Realizability Models Separating Various Fan Theorems

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Abstract. We develop a realizability model in which the realizers are the reals not just Turing computable in a fixed real but rather the reals in a countable ideal of Turing degrees. This is then applied to prove several separation results involving variants of the Fan Theorem.
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1 Introduction

Certain constructions in computability theory lend themselves well to realizability, the latter being based on an abstract notion of computation. A coarse example of this is the notion of a Turing computable function itself, as the collection of Turing machines makes an applicative structure and so provides an example of realizability. This model is closely tied to Turing computation, naturally enough, and so provides finer examples. Consider Weak König's Lemma, WKL, which is among constructivists more commonly studied in its contrapositive form, the Fan Theorem FAN. (For background on realizability, the Fan Theorem, and constructive mathematics in general, there are any of a number of standard texts, such as [3, 23, 25].) Kleene's well-known eponymous tree is a computable, infinite tree of binary sequences with no computable path. In the context of reverse mathematics, this shows that RCA_0 does not prove WKL. Within the realizability model, the same example shows IZF does not prove FAN.

Perhaps a word should be said on the choice of the ground theory. For the classical theory, it's RCA_0, while for the constructive, it's IZF. The former is notably weak, the latter strong. Why is that? And why those theories in particular? This is not the place to discuss the particular choice of RCA_0. As for IZF,

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its use is of secondary importance. The point is that much of reverse classical mathematics is to show the equivalence of various principles, for which a weaker base theory provides a stronger theorem. Even for independence results, which on the surface would be better over a stronger base, are often of the form that a weaker theory does not imply a stronger one, where the latter easily implies the former; clearly here, the base theory is the weaker of the two. Also for those cases where the independence result desired is an incomparability, the principles in question are all weak set existence principles, weak in the sense that they use a tiny fragment of ZF, and hence a weak base theory is needed, to keep the theories in question from being outright provable. In contrast, reverse constructive mathematics studies not set existence principles, but rather logical principles. Instead of fragments of ZF, the subject is fragments of Excluded Middle. Especially when discussing independence, when a stronger base theory gives a stronger theorem, in order to highlight that it really is the logic that is up for grabs, and not set existence, strong principles of set theory are taken as the base theory. IZF is used here, since it is the simplest and most common constructive correlate to ZF, the classical standard. Even for equivalence theorems, there would still be a tendency to work over IZF, since the degree to which a result depended on the IZF axioms is the degree to which the result is ultimately classical. In practice, if any theorem needs less than full IZF, what is actually used could be read off from the proof anyway.

Returning to realizability, the picture is not quite so rosy when it comes to other constructions. A case in point is the distinction between WKL and WWKL, Weak Weak König's Lemma. WWKL states that for every binary tree (of finite, 0-1 sequences) with no path there is a natural number n such that at least half the sequences of length n are not in the tree. This principle has been studied in reverse mathematics, both classical [22] and constructive [20]. Yu and Simpson [26] showed that WWKL does not imply WKL (over RCA_0). That's not so simple as merely taking the computable sets; while that would falsify WKL, as discussed above, it invalidates WWKL too. What they do is to extend the computable sets by a carefully selected real R_0 (implicitly closing under Turing reducibility) which provides a path through all the "bad" trees while not destroying the Kleene tree counter-example. That's not enough, though, because the construction of a counter-example to WWKL relativizes. So while R_0 kills off the bad computable trees, it introduces new bad trees of its own. Hence the construction must be iterated: $R_1, R_2, ...$ In the end, the union of the (reals computable in the) R_n's suffices.

The statement of WWKL carries over just fine to a constructive setting, where we will call it the Weak Fan Theorem W-FAN, as well as the question of whether W-FAN implies FAN. For better or worse, the construction though doesn't. One might first think to use the Yu-Simpson set of reals as the set of realizers. An immediate problem is that we need an applicative structure: the realizers need to act on themselves. It is immediate and routine to view these reals as functions from the naturals to the naturals – that's a trivial identification these days. It doesn't help though. If the realizers are those functions, well, those functions act on naturals, not on functions. What we would need would be integer codes for those functions. The realizers from this point of view would be that set of naturals, which as need be could be taken to be functions. But here's the problem: what code do you give a function which showed up in some increasing tower? If you were looking at those functions computable in some fixed oracle, you could consider all the naturals as each coding such a machine. If instead your oracle is continually changing, it's not clear what to do and still maintain an applicative structure. The fixes we tried did not work, as we were warned.

