $\mathbf{V}$[G]) as $\{ \overline{\alpha} \mid \alpha < \omega_1^{CK} \}$, where $\overline{\alpha} \in \mathbf{M}$ is $\{ \overline{\beta} \mid \beta < \alpha \}$. We need to show that for $\alpha < \omega_1^{CK}$ $\overline{\alpha} \in \mathbf{M}$. Say that an $\mathbf{M}$-ordinal is standard if it's 0, the successor of a standard ordinal, or an $\overline{\omega}$- limit of standard ordinals. Notice that this definition is internal to $\mathbf{M}$. By the Second Recursion Theorem, there is a Σ definition of standardness in $\mathbf{M}$. Inductively on $\alpha \geq \omega$, the standard ordinals in $\mathbf{M}_\alpha$ are a subset of $\overline{\alpha}$. To show that for $\alpha < \omega_1^{CK}$, $\overline{\alpha} \in \mathbf{M}$, assume inductively that for $\beta < \alpha$ $\overline{\beta} \in \mathbf{M}$. By the admissibility of the ambient universe (i.e. $L_{\omega_1^{CK}}[G]$), there is a γ such that for each $\beta < \alpha$ both $\overline{\beta}$ and a witness that $\overline{\beta}$ is standard are in $\mathbf{M}_\gamma$. (Here, a witness to standardness is a witness to the Σ definition of standardness.) Then either $\overline{\alpha}$ is definable over $\mathbf{M}_\gamma$ as the set of standard ordinals, or $\overline{\alpha}$ is already in $\mathbf{M}_\gamma$. Hence $\overline{\omega_1^{CK}} \subseteq \mathbf{M}$.

In order to have a resolution along $\overline{\omega_1^{CK}}$, we need it Δ_1 definable. It's already Σ definable, as the set of standard ordinals. But the witnesses to standardness are not really far away from $\overline{\alpha}$ itself. That is, classically, a witness to the standardness of α, consisting of suitable predecessors and cofinal ω-sequences, could be found in $L_{\alpha+\omega}$. So to see whether $\overline{\alpha}$ is standard in $\mathbf{M}$, look in $L_{\overline{\alpha}+\overline{\omega}}$. The function $\alpha \mapsto L_\alpha$ is total in $\mathbf{M}\models$ IKP (see [7]), so the Σ_1 definition of "α is not standard" is "there is a set $L_{\alpha+\overline{\omega}}$ and there is no witness in that set that α is standard". The Σ definition of standardness is equivalent to a Σ_1 formula. Now we need to check that this suffices for the notion of Δ_1 defined earlier. Recall that toward the beginning of section 2 a Δ_1 property of a variable x was defined as being given by a pair of Δ_0 formulas $\phi(x, y), \psi(x, y)$ with free variables x, y such that

$$\forall x[\exists y\neg\neg(\phi(x, y) \vee \psi(x, y)) \ \wedge \forall y\neg(\phi(x, y) \wedge \psi(x, y))].$$

So let $\phi(x, y)$ be

$$x \in ORD \ \wedge \ y = \langle z_0, z_1\rangle \ \wedge \ z_1 \textit{ witnesses that } z_0 = \mathbf{L}_{x+\overline{\omega}}$$

$$\wedge \ \exists w \in z_0 \ w \textit{ witnesses that } x \textit{ is standard},$$

and $\psi(x, y)$ be

$$(x \notin ORD) \ \vee \ (x \in ORD \ \wedge \ y = \langle z_0, z_1\rangle \ \wedge \ z_1 \textit{ witnesses that } z_0 = \mathbf{L}_{x+\overline{\omega}}$$

$$\wedge \ \forall w \in z_0 \ w \textit{ does not witness that } x \textit{ is standard}),$$

where "$x+\alpha$" is defined inductively on α even for non-ordinals x in the obvious way. For ordinals x there is certainly a y as desired, and it is also clear that $\forall x \ \neg\neg(x \in ORD \ \vee \ x \notin ORD)$. The challenge presented by the formalism as set up is that the y must be chosen before it is decided whether x is an ordinal or not. Here we have to use the development from the penultimate paragraph of section 4.1.2, in which iterations of definability along well-founded (and not necessarily well-ordered) lengths were developed. Let z_0 be $(x + \overline{\omega})$-def($\mathbf{0}$) and z_1 an appropriate construction of such. If x ever turns out to be an ordinal, then z_0 is automatically $\mathbf{L}_{x+\overline{\omega}}$. With this ϕ and ψ, the notion "α is standard" is shown to be Δ_1 definable.

From here, the resolution of $\mathbf{M}$ is its very definition. L_ξ is needed as a parameter, which is no problem because it exists as a set in $\mathbf{M}$: L_ξ is definable over $\mathbf{M}_0 = L_\xi$ as $\{ x \mid x = x \}$, and so is in $\mathbf{M}_1$. "$Y = \bigcup X$" is a Σ relation, as is "Y = def(X)" (for details on the latter, see [7]). So $\alpha \to \mathbf{M}_\alpha$ is definable via

Σ recursion as a Σ relation, hence Σ_1 also. Any Σ_1 definable function is Δ_1, even with the notion of Δ_1 operable here, for the same reason as in the classical context, as follows. Suppose the relation f(x) = y is definable as $\exists z\ \phi(x, y, z)$. Then let $\psi(x, y, w, z)$ be $\phi(x, w, z)\ \wedge\ y \neq w$. Then indeed

$$\forall x, y[\exists w, z\neg\neg(\phi(x, y, z) \vee \psi(x, y, w, z))\ \wedge \forall w, z\neg(\phi(x, y, z) \wedge \psi(x, y, w, z))].$$

