The Kripke Schema in Metric Topology

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March 2, 2012

Abstract

Kripke’s schema with parameters turns out to be equivalent to each of the following two statements from metric topology: every open subspace of a separable metric space is separable; every open subset of a separable metric space is a countable union of open balls. Thus Kripke’s schema serves as a point of reference for classifying theorems of classical mathematics within Bishop-style constructive reverse mathematics.

In this paper we show that a certain version of Kripke’s schema with parameters is equivalent to either of the following two statements from metric topology: every open subspace of a separable metric space is separable; every open subset of a separable metric space is a countable union of open balls. By so doing we use Kripke’s schema as a point of reference for classifying theorems of classical mathematics within the informal variant of the constructive reverse mathematics programme put forward by Ishihara [7, 8]. As for the latter, the overall framework of the present note is Bishop-style constructive mathematics [1, 2, 3, 4, 10], which can be thought of as mathematics carried out with intuitionistic logic [11].

The Kripke Schema can be stated as follows [3]:

For each proposition $P$ there is an increasing binary sequence $(a_n)$ such that $P$ holds if and only if $a_n = 1$ for some $n$.

Clearly the Kripke schema follows from the law of excluded middle: set $a_n = 1$ (respectively, $a_n = 0$) for every $n$ whenever $P$ (respectively, $\neg P$) holds.

\footnote{For more references and other authors on constructive reverse mathematics see [9].}
However, Kripke’s schema was intended to capture the essence of Brouwer’s creating subject, an idealised mathematician (IM) living forever in discrete time. One can tell whether at time \( n \) the IM has already proved a proposition \( P \), whence one can define the increasing binary sequence \((a_n)\) by setting \( a_n = 1 \) precisely when the IM has proved \( P \) at time \( n \) or before. Brouwer conceived the creating subject in order to refute Markov’s principle within intuitionistic mathematics. Indeed, in the presence of Kripke’s schema, Markov’s principle implies the law of excluded middle.

A subset \( Y \) detachable from a set \( X \) if for each \( x \in X \) either \( x \in Y \) or \( x \notin Y \). A set \( S \) is countable [10] if there is a surjective mapping \( D \to S \) from a detachable subset \( D \) of \( \mathbb{N} \) onto \( S \). Note that a detachable subset of a countable set is countable.

With this notion of a countable set, which includes the empty set, a parametrised version of Kripke’s schema [13] can be stated as follows [12]:

**KS\( _\omega \)**. *Every subset of \( \mathbb{N} \) is countable.*

In fact a subset \( S \) of \( \mathbb{N} \) is countable precisely when it is simply existential: that is, \( S = \pi_1 (E) \) for a detachable subset \( E \) of \( \mathbb{N} \times \mathbb{N} \), where \( \pi_1 \) denotes the first projection. We refer to [12] for more on Kripke’s schema including further references.

A set \( X \) is discrete if for every pair \( x, y \in X \) either \( x = y \) or \( x \neq y \). Clearly, every subset of a discrete set is discrete as well. We say that a subset \( Y \) of a set \( X \) is proper if there is \( x \in X \) with \( x \notin Y \). The equivalence of KS\( _\omega \) with item 1 of the following lemma has been observed in [12].

**Lemma 1** Each of the following items is equivalent to KS\( _\omega \):

1. Every subset of a countable set is countable.
2. Every subset of a discrete countable set is countable.
3. Every proper subset of a countable set is countable.
4. Every proper subset of a discrete countable set is countable.

A metric space is separable if it has a countable dense subset. As an easy exercise in metric topology, the proof of the next lemma is left to the reader; for the first part see, for example, the proof of [6, VIII 7.2 (2)].
Lemma 2 Let $A$ be an open subset of a metric space $X$.

1. If $S$ is a dense subset of $X$, then $S \cap A$ is dense in $A$.

2. If $T$ is a dense subset of $A$, then for each $a \in A$ there are $t \in T$ and $r \in \mathbb{Q}^+$ with
   \[ a \in B_r(t) \subseteq A \]

On a discrete set $X$ one can define the usual discrete metric. Recall that, with this metric on $X$, if $B_r(a)$ is the open ball $B_r(a)$ with center $a \in X$ and radius $r > 0$, then $B_r(a) = \{a\}$ if $r \leq 1$, and $B_r(a) = X$ if $r > 1$. In particular, if an open ball is a proper subset, then it is a singleton.

Lemma 3 Let $X$ be a discrete set with the discrete metric.

1. A subset of $X$ is countable if and only if it is separable.

2. A proper subset of $X$ is countable if and only if it is a countable union of open balls.

Lemma 4 KS$_\omega$ is equivalent to the statement

(*) Every open, separable subset of a metric space is a countable union of open balls with rational radii.

Proof. Let $T$ be a countable dense subset of the open subset $A$ of a metric space $X$. Note first that

\[ B = \{B_r(t) : r \in \mathbb{Q}^+, \ t \in T\} \]

is a countable set, as it is indexed by the countable set $\mathbb{Q}^+ \times T$. Suppose KS$_\omega$ true. Then the subset

\[ B_A = \{B \in B : B \subseteq A\} \]

of $B$ is countable, and by Lemma 2 we have $A = \bigcup B_A$.

Conversely, let $S$ be a subset of $\mathbb{N}$ and consider the subset $A$ of $\mathbb{Q}$, with the usual metric, consisting of $S \cup (\mathbb{Q} \setminus \mathbb{N})$. As $\mathbb{Q} \setminus \mathbb{N}$ is dense, the subset $A$ is separable. As each element $s \in S$ is contained in the ball $B_1(s)$, which
is totally contained in \( A \), the subset \( A \) is open. If \( A \) is a countable union of open balls with rational radii, then \( S \) is countable. ■

At the end of the proof, no appeal to countable choice is required: if \( B \) is an open ball with rational radius that is contained in \( A \), then \( B \cap S \) is finite. Compare condition \((*)\) in Lemma 4 to its special case “every separable metric space is a countable union of open balls”, which is provable without \( \text{KS}_\omega \) along the lines of the foregoing proof. Also, \((*)\) trivially holds for discrete sets with the discrete metric.

**Proposition 5** Each of the following statements is equivalent to \( \text{KS}_\omega \):

1. Every open subspace of a separable metric space is separable.

2. Every open subset of a separable metric space is a countable union of open balls.

**Proof.** We think of \( \text{KS}_\omega \) as characterised in Lemma 1. In particular \( \text{KS}_\omega \) is nothing but statement 1 applied to the discrete metric on any discrete set. Conversely, the general form of statement 1 follows from \( \text{KS}_\omega \) in view of Lemma 2. To see that \( \text{KS}_\omega \) implies statement 2, we also use it in the form of statement 1, and apply Lemma 4. By Lemma 3, \( \text{KS}_\omega \) follows from statement 2. ■

Note that in computable analysis [14] an open subset of a (separable) metric space can be characterised as a countable union of open balls.

**Acknowledgements** This topic was prompted by the investigations [5] of Beppo Levi’s approximation principle Bruni and Schuster have undertaken.

**References**


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