The same issue comes up with another distinction around FAN, namely the distinction between FAN and WKL. In order to show their inequivalence, one might want to come up with a model of FAN falsifying WKL. This has already been done using K_2 realizability, Kleene's notion of functional realizability. It is another matter to prove this theorem via K_1 realizability. The problem is as above: every Turing degree has an infinite binary tree with no path of the same or lesser degree (the Kleene tree relativized to that degree). One could take a set of degrees, any of the trees of which have a path in some other degree. The problem here is how to turn that into a realizability structure.

The goal of this paper is a realizability model in which the realizers code functions from the natural numbers to themselves with no highest Turing degree among them. As corollaries to this method we get the two results just cited.

2 The Main Construction

The main idea here is to build a Kripke model, and then within that a realizability model, which has sometimes been called relative realizability [25]. This kind of construction was apparently first suggested by de Jongh [10]. Variants have been used by several people: having the Kripke partial order consist of only two points and the realizers be at $\perp$ certain computable objects which are then injected into a full set of realizers [1, 2], or using instead of Kripke semantics either double-negation [14] or a kind of Beth [24] semantics. For more detail on all of this, see the last two sections of [25].

To help keep things simple, we assume ZFC in the meta-theory. For most of this work, neither classical logic nor the Axiom of Choice is necessary, but we will not be careful about this.

Let the underlying Kripke partial order be $\omega^{<\omega}$. Let M be the full Kripke model built on that p.o. Intuitively, that means throw in all possible sets. More formally, a set in the model is a function f from $\omega^{<\omega}$ to the sets of the model (inductively) which is non-decreasing (i.e. if $\sigma \subseteq \tau$ then $f(\sigma) \subseteq f(\tau)$). Equality and membership are defined by a mutual induction. On general principles, M $\models$ IZF. Moreover, the ground model V has a canonical image in M: given $x \in V$, let $\check{x}$ be such that $\check{x}(\sigma) = \{\check{y} \mid y \in x\}$. We often identify x with $\check{x}$, the context hopefully making clear whether we're in V or in M. For slightly more detail on the full model, defined over any partial order, see for instance [17].

Within M, we will identify a special set R of natural numbers, based on a prior sequence R_n of reals ($n \in \omega$). We assume the R_n are of strictly increasing

Turing degree. At node $\bot = \langle\rangle$, R looks empty: $\bot \not\models s \in R$; equivalently, $R(\langle\rangle) = \emptyset$. Suppose inductively $R(\sigma)$ is defined, where σ has length n. Let R^i_n list all the reals that differ from R_n in finitely many places. Let $R(\sigma^\frown i)$ be $R(\sigma) \cup \{\langle n, s\rangle \mid s \in R^i_n\}$. In words, at level $n+1$, beneath each node on level n, put into the n^{th} slice of R all of the finite variations of R_n, spread out among all the successors. So R is a kind of join of the R_n's, just not all at once.

Because R is (in M) a real, it makes sense to use R as an oracle for Turing computation. At $\bot$, if a computation makes any query $s \in R$ of R, there are some nodes at which s is in R and others where it is not, so the oracle cannot answer and the computation cannot continue. This follows from the formal model of oracle computability: a run of an oracle machine is a tuple of natural numbers coding a correct computation; the rule for extending a tuple when the last entry is an oracle call is that the next entry must contain the right answer; if there is, at a node, no right answer, then there can be no extending tuple. Hence the only convergent computations are those that make no oracle calls, and the only R-computable functions are the computable ones. More generally, at node σ of length n, any query of the form $\langle k, s\rangle \in R$ with $k \geq n$ will be true at some future nodes and false at others, hence unanswerable at σ. The computable functions at σ are those computable in R_{n-1}.