Hence **M** is resolvable. ▌

Lemma 4.2.4 **M** $\models$ *V=L.*

proof: First observe that $L^{\mathbf{M}} \models$ IKP. Extensionality and Foundation hold, because $L^{\mathbf{M}}$ is a definable sub-class of a model of Extensionality and Foundation. Pairing is simple: if x $\in L_\alpha$ and y $\in L_\beta$ then x, y $\in L_{\alpha\cup\beta}$, and {x, y} $\in$ def($L_{\alpha\cup\beta}$) = $L_{(\alpha\cup\beta)+1}$. Union is similar: if x $\in L_\alpha$ then $\bigcup x \in$ def(L_α) = $L_{\alpha+1}$. $\emptyset \in$ def(L_0) = L_1, and $\overline{\omega} \in$ def($L_{\overline{\omega}}$) = $L_{\overline{\omega}+1}$ to satisfy Infinity; for details on the latter, see [7]. For Δ_0 Comprehension, consider { x $\in$ X | ϕ(x) }, ϕ a Δ_0 formula. Let α be such that L_α contains X as well as ϕ's parameters. Then { x $\in$ X | ϕ(x) } $\in$ def(L_α) = $L_{\alpha+1}$. Even Δ_0 Bounding is smooth. Suppose $\forall x \in X\ \exists y \in L^{\mathbf{M}}\ \phi(x, y),\ \phi\Delta_0$. In **M**, $\forall x \in X\ \exists\alpha\ \exists y \in L_\alpha\ \phi(x, y)$. By Σ Bounding in **M**, let A be a set of ordinals such that $\forall x \in X\ \exists\alpha \in A\ \exists y \in L_\alpha\ \phi(x, y)$. Let β = A $\cup \bigcup$A. $\forall x \in X\ \exists y \in L_\beta\ \phi(x, y)$, and $L_\beta \in L^{\mathbf{M}}$, showing Δ_0 Bounding.

For any $\alpha < \omega_1^{CK}$, since $\overline{\alpha} \in \mathbf{M}$, $L_{\overline{\alpha}} \in \mathbf{M}$. $\overline{\alpha}$ is definable over $L_{\overline{\alpha}}$, so $\overline{\alpha} \in L^{\mathbf{M}}$. Also, {$B_i$ | $i \in \omega$} $\in$ **M**, as is $\hat{P}$, so $\xi \in$ **M**, and $L_\xi \in L^{\mathbf{M}}$. Finally, induction shows that any admissible set containing L_ξ as well as each $\overline{\alpha} < \overline{\omega_1^{CK}}$ must contain all of **M**. The crucial fact needed in this induction is that the function X $\mapsto$ def(X) is Δ_1 definable, provably in IKP. This was essentially done in [7], even though it was not flagged as such. There it was shown, in IZF, that this function in Δ_1 over all $\beta > \alpha_{aug} + 7$, where X $\in L_{\alpha_{aug}}$ and α_{aug} is defined inductively as $\bigcup\{\gamma_{aug} \mid \gamma \in \alpha\} \cup \{\alpha\} \cup (\omega + 1)$. (The purpose of this definition is to get ω into α, hereditarily, as the carrier of syntax.) The proof is that a witness to "Y = def(X)" is definable over $L_{\alpha_{aug}+7}$, so IZF isn't necessary; all you need is some ordinal containing as a member $\alpha_{aug} + 7$. Moreover, the recursion theorem shows that in IKP + Infinity the function $\alpha \mapsto \alpha_{aug} + 7$ is total. So in IKP, X $\mapsto$ def(X) is Δ_1. This allows the inductive proof that $\mathbf{M}_\alpha \in L^{\mathbf{M}}$, $\alpha \in \omega_1^{CK}$. ▌

Lemma 4.2.5 **M** $\models \neg\Sigma_1$ *DC.*

proof: This is where we will use the particular choice of the G_i's. Consider the relation ϕ(X, Y) "if X is an n-tuple of distinct elements of {B_i | $i \in \omega$}

then Y is an end-extension of X also of distinct elements". Any function f with domain ω satisfying ϕ(f(i), f(i+1)) (and starting from $\emptyset$) would easily produce an ω-sequence through $\{G_i \mid i \in \omega\}$ of distinct elements, of which there are none in $\mathbf{V}$[G], much less $L_{\omega_1^{CK}}$[G], to say nothing of $\mathbf{M}$. ▌

4.3 Π_2-Reflection + IKP + V=L + $\neg\Sigma_1$DC + ¬Resolvability

Recall the definition of a resolution (see section 4.2) as being a Δ_1-definable function f : O$\rightarrow\mathbf{V}$ such that $\mathbf{V} = \bigcup$rng(f) and O $\subseteq$ ORD is Δ_1-definable and linearly ordered. In the last section, the argument works just as well for this definition as for the classical one (O = ORD, of course not mentioning the requirement of the linear order). Here, though, it wouldn't. If $\mathbf{V}$ = L then f(α) = L_α is a classical resolution of $\mathbf{V}$. Hence the need here for a non-classical notion of a resolution.

The underlying partial order is $2^{<\omega}$, as before.

Let G and H be countable sets of mutual Cohen generics. The ambient universe will be $\bigcup_{H_{fin}} L_{\omega_1^{CK}}[G \cup H_{fin}]$, where H_{fin} ranges over the finite subsets of H. Within this universe the full intuitionistic model $\mathbf{M}_{big}$ can be defined inductively, just like forcing: a set at stage α is a function that, for each node $\sigma \in 2^{<\omega}$, picks out a collection of sets from stages $< \alpha$ (and this collection must grow as σ grows).

