On Extensions of Supercompactness

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Abstract

We show that, in terms of both implication and consistency strength, an extendible with a larger strong cardinal is stronger than an enhanced supercompact, which is itself stronger than a hypercompact, which is itself weaker than an extendible. All of these are easily seen to be stronger than a supercompact. We also study C^n-supercompactness.
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1 Introduction

In [1] and [2] were first defined some large cardinal properties that are extensions of supercompactness, called *hypercompactness* and *enhanced supercompactness* respectively. Their purpose there was as hypotheses in consistency theorems. In [8] both of those were shown to be weaker than a Vopěnka cardinal in terms of consistency strength. The original goal of this work was to weaken the Vopěnka hypothesis by showing that the existence of extendibles implies the consistency of the existence of hypercompacts and enhanced supercompacts, or show that extendibles are themselves hypercompact and enhanced supercompact, or both. We also wanted to compare them to each other. We got a hypercompact to be weaker than an extendible and weaker than an enhanced supercompact, and we got the latter to be weaker than a strengthening of an extendible, all in both senses of weaker (i.e. consistency and implication). We conjecture that enhanced supercompactness is still weaker than a simple extendible. Along the way, we also demonstrate the (quantifier) complexity of the definitions of those cardinals.

The other strengthening of the notion of supercompactness in the literature of which we are aware is C^n*-supercompactness*, an example of a C^n-cardinal [3]. There it was shown that measurability and C^n-measurability are equivalent.

However, an important property of an ultrafilter embedding $j : V \to M$ is that M be closed under κ-sequences (where κ is the critical point of j); that is, if κ is measurable then κ is κ-supercompact. Although that fact of such embeddings is so often used, it is not taken as part of the definition of measurability, because it follows so easily. However, in the context of C^n-cardinals, such closure does not follow immediately, and is even false for Bagaria's C^n-measurability embeddings. We show that having an embedding in which M is κ-closed is strictly stronger than a simple C^n-embedding, that is, that κ being C^k-measurable for any natural number k does not imply that κ is κ-C^n-supercompact ($n \geq 3$).

2 Hypercompacts

Definition 2.1 *[8] A cardinal κ is α-hypercompact if for all $\beta < \alpha$ and for unboundedly many λ there is a $\mathcal{P}_\kappa\lambda$-supercompactness embedding $j : V \to M$ such that, in M, the cardinal κ is β-hypercompact. A cardinal is hypercompact if it is α-hypercompact for all α.*

The following two basic lemmas clarify some details about the hypercompactness hierarchy.

Lemma 2.2 *For any pair of ordinals $\xi < \alpha$, if κ is α-hypercompact, then κ is ξ-hypercompact.*

Proof: The proof follows immediately from Definition 2.1. In particular, to show that κ is ξ-hypercompact, it must be shown that for all $\beta < \xi$, and for unboundedly many λ, there is a $P_\kappa\lambda$-supercompactness embedding $j : V \to M$ such that in M, κ is β-hypercompact. But any such β is less than α, since $\beta < \xi < \alpha$. Therefore, since κ is α-hypercompact, it follows immediately that such an embedding j exists. ▮

Lemma 2.3 *For every limit ordinal δ, the cardinal κ is δ-hypercompact if and only if κ is β-hypercompact for every $\beta < \delta$.*

Proof: The forwards direction of the proof follows from Lemma 2.2. The converse follows immediately from Definition 2.1: if κ is β-hypercompact for every $\beta < \delta$, then for every $\beta < \delta$ and for every $\alpha < \beta$ (or equivalently, for every $\alpha < \delta$), for unboundedly many λ, there is a $P_\kappa\lambda$-supercompactness embedding such that κ is α-hypercompact in the target model. Hence, κ is δ-hypercompact. ▮

We would like to discuss a potential alternative definition for the hypercompactness hierarchy which may appear more natural to some readers, resulting from switching the quantifier order of Definition 2.1. We will then explain why we prefer Definition 2.1. Let us say that κ is α-shmypercompact if for unboundedly many λ there is a $\mathcal{P}_\kappa\lambda$-supercompactness embedding such that in the model M, for all $\beta < \alpha$, the cardinal κ is β-shmypercompact. Clearly, for all ordinals α, every α-shmypercompact cardinal is α-hypercompact, because the definition of shmypercompactness is stronger, requiring the embedding j to be chosen uniformly for all $\beta < \alpha$. When $m < \omega$, it can be shown that every m-hypercompact cardinal is m-shmypercompact, as follows. Suppose inductively that for all $n < m < \omega$, we have shown that (in V) every n-hypercompact cardinal is n-shmypercompact. Let κ be m-hypercompact. For an arbitrary λ, let j be an embedding witnessing that κ is $(m-1)$-hypercompact in the target model. By Lemma 2.2, j witnesses that κ is n-hypercompact for all $n < m$. Inductively, j witnesses that κ is n-shmypercompact in the target model for all $n < m$. Since λ was arbitrary, it follows that κ is m-shmypercompact in V.