In M, let App be the applicative structure of the indices of functions computable in R (using, of course, the standard way of turning such indices into an applicative structure). In M, let M[App] be the induced realizability model. On general principles, M[App] $\models$ IZF. The natural numbers of M[App] can be identified with those of M, so any set of such in M[App] can be identified with one in M. Furthermore, at any node, a decidable real in one structure corresponds to a decidable real in the other, and that can be identified with a real in the ambient classical universe. Henceforth these various reals will not even be distinguished notationally. For instance, if $\sigma \models$ "$M[App] \models$ "$T \subseteq \omega$ is decidable" " then we might refer to the real T in V.

For notational convenience, we will abbreviate $\sigma \models$ "$M[App] \models \phi$" as $\sigma \models_{App} \phi$.

Lemma 1. *For σ of length n and R a real, $\sigma \models_{App}$ "X is decidable" iff X is Turing computable in R_{n-1}.*

Proof. The statement "X is decidable" is $\forall m \in \omega\ m \in X \vee m \notin X$. A realizer r of the latter would be a function that, on input m, decides whether m is in or out of X. If in the course of its computation r asked the oracle any question of the form $\langle k, s\rangle$ with $k \geq n$ then the computation would not terminate at σ. So r can access only R_{n-1}, making X computable in R_n. The converse is immediate.

Lemma 2. *If $\sigma \models_{App}$ "X is an infinite branch through the binary tree" then $\sigma \models_{App}$ "X is decidable."*

Proof. To be an infinite branch means for every natural number m there is a unique node of length m. The realizer that X is an infinite branch has to produce that node given m.

Often people are concerned about the use of various choice principles. The independence results presented here are that much stronger because Dependent Choice holds in our models.

Proposition 1. $\langle\rangle \models_{App}$ *DC*

Proof. The same proof that DC holds in standard Kleene K_1 realizability works here.

3 D-FAN and W-D-FAN

When adapting the classical results to the current setting, we need an additional stipulation. All of the trees, and hence principles, we consider will be decidable: for all binary sequences b and trees T, either $b \in T$ or $b \notin T$. So, for instance, instead of the full Fan Theorem FAN, we will be considering the Decidable Fan Theorem: if a decidable tree in $\{0,1\}^{\mathbb{N}}$ has no infinite path, then the tree is finite. Also, Weak FAN, also known as WWKL, when applied to decidable trees, would read: if a decidable tree in $\{0,1\}^{\mathbb{N}}$ has no infinite path, then there is an n such that at least half of $\{0,1\}^n$ is not in the tree.

This brings us to an annoying point about notation. Decidable FAN has been referred to in various places as D-FAN, FAN_D, Δ-FAN, and FAN_Δ. So notation for Weak Decidable FAN could be any of those, with a "W" stuck in somewhere. To make matters worse, even though the statement of Weak FAN is a weakening of FAN and not of WKL, the name WWKL for it is already established in the classical literature, and so one could make a case to stick with it, and insert decidability ("D" or "Δ") somewhere in there. These same considerations apply to other variants of FAN, whether already identified (c-FAN, Π^0_1-FAN) or not. Whatever we do here will likely not settle the matter. Still, we have to choose something. It strikes us as confusing to distinguish between FAN and WKL, and then call a variant of FAN by a variant of WKL. Also, what if somebody someday wants to study the contrapositive of "WWKL"? Hence we stick with the name W-FAN. As for how to get in the decidability part, we choose the option that's the easiest to type: W-D-FAN.

Returning to the matter at hand, Yu-Simpson [26] construct a sequence X^n of reals of increasing Turing degree such that:

i) if T is a tree computable in X^n the branches through which form a set of positive measure, then a path through T is computable in X^{n+1}, and

ii) no path through the Kleene tree is computable in any X^n.

We apply the construction from the previous section, with R_n set to X^n. From this, the following lemmas are pretty much immediate.

Lemma 3. $\langle\rangle \models_{App}$ *W-D-FAN*

Proof. At any node, a decidable tree T is computable in some X^n. If in V the measure of the branches through T were positive, then there would be a branch computable through X^{n+1}. So no node could force that there are no branches through T. To compute a level at which half the nodes are not in T, just go through T level by level until this is found.