More precisely, suppose $\mathbf{M}$ is an extensional Kripke model (for the language of set theory) with underlying partial order P and transition functions $f_{\sigma\tau}$ within a classical universe $\mathbf{V}$. P($\mathbf{M}$), the power set of $\mathbf{M}$, can be defined similarly to the way def($\mathbf{M}$) was defined in section 4.1. A set in P($\mathbf{M}$) at a node σ is a function f with domain $P_{\geq\sigma}$ such that f(τ) $\subseteq \mathbf{M}_\tau$, and if x $\in$ f(σ) then $f_{\sigma\tau}$(x) $\in$ f(τ). The transition function $f_{\sigma\tau}$ is restriction: $f_{\sigma\tau}$(f) = f $\upharpoonright P_{\geq\tau}$. $\mathbf{M}$ can be embedded in P($\mathbf{M}$), with x $\in \mathbf{M}_\sigma$ being identified with the function f such that f(τ) = { y $\in \mathbf{M}_\tau$ | y $\in f_{\sigma\tau}$(x) }. By the extensionality of $\mathbf{M}$, this embedding is 1-1. $\sigma \models$ "$f \in g$" if f is the image of some x under this embedding, and x $\in$ g(σ). P($\mathbf{M}$) is easily seen to be extensional, and an end-extension of $\mathbf{M}$. Furthermore, this process of taking the power set can be iterated transfinitely: $\mathbf{M}_0 = \mathbf{M}$, $\mathbf{M}_{\alpha+1} = P(\mathbf{M}_\alpha)$, and $\mathbf{M}_\lambda = \overset{lim}{\rightarrow} \{\mathbf{M}_\alpha \mid \alpha < \lambda\}$ (or, more colloquially, $\bigcup_{\alpha<\lambda} \mathbf{M}_\alpha$). If $\alpha > \beta$ then $\mathbf{M}_\alpha$ is an extensional end-extension of $\mathbf{M}_\beta$.

In the current context, $\mathbf{V}$ is $\bigcup_{H_{fin}} L_{\omega_1^{CK}}[G \cup H_{fin}]$. $\mathbf{V} \models$ KP, but $\mathbf{V}$ does <u>*not*</u> satisfy Power Set, so some care must be taken in the construction of P($\mathbf{M}$). We start with $\mathbf{M}$ being the empty structure – $\mathbf{M}_\sigma = \emptyset$ – but we do not claim that each $\mathbf{M}_\alpha$ is a set. Rather, the generation of the $\mathbf{M}_\alpha$-hierarchy is given by a Σ_1 induction, so the relation "f $\in (\mathbf{M}_\alpha)_\sigma$" is Σ_1. Let $\mathbf{M}_{big}$ be $\mathbf{M}_{\omega_1^{CK}}$. The final model $\mathbf{M}$ is $L^{\mathbf{M}_{big}}$; that this definition makes sense (i.e. that $\alpha \mapsto L_\alpha$ is a total Δ_1-definable function via the standard definitions) is implied by the following lemma.

Lemma 4.3.1 $\mathbf{M}_{big} \models$ *IKP.*

proof: For Empty, the function f(σ) = $\emptyset$ is in $\mathbf{M}_1$. For Infinity, defining $\overline{\omega}$ as in the previous section, the function f(σ) = $\overline{\omega}$ is in $M_{\omega+1}$. For Pairing, if x, y $\in \mathbf{M}_\alpha$, let f(σ) in $\mathbf{M}_{\alpha+1}$ be {x, y}. For Union, if x $\in \mathbf{M}_\alpha$, then f(σ) = $\bigcup$x(σ) is also in $\mathbf{M}_\alpha$. Extensionality was already remarked as holding. For Δ_0 Comprehension, by end-extensionality a Δ_0 formula over $\mathbf{M}_{big}$ is equivalent to a Δ_0 formula in $\mathbf{V}$, and Δ_0 Comprehension in $\mathbf{V}$ suffices. For Foundation, suppose to the contrary that $\sigma \models \forall x(\forall y \in x\ \phi(y)\ \rightarrow\ \phi(x))$, but $\sigma \not\models \forall x \phi(x)$. Let α be the least ordinal such that for some x $\in \mathbf{M}_\alpha$, $\tau \geq \sigma\ \tau \not\models \phi(x)$. But if $\xi \models y \in x$ then $y \in \mathbf{M}_\beta$, so $\xi \models \phi(y)$. Hence $\tau \models \forall y \in x\ \phi(y)$, and $\tau \models \phi(x)$, a contradiction. Finally, for Δ_0 Bounding, suppose $\sigma \models \forall x \in A\ \exists y\ \phi(x, y)$, ϕ a Δ_0 formula. Then in $\mathbf{V}$ $\forall \tau \geq \sigma\ \forall x \in A_\tau\ \exists \alpha\ \exists y \in \mathbf{M}_\alpha\ \tau \models \phi(x, y)$. By Δ_0 Bounding in $\mathbf{V}$, the αs needed can be bounded, say by β, and again by Δ_0 Bounding in $\mathbf{V}$ a bounding set B in $\mathbf{M}_{\beta+1}$ can be constructed. ▮