When $\alpha = \omega$, this proof breaks down: we cannot use the same argument to show that every ω-hypercompact cardinal is ω-shmypercompact, because ω has no immediate predecessor. However, if κ is $(\omega+1)$-hypercompact, then the argument works again, but with a slightly different result. For a fixed λ, let j be the $P_\kappa\lambda$-embedding such that κ is ω-hypercompact in the target model. Then by lemma 2.3 and the argument in the previous paragraph, this same embedding witnesses (with respect to λ) that κ is ω-shmypercompact. So we see that every $(\omega + 1)$-hypercompact cardinal is ω-shmypercompact. Conversely, if κ is ω-shmypercompact, then for a fixed λ, there is a single $P_\kappa\lambda$ embedding such that κ is n-shmypercompact (and therefore n-hypercompact) in the target model for all natural numbers n. By Lemma 2.3, it follows that κ is ω-hypercompact in the target model, and so by Lemma 2.2, it follows that κ is $(\omega + 1)$-hypercompact.

We have seen now that an $(\omega + 1)$-hypercompact cardinal is equivalent to an ω-shmypercompact cardinal. We can continue the inductive argument up through the ordinals. In general, for successor ordinals $\alpha > \omega$, an $(\alpha + 1)$-hypercompact cardinal is the same as an α-shmypercompact cardinal, and at limit ordinals δ, the δ-hypercompact cardinals are characterized by Lemma 2.3. This fact explains one reason why we prefer Definition 2.1 over the shmypercompact definition. If we instead used the shmypercompact definition, then we would have to resort to awkward phrases such as "n-shmypercompact for all $n < \omega$" rather than the more concise and elegant "ω-hypercompact."

Another reason to prefer Definition 2.1 is that under this definition, degrees of hypercompactness are defined in closer analogy with the definition of the Mitchell rank of a measurable cardinal κ (see for instance [7, Definition 19.33]).

Additionally, Definition 2.1 is to be contrasted with what is called hypercompactness in [1] and excessive hypercompactness in [8], which posits the existence of embeddings for all $\lambda > \kappa$, rather than just for unboundedly many λ. In fact, no κ can be $(2^\kappa)^+$-excessively hypercompact [8], whereas it is consistent relative to a Vopěnka cardinal that there exists a fully hypercompact cardinal [8].

We continue the analysis of hypercompact cardinals with the following theorem.

Theorem 2.4 *κ is α-hypercompact iff for all $\lambda > \kappa + \alpha$ there is a $\mathcal{P}_\kappa\lambda$-*

supercompactness embedding, under which κ remains β-hypercompact ($\beta < \alpha$).

Proof: We consider only the non-trivial direction. Given $\beta < \alpha$ and $\lambda > \kappa + \alpha$, let $j : V \to M$ be a witnessing embedding; that is, a $\mathcal{P}_\kappa\mu$-supercompactness embedding, for some $\mu > \lambda$, such that κ is β-hypercompact in M. Let $j_\lambda : V \to N$ be the factor embedding of j induced by the seed $j"\lambda$; that is, j_λ is induced by the ultrafilter $\mathcal{U}$ on $\mathcal{P}_\kappa\lambda$ such that $X \in \mathcal{U}$ iff $j"\lambda \in j(X)$. Let $k : N \to M$ elementary be such that $j = k \circ j_\lambda$. By general facts about factor embeddings (see for instance [6], Lemma 22.12) the critical point of k is greater than λ, which is greater than both κ and α, and hence also β. By elementarity, $N \models$ "κ is β-hypercompact" iff $M \models$ "$k(\kappa)$ is $k(\beta)$-hypercompact.". ▮

Lemma 2.5 *Being hypercompact is Π_3 definable.*

Proof: κ is hypercompact iff κ is β-hypercompact for all β. So it suffices to show that being β-hypercompact is Δ_3.