Lemma 4. $\langle\rangle \models_{App} \neg$*D-FAN*

Proof. The Kleene tree provides a counter-example.

While we expect that even full W-FAN does not imply D-FAN, this model does not satisfy W-FAN. To see that, recall that any path through the binary tree is decidable, hence computable in some X^n. There are only countably many such paths. It is easy in V to construct a tree avoiding those countably many paths with measure (of the paths) being as close to 1 as you'd like. The internalization of such a tree in M[App] will not be decidable, but will be internally a tree with no paths.

4 FAN and WKL

The distinction between FAN and WKL is a strange case. Their relation is that WKL implies FAN, but not the converse. With some exaggeration, it seems as though everyone knows that but no one has proven it. [3] At the very introduction of non-classical logic, Brouwer himself must have realized this distinction, as he made a conscious choice which variant of this class of principles he accepted. Moreover, while he accepted FAN (having proven it from Bar Induction), it is easy to see that WKL implies LLPO, which Brouwer rejected. So while Brouwer did not provide what we would today consider a model of FAN + ¬WKL, we would like to honor him in the style of the Pythagoreans by attributing this result to him, whatever may actually have been going through his mind.

Such models have since been provided, for instance by Kleene, using his functional realizability K_2 [16, 21]. However, in neither of those sources is it mentioned that WKL fails. In [5], both FAN and WKL are analyzed into constituent principles, it is shown that WKL's components imply the corresponding FAN components, and it is nowhere stated that the converse does not hold. In [7], equivalents are given for what is there called FAN and WKL, although their FAN is actually D-FAN, and it is at least asked how much stronger WKL is than FAN. The one proof we have been able to find of some fan theorem not implying WKL is in [19], where once again the fan principle used is D-FAN. For what it's worth, that argument, like ours, uses relative realizability [8], albeit with K_2 realizability.

Below we give a full proof that FAN does not imply WKL. We would be interested in hearing of other extant proofs of such, and would find it amusing if there were none. What might be new here, if anything, is not the result itself, but rather the methodological point that this model is based on K_1. That is, while K_1 is usually used to falsify even D-FAN, its variant below validates full FAN.

Theorem 1. *(Brouwer) FAN does not imply WKL.*

[3] Thanks are due here to Hannes Diener for first pointing this out to us and Thomas Streicher for useful discussion.

Proof. Take a countable ω-standard model of WKL_0 (see [22], ch. VIII). There is a sequence X^n of reals of increasing Turing degree such that the reals in this model are exactly those computable in some X^n. This induces a model M[App] as in section 2 above (with R_n being set to X^n).

To see that WKL fails, suppose to the contrary $\sigma \models_{App}$ "$n \Vdash_r WKL$". So if $\sigma \models_{App}$ "$e \Vdash_r T$ is an infinite binary tree" then, at σ, $\{n\}(e)$ must compute a path through T. But a path through a tree is not computable in the tree, as is standard, by considering the Kleene tree. This shows moreover that WKL for decidable trees fails.

To show FAN, suppose at some node σ in the Kripke partial order r realizes that B is a bar. We must show how to compute a uniform bound on B. To do this, we will build a decidable subset C of B. Inductively at stage n, suppose we have decided C on all binary sequences of length less than n. Consider each binary sequence $\bar{b}$ of length n (except if $\bar{b}$ extends something in C of shorter length, in which case what happens with $\bar{b}$ just doesn't matter anymore). Consider the path $P_{\bar{b}}$ which passes through $\bar{b}$ and is always 0 after that. Applying r to $P_{\bar{b}}$ produces a sequence b^+ in B on $P_{\bar{b}}$. If b^+ has length at most n, include in C all extensions of b^+ of length n, else just include b^+ in C. After doing this for all $\bar{b}$ of length n, anything of length n not put into C is out of C. This procedure terminates only when we have a uniform bound on C, hence on B. If this never terminates, we have a decidable, infinite set of binary sequences not in C computable from r. Hence at any child τ of σ there will be an infinite path P avoiding C. Applying r to P produces an initial segment of P in B, say b. This computation itself used only an initial segment of P, say c. Letting n be the maximum length of b and c, at stage n no initial segment of either has been put into C, by the choice of P. So the procedure would consider the path through b and c which is all 0s afterwards. This would then put the longer of b and c into C, contradicting the choice of P. So this procedure must terminate, producing a bound for B.