Corollary 4.3.2 $\mathbf{M} \models$ *V=L.*

proof: This is the basic result of [7]. ▮

Lemma 4.3.3 $\mathbf{M} \models$ *IKP +* Π_2 *Reflection.*

proof: The minor axioms of IKP are all easy. (For Extensionality, use the fact that $\mathbf{M}$ is a transitive sub-class of the extensional $\mathbf{M}_{big}$. For Foundation, use the definability of $\mathbf{M}$ inside of the well-founded $\mathbf{M}_{big}$.) Δ_0 Comprehension follows from the definition of L and the absoluteness of Δ_0 formulas, just as in the classical case. Δ_0 Bounding follows from Π_2 Reflection. For Π_2 Reflection, suppose $\mathbf{M} \models \forall x\ \exists y\ \phi(x, y)$, ϕ a Δ_0 formula, and A is a set in $\mathbf{M}$. We will enlarge A to a transitive model of the same sentence. Work in the ambient universe $\mathbf{V}$. All of the parameters in sight (i.e. A and those of ϕ) are members of $L_{\omega_1^{CK}}[G \cup H_{fin}]$ for a fixed H_{fin}. Moreover, for any $\Delta_0(\mathbf{M})$ formula ϕ, there is an equivalent $\Delta_0(\mathbf{V})$ formula ϕ^*, exactly as in 4.2.2. By standard symmetry arguments, no more members of H are needed to find witnesses y in $\mathbf{V}$. So for each x in A let α_x be an ordinal such that a witness for x is in $L_{\alpha_x}[G \cup H_{fin}]$. Moreover, $L_{\alpha_x}[G \cup H_{fin}]$ should also include all the relevant evidence that the witness is in $\mathbf{M}$. Using the admissibility of the ambient universe, let α_0 bound the α_x's needed. Of course, this must be iterated to account for the new sets in $L_{\alpha_0}[G \cup H_{fin}]$ which are in $L^{\mathbf{M}}$ (via an ordinal also in $L_{\alpha_0}[G \cup H_{fin}]$). Iterate

ω-many times. Let α be $\bigcup \alpha_n$, and β be {x | x is an ordinal in $\mathbf{M}_{big}$} over L_α[G $\cup$ H_{fin}]. L_β reflects ϕ. ▮

Lemma 4.3.4 **M** $\models \neg\Sigma_1 DC$.

proof: As in 4.2.5, there can be no sequence of distinct members of {B(R) | $R \in$ G}, which is a set in L. ▮

Lemma 4.3.5 **M** $\models \neg$*Resolvability.*

proof: This argument is somewhat technical, but since the idea is rather simple we will give a quick overview of it. Suppose there were such a resolution. Since **M** is definable in the ambient universe, so is this resolution. Any definition can use only finitely many parameters, in particular only finitely many members of H. Consider branches of the form B(R), R $\in$ H. Which ordinal each gets attached to is determined by some forcing condition. Furthermore, beyond the finitely many parameters, such a forcing condition need mention only (that is, is an initial segment of) R. By mutual genericity, for every initial segment of a real there are infinitely many R's that share that initial segment. So any forcing condition attaching B(R) to an ordinal forces infinitely many different B(R)'s to that same ordinal. Therefore there is a set which contains infinitely many B(R)'s, R $\in$ H, a contradiction.

The following proof makes essential use of the definability of the forcing relation "p $\Vdash \phi$" over the ground model $L_{\omega_1^{CK}}$[G] for the forcing partial order to get the set H. The forcing partial order is { p | $p : \omega \to 2^{<\omega}$, supp(p) finite}, where supp(p), the support of p, is { n $\in \omega$ | $p(n) \neq \emptyset$}. The resulting generic is (trivially identifiable with) a function f_H with domain ω and range H. We will assume that the forcing language has a constant H_n for each $n \in \omega$, which is interpreted in the generic extension as $f_H(n)$, as well as constants for f_H itself and for the range of f_H, for which we will ambiguously use the symbols f_H and H respectively (f_H and H being also sets in the generic extension; in effect, the constant H will be interpreted as the set H and similarly for f_H; whether the constant symbol or its interpretation is meant every time we say "H" of "f_H" should be clear from the context). To name the elements in the generic extenion $L_{\omega_1^{CK}}$[G, H] it suffices to consider the language generated by the symbols H and the H_n's; f_H is not necessary. To name the elements in **M** it suffices to consider the language generated by the H_n's; H is not necessary. (Recall the definition of **V**, over which **M** was defined.)

M is a structure definable over $L_{\omega_1^{CK}}$[G, H], and so the truth predicate over **M**, "$\sigma \models \phi$", is also definable over $L_{\omega_1^{CK}}$[G, H]. It follows that the forcing relation p $\Vdash$ "$\sigma \models \phi$" is definable over $L_{\omega_1^{CK}}[G]$. If the reader feels uncomfortable with the indirectness hidden behind the simple notation (**M** being a definable sub-class of $\mathbf{M}_{big}$, itself a union not of sets but of definable classes), then to save the trouble of the detailed argument needed to prove the definability of the forcing relation, with just a tad more set-theoretic power we can take the ground model to be $L_{\omega_2^{CK}}[G]$, ω_2^{CK} the first admissible ordinal after ω_1^{CK}. (Our claim is, ultimately, that the results here are theorems of ZF.) In $L_{\omega_2^{CK}}[G, H]$ **M** is a set, so the forcing relation p $\Vdash$ "$\sigma \models \phi$" for statements about **M** is definable, Δ_1 even. So take the ground model **U** to be $L_{\omega_1^{CK}}[G]$ or $L_{\omega_2^{CK}}[G]$, whichever you feel more comfortable with.

Suppose for a contradiction that Resolvability were to hold in **M**: $\sigma \models$ "f is a resolution". Let p $\in f_H$ force such: p $\Vdash$ "$\sigma \models$ f is a resolution". Let supp(f) be $\{$ n $\mid$ H_n occurs in the definition of f$\}$. f mentions only finitely many paramters, each of which is ultimately evaluated in $L_{\omega_1^{CK}}[G \cup H_{fin}]$ for some finite $H_{fin} \subseteq$ H. Hence supp(f) is finite, and without loss of generality supp(f) $\cup$ supp(σ) $\subseteq$ supp(p).

Not only does **M** satisfy the Axiom of Foundation, it is even well-founded in the sense of **U**[H]; that is, the relation $(\sigma, x) < (\tau, y)$ if $(\sigma \geq \tau \wedge \sigma \models$ "$x \in y$") is well-founded in **U**[H]. It would be shown inductively on γ that each $\mathbf{M}_\gamma$ has this property in **U**[H], hence $\mathbf{M}_{big}$ too as the union of the $\mathbf{M}_\gamma$'s, and finally **M** as a submodel of $\mathbf{M}_{big}$. Therefore, not only do we have that $\sigma \models$ "dom(f) is a well-founded linear order", but also $\{\ \alpha \mid \sigma \models$ "$\alpha \in dom(f)$"$\}$ (a collection in **U**[H] of Kripke sets) is also a well-founded linear order (with the order relation being that induced by **M**). We will refer to this set as ext_σ(dom f), the externalization of the domain of f as σ.