The β-hypercompactness of κ will be witnessed by a sequence $HPC_\lambda \subseteq (\beta + 1) \times \lambda$ for $\lambda \leq \kappa + 1$. Each HPC_λ will be so defined as to contain $\langle \alpha, \mu \rangle$ iff μ is α-hypercompact. So κ will be β-hypercompact iff $\langle \beta, \kappa \rangle \in HPC_{\kappa+1}$. We need to check the definability of such an HPC-sequence.

By the previous theorem, μ is α-hypercompact iff for all $\gamma < \alpha$ and for all $\rho > \mu + \alpha$ there is a supercompactness embedding j induced by an ultrafilter on $\mathcal{P}_\mu\rho$ such that, in M, μ is γ-hypercompact. If we define the HPC-sequence right, then μ's γ-hypercompactness will be witnessed by $\langle \gamma, \mu \rangle \in j(HPC_\mu)$.

Hence we define the HPC-sequence by:

$\forall \lambda \leq \kappa + 1 \ \forall \alpha \leq \beta \ \forall \mu < \lambda [\langle \alpha, \mu \rangle \in HPC_\lambda \leftrightarrow (\forall \gamma < \alpha \ \forall \rho > \mu + \alpha \ \forall X$ if $X = V_{\rho+2}$ then there is a normal fine ultrafilter on $\mathcal{P}_\mu\rho$ in X inducing j such that $\langle \gamma, \mu \rangle \in j(HPC_\mu))]$.

To say that a set is an initial segment of the V-hierarchy is Π_1, and so the part in parentheses is Π_2. The formula in brackets is a Boolean combination of Π_2 formulas, and so is Δ_3. The quantifiers in front are bounded, and as such do not raise the quantifier complexity.

Finally, κ is β-hypercompact iff there is an HPC-sequence as defined above and $\langle \beta, \kappa \rangle \in HPC_{\kappa+1}$ iff for every HPC-sequence if it satisfies the definition above then $\langle \beta, \kappa \rangle \in HPC_{\kappa+1}$, which is Δ_3. ▮

Theorem 2.6 *If κ is extendible then κ is hypercompact.*

Proof: We need to refine the notion of α-hypercompactness. Whereas α-hypercompactness refers to $\mathcal{P}_\kappa\lambda$ embeddings for (essentially) all λ, γ-supercompactness refers to a single $\mathcal{P}_\kappa\gamma$ embedding. Hence we say that κ is $\gamma - \alpha$-hypercompact if for all $\beta < \alpha$ there is a $\mathcal{P}_\kappa\gamma$ embedding $j : V \to M$ such that, in M, κ is β-hypercompact.

Let κ be extendible. So κ is supercompact, i.e. 1-hypercompact. Assume inductively κ is β-hypercompact for $\beta < \alpha$. We would like to get κ to be $\gamma - \alpha$-hypercompact, for any fixed $\gamma > \kappa + \alpha$. Let $\eta > \gamma$ be such that $V_\eta \prec_{\Sigma_3} V$. Let $j : V_{\eta+\omega} \to V_{\zeta+\omega}$ be given by the extendibility of κ. As explained to us by Joel Hamkins [5], j can be extended to an elementary embedding from V to M, which we will also call j. (Briefly, each $x \in V_{\zeta+\omega}$ induces an ultrafilter back in $V_{\eta+\omega}$, which induces an embedding from all of V. Those ultrafilters taken together form an extender, and the desired $j : V \to M$ is the extender embedding.)

