The Weak Fundamental Theorem of Algebra

Robert Lubarsky
Fred Richman
Florida Atlantic University

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Is it ever true constructively?

– Over a discrete field.

– Under Countable Choice.

– When the coefficients are Cauchy reals. (Ruitenburg)

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– The Euclidean algorithm.

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Theorem

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- the distance $d(z, S) = \inf_{x \in S} d(z, x)$ may not be defined, but the *quasi-distance* $\delta(z, S) = \sup_{x \in S} d(z, x)$ is;

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- S may not be finite, but it's *quasi-finite*; and
- there is a Riesz space of functions on S.

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Example: For $a = \Sigma_i a_i x^i$ and $b = \Sigma_j b_j x^j$, the Sylvester matrix is

$$\begin{pmatrix} a_2 & a_1 & a_0 & 0 & 0 & 0 \\ 0 & a_2 & a_1 & a_0 & 0 & 0 \\ 0 & 0 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & 0 & a_2 & a_1 & a_0 \\ b_4 & b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & b_4 & b_3 & b_2 & b_1 & b_0 \end{pmatrix}.$$

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This doesn't generalize well to arbitrary rings. Example: Over $\mathbb{Z}_8$, $x^2 + 4$ and $x^2 + 4x$ have a resultant of 0 but no non-trivial common factors. Hence:

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For a and b monic, $Res(a, b)$ is a unit iff a and b are comaximal.

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Corollary

For a and b monic polynomials over $\mathbb{C}$, Res(a, b) is a unit iff a and b are comaximal iff there is a positive distance between the roots of a and b.

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