So let q $\leq$ p, q $\in f_H$, α, and H_i be such that
q $\Vdash$ "[α is the least member of ext_σ(dom f) such that for some X $\in$ H
$\sigma \models (\bigwedge_{n \in supp(p)} B(H_n) \neq B(X) \ \wedge \ B(X) \in f(\alpha))$]
$\wedge$ [$\sigma \models (\bigwedge_{n \in supp(p)} B(H_n) \neq B(H_i) \ \wedge \ B(H_i) \in f(\alpha))$]".

Without loss of generality supp(α) $\cup\{i\} \subseteq$ supp(q). Consider j $\notin$ supp(q). Let ρ_j be the permutation of ω which interchanges i and j. ρ_j can be extended to a permutation of terms in the forcing language which interchanges all occurrences of H_i with H_j. This permutation we denote also as ρ_j. Such a permutation on terms induces a permutation on sentences (by merely permuting the parameters), and this permutation is truth-preserving (for the language describing **U**[H], i.e. without f_H):

$$r \Vdash \phi \ \leftrightarrow \ r \circ \rho_j \Vdash \rho_j(\phi).$$

(For a development of focing and symmetric models, see [5] for instance.) The sentence forced by q above is in this restricted language, hence
q $\circ\rho_j$ $\Vdash$ "[$\rho_j(\alpha)$ is the least member of ext_σ(dom f) such that
for some X $\in$ H $\sigma \models (\bigwedge_{n \in supp(p)} B(H_n) \neq B(X) \ \wedge \ B(X) \in f(\rho_j(\alpha)))$]
$\wedge$ [$\sigma \models (\bigwedge_{n \in supp(p)} B(H_n) \neq B(H_j) \ \wedge \ B(H_j) \in f(\rho_j(\alpha))$]".

(Of course we are using the fact that $\rho_j(f) = f$.)

Let r $\leq$ q, q$\circ\rho_j$. Since r forces that both α and $\rho_j(\alpha)$ are the least **M**-ordinals with a certain property, r $|\!\!\vdash \alpha = \rho_j(\alpha)$. Hence r $|\!\!\vdash$ "$B(H_i), B(H_j) \in f(\alpha)$".

Let J $\subseteq \omega\setminus$ supp(q), |J| = n finite. Let q $\circ\rho_J$ be { q $\circ\rho_j$ | j $\in$ J } and r_J be inf(q $\circ\rho_J \cup$ {q}). Then r_J $|\!\!\vdash$ "$B(H_i) \in f(\alpha) \ \wedge \ \bigwedge_{j\in J} B(H_j) \in f(\alpha)$". The set R_n = { r_J | J $\subseteq \omega\setminus$ supp(q), |J| = n } is definable and pre-dense in the forcing partial order, so $f_H\cap R_n \neq \emptyset$. This means that f($\alpha$) contains at least n+1-many distinct Kripke sets of the form B(H_j). Since this holds for all n, f(α) contains infinitely many such Kripke sets. This, though, is a contradiction. According to the definition of a resolution, $\sigma \models$ "f(α) is a set", and each Kripke set in **M** is in $L_{\omega_1^{CK}}$[G $\cup$ H_{fin}] for some finite H_{fin}. ∎

4.4 IKP + V=L + $\neg\Pi_2$-Reflection

This construction is weaker than the others in that the model produced will not satisfy $\neg\Pi_2$-Reflection, but merely will not satisfy Π_2-Reflection. In fact it will be a model of $\neg\neg\Pi_2$-Reflection. This suffices to show that IKP + V=L does not prove Π_2-Reflection, but it is still open whether IKP + V=L proves $\neg\neg\Pi_2$-Reflection.

The partial order of this model will be ω-many copies of $2^{<\omega}$ next to each other, with two consecutive bottom elements. That is, there is a first node $\bot\bot$, followed by another node $\bot$, itself followed by ω-many incompatible nodes $\bot_n$, each of which is the bottom node of a copy of $2^{<\omega}$. We will really be working with the subtree above $\bot$; the role of bottom is merely that $\bot \models$ "$1_{\bot\bot} = 1$", i.e. $\bot \models$ "$1 \in \hat{P}$". Let $\hat{P}_n$ be the nth subtree, internally: $[\sigma \models y \in \hat{P}_n] \leftrightarrow [\exists\tau$ compatible with $\bot_n$ $\ \sigma \models y = 1_\tau]$; notice that for σ incompatible with $\bot_n$, $\sigma \models \hat{P}_n = \{1\}$. (By way of notation, let the numbering of these trees start with 1 instead of 0, because 0 as a subscript is already in use, to indicate including 0 as a member.)

We need the ω-sequence $< \hat{P}_n \mid n \geq 1 >$ in the model. To this end, let α_n be $\hat{P}_n \cup \overline{n+1}$, where, as before, $\overline{n+1}$ is the internalization of n+1. Notice that α_n is an ordinal, and $L_{\alpha_n} = \hat{P}_n \cup L_{\overline{n+1}}$ (using lemma 4.1.1 here). Let α be $\hat{P} \cup \overline{\omega} \cup \{T_{n0} \mid n \geq 1\} \cup \{\alpha_n \mid n \geq 1\}$. So

$$L_\alpha = \hat{P} \cup L_{\overline{\omega}} \cup \bigcup\{def(T_{n0}) \mid n \geq 1\} \cup \bigcup\{def(\hat{P}_n \cup L_{n+1}) \mid n \geq 1\}.$$