By hypothesis, κ is β-hypercompact in V ($\beta < \alpha$). We claim that by a careful examination of the definition of β-hypercompactness from the previous lemma one can see that $V_\zeta \models$ "κ is β-hypercompact." In some detail, notice that the definition of $\langle \alpha, \mu \rangle \in HPC_\lambda$ begins (after the bounded quantifier) with a universally quantified ρ. When interpreted in V_ζ instead of V, this much is even easier to satisfy, as there are fewer ρ's to consider, so if anything $(HPC_\lambda)^{V_\zeta}$ will be a superset of HPC_λ. The next part of the definition is about initial segments of V (namely $V_{\rho+2}$). There is agreement between V and V_λ on what's an initial segment of V. The final part of the definition is whether $\langle \gamma, \mu \rangle \in j(HPC_\mu)$. This is a positive assertion (that is, in the scope of no negations), and so if true in V then true in V_ζ. To be more formal about it, we would show that $HPC_\lambda \subseteq (HPC_\lambda)^{V_\zeta}$, inductively on λ. Assuming that this is true for all $\mu < \lambda$, if $\langle \alpha, \mu \rangle \in HPC_\lambda$, then for all relevant ρ (i.e. between $\mu + \alpha$ and ζ) there is a j in V, and hence also in V_ζ, such that $\langle \gamma, \mu \rangle \in j(HPC_\mu)$, and hence (inductively) also a member of $j(HPC_\mu^{V_\zeta})$. In V_ζ therefore $\langle \alpha, \mu \rangle$ also satisfies the definition of being in HPC_λ.

Now we use the specific choice of η. Since $V_\eta \prec_{\Sigma_3} V$ and $\zeta = j(\eta)$, by elementarity $V_\zeta \prec_{\Sigma_3} M$. By the definability of β-hypercompactness, κ is β-hypercompact in M. Now take a $\mathcal{P}_\kappa\gamma$ factor embedding j_U of j, with $k : M_U \to M$ elementary and $j = k \circ j_U$. As is standard, $k(\kappa) = \kappa$ and $k(\beta) = \beta$, and so by elementarity κ is β-hypercompact in M_U. This shows the $\gamma - \alpha$-hypercompactness of κ. This works for all sufficiently large γ, so κ is α-hypercompact. ▌

Theorem 2.7 *Extendibility has a greater consistency strength than hypercompactness. In particular, if κ is extendible, then for all limit $\lambda > \kappa$, $V_\lambda \models$ "κ is hypercompact".*

Proof: By the previous theorem, κ is hypercompact, and as argued in the previous proof, that fact reflects down to all initial segments of V. Moreover, if there in an extendible, then there are unboundedly many inaccessibles. To see this, let $\eta \in \mathrm{ORD}$ be arbitrary, and $j : V_{\kappa+\eta} \to V_\lambda$ be elementary with $j(\kappa) > \eta$. Since $V_{\kappa+\eta} \models$ "κ is inaccessible," $V_\lambda \models$ "$j(\kappa)$ is inaccessible." Inaccessibility is absolute between V and its initial segments. So there are unboundedly many inaccessibles, and hence unboundedly many models of ZFC. ▌

In fact, by using this reflection property of hypercompactness, we can improve an earlier result.

Proposition 2.8 *Being hypercompact is* Π_2 *definable and not* Σ_2 *definable.*

Proof: The Π_3 definition of hypercompactness given earlier used the correct HPC-sequence. As argued above, if the definition of the HPC-sequence were interpreted in an initial segment of V, any mistake it might make would be in being too liberal, and evaluating as β-hypercompact ordinals that really weren't, not the other way around. So in fact κ is β-hypercompact if for all ρ and X (if X is an initial segment of V large enough to evaluate the transitive collapses of sufficiently large initial segments of the images of elementary embeddings given by normal fine ultrafilters on $\mathcal{P}_\kappa\rho$, and if HPC_λ is the HPC-sequence as interpreted in X, then $\langle\beta,\kappa\rangle \in HPC_{\kappa+1}$).

If hypercompactness were Σ_2, then the existence of a hypercompact is also Σ_2, and so if there were one in V there would be one in V_κ, where κ is the least supercompact, because $V_\kappa \prec_{\Sigma_2} V$ (see [6], Prop. 22.3). This clearly cannot happen, as hypercompacts are supercompact. ▮

3 Enhanced Supercompacts

Definition 3.1 *[2] A cardinal* κ *is enhanced supercompact if it is supercompact, and there is a* $\theta > \kappa$ *which is strong, and for all* $\lambda > \theta$ *there is a* λ*-supercompactness embedding* $j : V \to M$ *for* κ *such that, in* M, θ *remains strong.*

Proposition 3.2 *Being enhanced supercompact is* Σ_3 *definable and not* Π_3 *definable.*

Proof: Direct from the definition, κ is enhanced supercompact if $\exists\theta(\theta > \kappa \wedge \kappa$ is supercompact $\wedge\ \theta$ is strong $\wedge\ \forall\lambda > \theta$ ($\exists j : V \to M$ j is a λ-supercompactness embedding for κ and $M \models$ "θ is strong")). As is standard, being supercompact and being strong are Π_2 ([6], p. 302 and 360). The formula in parentheses can be written as "$\forall X$ if $X = V_{\lambda+5}$ then there is a normal ultrafilter U in X such that $\{x \in \mathcal{P}_\kappa\lambda \mid V_\theta \models$ "o.t.$(x \cap \theta)$ is strong"$\} \in U$."