5 Questions

1. The second construction was developed for an entirely different purpose.

One way of stating FAN is that every bar is uniform. Weaker versions of FAN can be developed by restricting the bars to which the assertion applies. For instance, Decidable FAN, D-FAN, states that for every decidable set B (i.e. for all b either $b \in B$ or $b \notin B$), if B is a bar then B is uniform. Constructively, decidability is a very strong property; in fact, it is the strongest hypothesis on a bar yet to be identified. D-FAN is trivially implied by FAN; it has long been known that D-FAN is not provable in IZF (via the Kleene tree, described in any standard reference, such as [3, 23]; see [18] for a different proof). A somewhat milder restriction on a bar B is that it be the intersection of countably many decidable sets; that is, B is Π^0_1 definable. Between decidable and Π^0_1 bars are c-bars: if there is a decidable set B′ such that $b \in B$ iff for every c extending b $c \in B'$, then B is called a c-set, and if it's a bar to boot then it's called a c-bar. Often this definition seems at first unnatural and rather technical. All that matters at the moment is that this is a weaker condition than decidability: every decidable bar is a c-bar. c-FAN is the assertion that every c-bar is uniform. Such principles occur naturally in reverse constructive mathematics [15, 4, 12], and are all inequivalent [18].

The first proof that D-FAN does not imply c-FAN, by Josef Berger [6], went as follows. Classically, for X any collection of bars, X-FAN and X-WKL are equivalent (as contrapositives). Furthermore, the Turing jump of a real R can be coded into a tree computable in R, so that c-WKL implies that jumps always exist. Hence over RCA_0 c-WKL implies ACA. D-FAN and WKL (which, in the setting of limited comprehension, is just D-WKL) are equivalent. So if D-FAN implied c-FAN, constructively or classically, then WKL would imply ACA, which is known not to be the case [22].

A limitation of this argument is that it works over a very weak base theory. It leaves open the question of whether D-FAN implies c-FAN over IZF. While this has been settled [18], a question of method still remains open. Could Berger's argument be re-cast to provide an independence results over IZF? The obvious place to look seemed to be a model of WKL + $\neg$ACA, using the functions there, which necessarily have no largest Turing degree, as realizers. Our analysis of such a model did not achieve that goal. Is there another way of turning such a model into a separation of D- and c-FAN?

More generally, could there be any realizability model separating D- and c-FAN? All of the realizability models we know about either falsify D-FAN or satisfy full FAN. Perhaps that's because of the difficulty of realizing that something is a bar. That is, to falsify any version of the FAN, one might well want to provide a counter-example, which would be a non-uniform bar. If a bar is not uniform, realizing the non-uniformity would typically be trivial, as nothing could realize that it is uniform, which suffices. Realizing that a set is a bar is different: given a binary path, you'd have to compute a place on that path and realize that that location is in the alleged bar. If this set is decidable, that's easy: continue along the path, checking each node on the way, until you're in it. If the set cannot be assumed decidable, it is unclear to us how to realize that it's a bar. This is something we would like to see: a way of realizing a non-decidable set being a bar.

2. The differences among D-FAN, c-FAN, Π^0_1-FAN, and FAN have to do with the hypothesis; they apply to different kinds of bars. In contrast, the difference between FAN and W-FAN has to do with the conclusion, with whether the bar is uniform or half-uniform, to coin a phrase. So these variants can be mixed and matched. There are D-FAN, c-FAN, Π^0_1-FAN, FAN, and also W-D-FAN, W-c-FAN, W-Π^0_1-FAN, and W-FAN. Clearly any version of FAN implies its weak cousin. Other than that, we conjecture there is complete independence between the variants of FAN and the variants of Weak FAN. This is, we conjecture W-FAN does not imply D-FAN, and conjoined with D-FAN does not imply c-FAN, and so on. Furthermore, we expect that D-FAN, while of course implying W-D-FAN, does not imply W-c-FAN, and c-FAN does not imply W-FAN, and so on for other variants that might appear.

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