Over L_α let ϕ(x, y) iff

$$\begin{aligned} 0 \neq x \in \overline{\omega} \quad & \wedge \quad 0 \neq y \subseteq 1 \\ & \wedge \quad \exists z(x, y \in z) \\ & \wedge \quad \forall z(x+1 \in z \wedge y \in z \rightarrow y = 1). \end{aligned}$$

We claim that $\forall x \perp \models$ "$\forall y\ \phi(x,y) \leftrightarrow y \in \hat{P}_x$". To see this, go through the members of L_α. There is no z in $\hat{P}$ with a non-0 member. In $L_{\overline{\omega}}$ the only possible y (i.e. the only non-0 subset of 1) is 1 itself, and indeed for all x $\geq$ 1 ϕ(x, 1); since $\perp \models 1 \in \hat{P}_x$, this is fine. The only members of $\overline{\omega}$ also in $\hat{P}_{n0}$ are 0 and 1, so if z $\in$ def($\hat{P}_{n0}$) has a legal x as a member then x = 1. In fact this is subsumed by the final case: the common members of $\overline{\omega}$ and ($\hat{P}_n \cup L_{\overline{n+1}}$) are 0, 1, ... , $\overline{n}$; the non-0 subsets of 1 are exactly the members of T_n; so the appropriate $< x, y >$ candidates are $< m, y >$, y $\in \hat{P}_n$ and m$\leq$n. If m<n and $\neg\sigma \models y = 1$ then {m+1, y} shows that $\neg\sigma \models \phi(m, y)$. Hence the only time $\sigma \models \phi(x, y)$ is when $\sigma \models y \in \hat{P}_x$.

Keeping in mind that $\hat{P}_x \in L_\alpha$, the functional relation "f($\overline{n}$) = $\hat{P}_{\overline{n}}$" (as a two-place relation $\psi(\overline{n}, Z)$) can be defined over L_α as "$\forall y\ y \in Z \leftrightarrow \phi(\overline{n}, y)$". There is a small integer m such that this function f is actually a member of $L_{\alpha+\overline{m}}$. (L_α is available as a parameter already in $L_{\alpha+1}$, and all else you need are some ordered pairs.)

Next comes an extension of the notion of a branch to include a branch through $\hat{P}_n$. If Q $\subseteq \hat{P}$ is closed downwards (i.e. closed under taking supersets), then by requiring in the definition of a branch that B $\subseteq$ Q and restricting the γ's in clause (3) to Q, we get a branch through Q. The proof of lemma 4.1.7 goes through with Q in place of $\hat{P}$. In the case at hand, any branch through $\hat{P}_n$, externally at some σ compatible with $\perp_n$, is a branch through $\hat{P}_\sigma$, and externally at some σ incompatible with $\perp_n$ is just {1}. Let R be any real, and B_n the corresponding branch through $\hat{P}_n$.

Let W be a recursive ordering of ω which is not well-founded but which has no hyperarithmetic infinite descending sequence. For the convenience of the reader, we review some basic facts about W which will be used.

For every i there is a W-least j $\leq_W$ i such that the W-interval $[j,i]_W$ has order-type some $\alpha < \omega_1^{CK}$. To see this, work in $L_{\omega_1^{CK}}$. Define f(0) = $\langle i, 0, 0\rangle$, f(n+1) = the L-least triple $\langle j, \alpha, g\rangle$ such that $j <_W$ proj$_0$(f(n)) and g is an order-isomorphism between $[j,i]_W$ and the ordinal α. Set j_n = proj$_0$(f(n)). f is a Σ_1-definable function on an initial segment of ω. By the admissibility of $L_{\omega_1^{CK}}$, f $\in L_{\omega_1^{CK}}$. If dom f = ω then i = $j_0 >_W j_1 >_W$ ... would be a hyperarithmetic infinite descending sequence, in contrast to the choice of W. Let n = max(dom f). Then $[j_n,i]_W$ is isomorphic to some $\alpha \in \omega_1^{CK}$, but for all j < j_n $[j,i]_W$ is not well-founded. Call this j_n corresponding to a given i 0_i ("0 in the sense of i").

In the other direction, $\forall i \in \omega,\ \alpha \in \omega_1^{CK}\ \exists j_\alpha \in \omega$ such that $[i, j_\alpha)_W$ has order-type α (except possibly for those i's belonging to a final segment of W of order-type some $\beta < \omega_1^{CK}$). To see this, for a given i, if this fails, let α be the least ordinal for which this fails. By the admissibility of $L_{\omega_1^{CK}}$, $\{j_\beta \mid \beta < \alpha\} \in L_{\omega_1^{CK}}$. If this set is a final segment of W, then the end-segment starting at 0_i (as constructed above) is an end-segment of order-type γ = o.t.$[0_i, i)_W + \alpha < \omega_1^{CK}$. In the other case, there is a j $>_W$ j_β for all $\beta < \alpha$. In fact, there is a least such j, or else the function f(0) = j > j_β (for all β simultaneously) arbitrary, f(n+1)

= the least (in the standard ordering of ω) j such that j $<_W$ f(n) and j $>$ j_β ($\forall\beta$) would produce a hyperarithmetic infinite descending sequence. This least such j witnesses that [i,j) has order-type α, in contrast to the choice of α.