Suppose $\phi(x)$ were Π_3 and provably equivalent to x being enhanced supercompact. Let κ be enhanced supercompact. Then $\phi(\kappa)$ holds in V, and moreover reflects down to any Σ_2 substructure of V that includes κ. Let λ be the least ordinal greater than κ such that $V_\lambda \prec_{\Sigma_2} V$. Then $V_\lambda \models$ "κ is enhanced supercompact." Hence $V_\lambda \models$ "there is a strong cardinal θ greater than κ." That means that $V_\lambda \models$ "$V_\theta \prec_{\Sigma_2} V$," so $V_\theta \prec_{\Sigma_2} V_\lambda$, and $V_\theta \prec_{\Sigma_2} V$, contradicting the choice of λ. ▮

Theorem 3.3 *If κ is extendible and $\theta > \kappa$ is strong then κ is enhanced supercompact.*

Proof: Recall from Theorem 2.7 that if κ is extendible then the inaccessibles are cofinal in ORD. So letting $\beta > \theta$ be inaccessible, it suffices to show that $V_\beta \models$ "κ is enhanced supercompact, with witnessing strong θ."

Let $j : V_\beta \to V_\lambda$ be a β-extendibility embedding for κ. For α between θ and β, let k be the α-supercompactness factor embedding of j from V_β to M, with i the elementary embedding from M to V_λ such that $j = i \circ k$. It suffices to show that θ is strong in M.

If θ were not strong in M, then $i(\theta)$ would not be strong in V_λ. The critical point of i is greater than α, so $i(\theta) = \theta$. But being a strong cardinal is witnessed by the existence of an extender for each ordinal, and so reflects from V to V_λ. Hence θ is strong in V_λ. ∎

Theorem 3.4 *If κ is extendible and $\theta > \kappa$ is strong then $V_\kappa \models$ "There is an enhanced supercompact."*

Proof: By the preceding theorem, κ itself is enhanced supercompact. The existence of enhanced supercompacts is Σ_3. By the Σ_3-elementarity of extendibles ([6] Prop. 23.10), since there is an enhanced supercompact in V there is one in V_κ. ∎

So enhanced supercompactness is strictly weaker both implicationally and in consistency than an extendible with a larger strong. Is this larger strong cardinal necessary?

Conjecture 3.5 *Enhanced supercompactness is strictly weaker both implicationally and in consistency than extendibility.*

That the preceding theorems do not already prove this conjecture is the purpose of the following.

Proposition 3.6 *If κ is extendible and $\theta > \kappa$ is strong then $V_\theta \models$ "κ is extendible."*

Proof: The property of being extendible is Π_3 definable ([6], p. 318). Strong cardinals provide Σ_2-elementary substructures of V, and it's easy to see that Π_3 statements reflect down to Σ_2-elementary substructures. ∎

Now we want to compare enhanced supercompactness with hypercompactness, from the previous section.

Theorem 3.7 *Enhanced supercompactness has a greater consistency strength than hypercompactness. In particular, if κ is enhanced supercompact, as witnessed by a strong $\theta > \kappa$, then $V_\theta \models$ "κ is hypercompact".*

Proof: Suppose κ is enhanced supercompact, as witnessed by a strong $\theta > \kappa$. Let β be less than θ. Assume inductively that for all $\alpha < \beta, \kappa$ is α-hypercompact. By the strongness of θ, we have that $V_\theta \prec_{\Sigma_2} V$, and it follows directly that Δ_3 properties persist between V_θ and V. So, in V_θ, κ is α-hypercompact.

By κ's enhanced supercompactness, for $\lambda > \theta$, let $j : V \to M$ be a λ-supercompactness embedding (critical point κ, $j(\kappa) > \lambda$) with θ strong in M. Since $M_\theta = V_\theta, M_\theta \models$ "κ is α-hypercompact." By Δ_3 elementarity, $M \models$ "κ is α-hypercompact." So j witnesses that (in V) κ is β-hypercompact.