Combining these two results, we see that every i is contained in a maximal interval of order-type an ordinal, and this interval starts at $0_i \leq_W$ i and has order-type ω_1^{CK} (or some $\beta < \omega_1^{CK}$ on a final segment). We will call this interval i's standard neighborhood, with the notation sn(i). If $i_1 >_W i_0$ is not in i_0's standard neighborhood, then $\forall j \in \mathrm{sn}(i_0)\ i_1 >_W j$; we will use the notation $i_1 >_W \mathrm{sn}(i_0)$ in this case and say that i_1 is a non-standard distance above i_0. Notice that in this case there is an $\hat{i}$ such that $i_1 >_W \mathrm{sn}(\hat{i})$ and $\hat{i} >_W \mathrm{sn}(i_0)$. To see this, notice that $\{\ j \mid \mathrm{sn}(i_0) <_W j <_W 0_{i_1}\ \}$ is non-empty, else $\mathrm{sn}(i_0)$ would be a member of $\mathrm{L}_{\omega_1^{CK}}$ (definable as $[0_{i_0}, 0_{i_1})_W$) and f(j) = order-type($[0_{i_0}, j)_W$) would then be a total function from $\mathrm{sn}(i_0)$ onto ω_1^{CK}. Let $\hat{i}$ be any element such that $\mathrm{sn}(i_0) <_W \hat{i} <_W 0_{i_1}$.

With these preliminaries behind us, we can now return to the construction of the model **M**. Given a final segment of W with least element i, let ξ_i be $\{\mathrm{B}_{j0} \mid j \geq_W i\} \cup \alpha \cup \{\alpha\}$. As in section 4.2, iterate definability ω_1^{CK}-many times on top of L_{ξ_i}, to get $\mathbf{M}_i$. As has already been observed, lemmas 4.2.2 - 4.2.4 did not use any particular facts about the reals used to define the branches, and so in the following we will freely omit details of the proofs, referring the reader to the proofs of these previous lemmas.

M will be $\bigcup_{i\in I} \mathbf{M}_i$, for some final segment I of W with no least element. The exact choice of I will be given in lemma 4.4.8, the only place where it's used.

Lemma 4.4.1 $\mathbf{M}_i \models$ *IKP.*

proof: As in lemma 4.2.2. ▌

Lemma 4.4.2 $L_\alpha \in L^{\mathbf{M}_i}$.

proof: α_n is definable over L_{α_n} as the set of all ordinals, so $\alpha_n \in \mathrm{L}_\alpha$ and $\alpha \subseteq \mathrm{L}_\alpha$. Since $\mathbf{M}_i$ is admissible, $\gamma = \{\ \beta \in L_\alpha \mid \mathrm{def}(\mathrm{L}_\beta) \subseteq \mathrm{L}_\alpha\ \} \in \mathbf{M}_i$. $\alpha \subseteq \gamma$, so $\mathrm{L}_\alpha \subseteq \mathrm{L}_\gamma$. By the definition of γ, $\mathrm{L}_\gamma \subseteq \mathrm{L}_\alpha$. ▌

Corollary 4.4.3 $f \in \mathbf{M}_i$.

proof: f $\in \mathrm{L}_{\alpha+\overline{n}}$, and since $\mathrm{L}_\gamma = \mathrm{L}_\alpha$, $\mathrm{L}_{\gamma+\overline{n}} = \mathrm{L}_{\alpha+\overline{n}}$. ▌

Lemma 4.4.4 *In* $\mathbf{M}_i$*, if* $\bot \models$ *"B is a branch through f(j)" then* $j \geq_W i$*.*

proof: Suppose j $<_W$ i. Consider $\mathbf{M}_i$ at node $\bot_j$. For any definition $\phi(x)$ at any stage in the construction of $\mathbf{M}_i$, consider the structure which consists of all of the relevant information: at first this includes the stage α at which ϕ is being evaluated, $(\mathbf{M}_i)_\alpha$, the formula ϕ, and ϕ's parameters, but these parameters can themselves be unpacked, until the only parameters left are from $\hat{P}_j$. This uses of course that at $\bot_j$ all of the branches included in ξ_j are $\{1\}$. The parameters are all of the form 1_σ. Let τ be any node beyond or incompatible with all such σ's. Then $\tau \models \phi(1_\rho)$ for some ρ extending τ iff $\tau \models \phi(1_\rho)$ for every ρ extending τ of the same length. (The proof of this would use an automorphism of the model interchanging any two such ρ's.) So ϕ could not possibly define a branch through $\hat{P}_j$. ▮

Lemma 4.4.5 *If* $k \leq_W i$ *then* $L_{\xi_i} \in L^{\mathbf{M}_k}$*.*

proof: In $\mathbf{M}_k$, let B_{full} be $\{ B \in L_{\xi_k} \mid$ B is a branch through f(j) for some j $\geq_W$ i $\}$. Letting δ be $\{ B_0 \mid B \in B_{full} \} \cup$ T, and letting γ be as in lemma 4.4.2, $L_{\xi_i} = L_{\delta \cup \gamma \cup \{\gamma\}}$. ▮

Corollary 4.4.6 *If* $k \leq_W i$ *then* $\mathbf{M}_i \subseteq \mathbf{M}_k$*.*

proof: As in lemma 4.2.3. ▮

Corollary 4.4.7 $\mathbf{M}_i \models$ *V=L.*

proof: As in lemma 4.2.4. ▮

Lemma 4.4.8 $\mathbf{M} \models$ *IKP.*

proof: As the union of transitive models of IKP (see lemma 4.4.1), $\mathbf{M}$ automatically satisfies Empty Set, Infinity, Union, and Δ_0-Comprehension. One of the more difficult axioms to check, surprisingly enough, is Pairing. That is the point of the penultimate corollary, which also implies Extensionality. Foundation holds because $\mathbf{M}$ is well-founded in $\mathbf{V}$.

The only other axiom is Δ_0 Bounding. The choice of I will be exactly to ensure this. The construction is really the same as in [8]; we replicate the main idea here to make this paper self-contained. Notice that for any choice of I any node other than $\perp$ forces IKP, because $\perp_i \models \mathbf{M} = \mathbf{M}_i$ if i $\in$ I, and $\perp_i \models \mathbf{M}$ = "$\mathbf{M}_i$ without the branch" which also models IKP, if i $\notin$ I. So we need concern ourselves only with functions forced to be total by $\perp$. There are countably many Δ_0 definitions (with parameters) which might define a total function on a set. We'll handle the nth, $\phi_n(x, y)$, at stage n. Assume inductively we have $\max_n$ a non-standard distance above $\min_n$ in W ($\max_0$ and $\min_0$ being arbitrary elements of W a non-standard distance apart). By the discussion about W (just before the lemmas of this section), let m be a non-standard distance above $\min_n$ and a non-standard distance below $\max_n$ (i.e. sn($\min_n$) < m and sn(m) < $\max_n$). In $\mathbf{M}_m$ either ϕ_n defines a total function or it doesn't. In the former case, $\mathbf{M}_m$ contains a bounding set (by lemma 4.4.1), which is also a bounding set in any model extending $\mathbf{M}_m$, so let $\max_{n+1}$ = m (and let $\min_{n+1} = \min_n$). In the latter case, ϕ_n is not total in $\mathbf{M}_m$, or, for that matter, in any transitive submodel thereof (by the absoluteness of Δ_0 formulas among transitive models). So let $\min_{n+1}$ = m (and let $\max_{n+1} = \max_n$). Let I = {i | $\exists$n i $\geq \max_n$}. ▮

Lemma 4.4.9 **M** $\models$ *V=L.*

proof: The union of models of V=L is also a model of V=L. ▮

Lemma 4.4.10 **M** $\not\models \Pi_2$*-Reflection.*

proof: The assertion "B is a branch through S" is Δ_0, hence absolute in all transitive sets containing the parameters. So if $\perp \models$ "B is a branch through f(j)" then j $\in$ I. Moreover, if j $\in$ I, then $\perp \models$ "there is a branch through f(j)". So $\perp \models$ "$\forall j, B_j\ \exists i, B_i$ if B_j is a branch through f(j) then either f(j) = {1} or (i$<_W$j and B_i is a branch through f(i))". But this does not reflect to any set, because each set is a member of some $\mathbf{M}_i$, which is constructible in $L_{\omega_1^{CK}}$, in which W has no infinite descending sequence. ▮

5 Questions

1. Is there a model of Σ_1 DC + IKP + V=L + ¬Resolvability?
2. Is there a model of IKP + V=L + ¬Π_2-Reflection?
3. Does IZF prove that admissible sets are unbounded (that is, for each set X is there a transitive admissible set containing X)? Does IZF prove the existence of an admissible set with ω? The usual constructions of admissible sets use

either reflection or the linearity of the ordinals, neither of which is available to us here.

4. Does IZF_{Rep} (that is, IZF with the Bounding Schema substituted by the Replacement Schema) prove IKP? With some better technology (see question 6) it should be possible to show that IZF_{Rep} - Power does not prove IKP. Also, [3] (with a minor alteration) shows that IZF_{Rep} does not prove Π_1 Bounding. But whether IZF_{Rep} proves Δ_0 Bounding is apparently unknown.

5. As has already been observed, least fixed points and greatest fixed points are closely connected classically, but not so intuitionistically. How much can they be separated? Can IKP be strengthened so that even more least fixed points become definable, without picking up Π Persistence or any greatest fixed points? Conversely, can Π Persistence by extended to get more greatest fixed points, without picking up IKP or least fixed points? There are connections between least and greatest fixed points and their categorical analogues, initial algebras and final co-algebras of functors, so these matters could be pursued from this angle; but not so much is known even there, and the connections are not yet so clear. These considerations suggest a similar yet simpler question, involving the first order quantifiers: has anyone investigated existential intuitionistic logic (with $\exists$, even $\neg\exists\neg$, but not $\forall$), or universal intuitionistic logic?

6. Are there other natural examples of the difference between IKP and Π Persistence?

7. Another variant on IKP yet to be mentioned is the weakening of Δ_0 Bounding to Σ_1 Replacement. It can be shown that classically Σ_1 Replacement does not imply Δ_0 Bounding; in fact, it is possible to build a classical model of full Replacement + $\neg\Delta_0$ Bounding. It would have been nice to have included in this paper an adaptation of this model to IL. However, this project brings up a host of new problems. All of the models presented here which use forcing do so only in an ambient classical universe to get some particular desired property. For instance, a failure of Σ_1 DC in a classical model is imported to become a failure of Σ_1 DC in the intuitionistic model. In order to get Δ_0 Bounding to fail, the forcing has to be done internally. Unfortunately, intuitionistic forcing technology is not yet well enough developed to carry such a burden. Certainly some work has already been done in this area; see [2] for instance. But not enough has been done yet to provide a robust enough theory to carry through the construction envisioned here. This should be possible, but would take us too far beyond the scope of the current paper.

References

[1] Jon Barwise, *Admissible Sets and Structures*, Springer- Verlag, 1975.

[2] Michael J. Beeson, *Foundations of Constructive Mathematics*, Springer-Verlag, 1985.

[3] Harvey Friedman and Andrej Scedrov, The lack of definable witnesses and provably recursive functions in intuitionistic set theories, *Advances in Math*, 1985, p. 1-13.

[4] Thomas Jech, *Multiple Forcing*, Cambridge University Press, 1986

[5] Thomas Jech, *Set Theory*, Academic Press, 1978

[6] Jim Lipton, Realizability, Set Theory, and Term Extraction, in *The Curry-Howard Isomorphism* (Ph. de Groote, ed.), Volume 8 of the "Cahiers du Centre de logique" (Université catholique de Louvain), Academia, 1995, pp. 257-364.

[7] Robert Lubarsky, Intuitionistic L, in *Logical Methods in Computer Science: The Nerode Conference*(Crossley et al., eds.), Birkhauser, 1993, p. 555-571.

[8] Robert Lubarsky, Sacks forcing sometimes needs help to produce a minimal upper bound, *Journal of Symbolic Logic*, 1989, p. 490-498.

[9] William Powell, Extending Godel's negative interpretation to ZF, *Journal of Symbolic Logic*, 1975, p. 221-229.