By induction, for all $\beta < \theta, \kappa$ is β-hypercompact. Hence $V_\theta \models$ "κ is hypercompact." ∎

Theorem 3.8 *If κ is enhanced supercompact then κ is hypercompact.*

Proof: Let $\theta > \kappa$ be the strong cardinal witnessing enhanced supercompactness. Let $\lambda > \theta, V_\lambda \prec_{\Sigma_3} V$, and $j : V \to M$ be a λ-strongness embedding for θ. As shown during the proof of the previous theorem, $V \models$ "$\forall\beta < \theta$ (κ is β-hypercompact)." By elementarity, $M \models$ "$\forall\beta < j(\theta)$ (κ is β-hypercompact)." In particular, $M \models$ "$\forall\beta < \lambda$ (κ is β-hypercompact)."

We claim that $M_\lambda \models$ "$\forall\beta$ (κ is β-hypercompact)." Choose $\beta < \lambda$. The witness to κ's β-hypercompactness in M_λ, if any, would be an HPC-sequence as defined in the previous section. The defining characteristic of some pair $\langle\alpha, \mu\rangle$ being in some HPC_γ is Π_2; more than that, the existential quantifier in this Π_2 definition is just to verify that a set is an initial segment of the V-hierarchy, and so reflects from M to M_λ whenever λ is a limit ordinal. Hence $HPC_\gamma \subseteq HPC_\gamma^{M_\lambda}$, which suffices.

Since $M_\lambda = V_\lambda$, $V_\lambda \models$ "$\forall\beta$ (κ is β-hypercompact)." By the choice of λ, κ is hypercompact. ∎

4 C^n-supercompacts

In [3], Bagaria gives a meta-definition of a C^n-large cardinal, where C^n is the proper class of ordinals λ such that $V_\lambda \prec_{\Sigma_n} V$.[1] If X is a large cardinal notion based on elementary embeddings, then κ is said to be C^n-X if there is an elementary embedding j as given by X, with critical point κ, and $j(\kappa) \in C^n$. Note that this is C^n as interpreted in V, even though the object $j(\kappa)$ leads one to think of M. For instance:

[1] Bagaria actually uses the notation $C^{(n)}$. We feel that C^n is a cleaner notation and are advocating the use of this cleaner notation.

Definition 4.1 *[3] A cardinal κ is C^n-measurable if there is an elementary embedding j with critical point κ such that $j(\kappa) \in C^n$. Similarly, κ is C^n-supercompact if for all λ there is a λ-supercompactness embedding j for κ with $j(\kappa) \in C^n$.*

It is often observed that κ is measurable iff κ is κ-supercompact. This is no longer obvious in the current context of C^n-cardinals, and as we will see not even true. What's up for grabs is whether the target model is closed under κ-sequences. While this is easily seen to be the case under an ultrafilter-induced embedding, in Bagaria's proof that measurability and C^n-measurability coincide he iterates that embedding, and it is easily seen that κ-closure does not persist past limit stages (by considering the ω-sequence $\kappa, j(\kappa), j(j(\kappa)), ...$).

The argument below revolves around a very similar notion, tallness:

Definition 4.2 *A cardinal κ is tall if for all θ there is an elementary embedding $j : V \to M$ with critical point κ, $j(\kappa) > \theta$, and ${}^\kappa M \subseteq M$. Similarly, κ is C^n-tall if for all θ there is an elementary embedding $j : V \to M$ with critical point κ, $j(\kappa) > \theta$, ${}^\kappa M \subseteq M$, and $\kappa \in C^n$.*

Note that if κ is tall then κ is κ-supercompact, and if κ is C^n-tall then κ is κ-C^n-supercompact. For an overview and background of results around tallness, see [4]; for other work on C^n-tallness, see [9].

Lemma 4.3 *If κ is κ-C^{n+2}-supercompact then κ is C^n-tall.*

Proof: As in [3], if there is such a supercompactness embedding then one such can be given by an extender. Suppose that κ were not C^n-tall, that is, the θ's for which there is a C^n-θ-tallness embedding were bounded. Hence $\exists\theta\ \forall E$ if E is an extender on κ then either $j_E(\kappa) \leq \theta$ or $j_E(\kappa) \notin C^n$ or ${}^\kappa M \not\subseteq M$. That statement is Σ_{n+2}, as the only part of it of any quantifier complexity (beyond the explicit quantifiers at the beginning) is the Δ_{n+1} definition of C^n. Because of that quantifier complexity, it is true in V_μ, where μ is the smallest member of C^{n+2} greater than κ. Moreover, the witness θ in V_μ is also (by elementarity) a witness in V. But taking j to be a κ-C^{n+2}-supercompactness embedding, $j(\kappa) \geq \mu$ (because $j(\kappa) \in C^{n+2}, j(\kappa) > \kappa$, and μ is the least such ordinal), and $\mu > \theta$. The existence of j contradicts the definition of θ, and so shows that κ is C^n-tall. ▮

Corollary 4.4 *κ is C^n-tall for all n iff κ is κ-C^n- supercompact for all n.*

Theorem 4.5 *There is no natural number k such that ZFC proves "if κ is C^k-measurable then κ is κ-C^3-supercompact."*

Proof: Measurability implies C^k-measurability for all k [3]. If κ is κ-C^3-supercompact then by Lemma 4.3 κ would be C^1-tall, and hence tall. So if this theorem were false, then measurability would imply tallness, which it does not [4]. ▮

Corollary 4.6 below follows immediately. Although weaker than Theorem 4.5, its statement is more elegant.

Corollary 4.6 *There is no natural number $k \geq 3$ such that ZFC proves "if κ is C^k-measurable then κ is κ-C^k-supercompact."*

5 Questions and Conjectures

Question 5.1 *What other large cardinals are known or suspected to lie between supercompacts and extendibles, and how are they related to one another and to hypercompacts and enhanced supercompacts?*

To repeat our earlier conjecture about enhanced supercompactness:

Conjecture 5.2 *Enhanced supercompactness is strictly weaker both implicationally and in consistency than extendibility.*

Regarding C^n-supercompactness, we can refer the reader to many questions about this in [3]. The one we find most intriguing is to get a consistency theorem for C^n-supercompactness relative to a standard large cardinal. Bagaria [3] was able to do this using an E_0-cardinal, also known as an I_3-cardinal, which is a very large upper bound. Later, Tsaprounis [9, Theorem 2.21] reduced this upper bound to an almost huge. Is there a good equiconsistency result? We think not. The C^n-large cardinals seem to have their own nature.

Conjecture 5.3 *The existence of a C^n-supercompact does not imply the consistency of an extendible, nor does the existence of an extendible imply the consistency of a C^n-supercompact.*

The following additional open question about C^k cardinals asks whether the result of Theorem 4.5 is optimal. (This question is actually two questions, one for each value of n.)

Question 5.4 *Let $n = 1$ or $n = 2$. Does there exist a natural number k such that, provably in ZFC, for all κ, if κ is C^k-measurable then κ is κ-C^n-supercompact?*

In the case $n = 0$, the answer is trivially yes, because V_α is Σ_0-elementary in V for all ordinals α. If κ is C^k-measurable, it follows that κ is measurable in the ordinary sense, and any elementary embedding by a normal measure on κ witnesses that κ is κ-C^0-supercompact.

References

[1] Arthur W. Apter, “Reducing the consistency strength of an indestructibility theorem,” Mathematical Logic Quarterly, 54 (3): 288 - 293, 2008.

[2] Arthur W. Apter, “Some applications of Sargsyan’s equiconsistency method,” Fund. Math., 216 (3): 207 - 222, 2012.

[3] Joan Bagaria, “$C^{(n)}$-cardinals,” Archive for Mathematical Logic, 51 (3-4): 213 - 240, 2012.

[4] Joel David Hamkins, “Tall cardinals,” Mathematical Logic Quarterly, 55 (1): 68 - 86, 2009.

[5] Joel David Hamkins, Math Overflow, answer to “extending elementary embeddings from initial segments of V to all of V,” Nov. 1, 2013, http://mathoverflow.net/questions/146567/.

[6] Akihiro Kanamori, **The Higher Infinite**, Springer, 2005.

[7] Thomas Jech, **Set Theory: The Third Millennium Edition**, Springer, 2003.

[8] Norman Lewis Perlmutter, “The large cardinals between supercompact and almost-huge,” Archive for Mathematical Logic, to appear. Preprint http://arxiv.org/abs/1307.7387.

[9] Konstantinos Tsaprounis, “Elementary chains and $C^{(n)}$-cardinals,” Archive for Mathematical Logic, 53 (1-2): 89 - 118, 2014.