The Kripke Schema in Metric Topology

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Abstract

Kripke's schema with parameters turns out to be equivalent to each of the following two statements from metric topology: every open subspace of a separable metric space is separable; every open subset of a separable metric space is a countable union of open balls. Thus Kripke's schema serves as a point of reference for classifying theorems of classical mathematics within Bishop-style constructive reverse mathematics.

In this paper we show that a certain version of Kripke's schema with parameters is equivalent to either of the following two statements from metric topology: every open subspace of a separable metric space is separable; every open subset of a separable metric space is a countable union of open balls. By so doing we use Kripke's schema as a point of reference for classifying theorems of classical mathematics within the informal variant of the constructive reverse mathematics programme put forward by Ishihara [7, 8].[1] As for the latter, the overall framework of the present note is Bishop-style constructive mathematics [1, 2, 3, 4, 10], which can be thought of as mathematics carried out with intuitionistic logic [11].

The *Kripke Schema* can be stated as follows [3]:

> *For each proposition P there is an increasing binary sequence (a_n) such that P holds if and only if $a_n = 1$ for some n.*

Clearly the Kripke schema follows from the law of excluded middle: set $a_n = 1$ (respectively, $a_n = 0$) for every n whenever P (respectively, $\neg P$) holds.

[1] For more references and other authors on constructive reverse mathematics see [9].

However, Kripke's schema was intended to capture the essence of Brouwer's *creating subject*, an idealised mathematician (IM) living forever in discrete time. One can tell whether at time n the IM has already proved a proposition P, whence one can define the increasing binary sequence (a_n) by setting $a_n = 1$ precisely when the IM has proved P at time n or before. Brouwer conceived the creating subject in order to refute Markov's principle within intuitionistic mathematics. Indeed, in the presence of Kripke's schema, Markov's principle implies the law of excluded middle.

A subset Y *detachable* from a set X if for each $x \in X$ either $x \in Y$ or $x \notin Y$. A set S is *countable* [10] if there is a surjective mapping $D \to S$ from a detachable subset D of $\mathbb{N}$ onto S. Note that a detachable subset of a countable set is countable.

With this notion of a countable set, which includes the empty set, a parametrised version of Kripke's schema [13] can be stated as follows [12]:

KS$_\omega$ *Every subset of* $\mathbb{N}$ *is countable.*

In fact a subset S of $\mathbb{N}$ is countable precisely when it is *simply existential*: that is, $S = \pi_1(E)$ for a detachable subset E of $\mathbb{N} \times \mathbb{N}$, where π_1 denotes the first projection. We refer to [12] for more on Kripke's schema including further references.

A set X is *discrete* if for every pair $x, y \in X$ either $x = y$ or $x \neq y$. Clearly, every subset of a discrete set is discrete as well. We say that a subset Y of a set X is *proper* if there is $x \in X$ with $x \notin Y$. The equivalence of KS$_\omega$ with item 1 of the following lemma has been observed in [12].

Lemma 1 *Each of the following items is equivalent to* KS$_\omega$*:*

1. *Every subset of a countable set is countable.*

2. *Every subset of a* discrete *countable set is countable.*

3. *Every* proper *subset of a countable set is countable.*

4. *Every* proper *subset of a* discrete *countable set is countable.*

A metric space is *separable* if it has a countable dense subset. As an easy exercise in metric topology, the proof of the next lemma is left to the reader; for the first part see, for example, the proof of [6, VIII 7.2 (2)].

Lemma 2 *Let A be an open subset of a metric space X.*

1. *If S is a dense subset of X, then $S \cap A$ is dense in A.*

2. *If T is a dense subset of A, then for each $a \in A$ there are $t \in T$ and $r \in \mathbb{Q}^+$ with*
$$a \in B_r(t) \subseteq A$$

On a discrete set X one can define the usual discrete metric. Recall that, with this metric on X, if $B_r(a)$ is the open ball $B_r(a)$ with center $a \in X$ and radius $r > 0$, then $B_r(a) = \{a\}$ if $r \leq 1$, and $B_r(a) = X$ if $r > 1$. In particular, if an open ball is a proper subset, then it is a singleton.

Lemma 3 *Let X be a discrete set with the discrete metric.*

1. *A subset of X is countable if and only if it is separable.*

2. *A proper subset of X is countable if and only if it is a countable union of open balls.*

Lemma 4 KS_ω *is equivalent to the statement*

(∗) *Every open, separable subset of a metric space is a countable union of open balls with rational radii.*

Proof. Let T be a countable dense subset of the open subset A of a metric space X. Note first that

$$\mathcal{B} = \{B_r(t) : r \in \mathbb{Q}^+ ,\ t \in T\}$$

is a countable set, as it is indexed by the countable set $\mathbb{Q}^+ \times T$. Suppose KS_ω true. Then the subset

$$\mathcal{B}_A = \{B \in \mathcal{B} : B \subseteq A\}$$

of $\mathcal{B}$ is countable, and by Lemma 2 we have $A = \bigcup \mathcal{B}_A$.

Conversely, let S be a subset of $\mathbb{N}$ and consider the subset A of $\mathbb{Q}$, with the usual metric, consisting of $S \cup (\mathbb{Q} \setminus \mathbb{N})$. As $\mathbb{Q} \setminus \mathbb{N}$ is dense, the subset A is separable. As each element $s \in S$ is contained in the ball $B_1(s)$, which is totally contained in A, the subset A is open. If A is a countable union of open balls with rational radii, then S is countable. ■

At the end of the proof, no appeal to countable choice is required: if B is an open ball with rational radius that is contained in A, then $B \cap S$ is finite. Compare condition $(*)$ in Lemma 4 to its special case "every separable metric space is a countable union of open balls", which is provable without KS_ω along the lines of the foregoing proof. Also, $(*)$ trivially holds for discrete sets with the discrete metric.

Proposition 5 *Each of the following statements is equivalent to* KS_ω*:*

1. *Every open subspace of a separable metric space is separable.*

2. *Every open subset of a separable metric space is a countable union of open balls.*

Proof. We think of KS_ω as characterised in Lemma 1. In particular KS_ω is nothing but statement 1 applied to the discrete metric on any discrete set. Conversely, the general form of statement 1 follows from KS_ω in view of Lemma 2. To see that KS_ω implies statement 2, we also use it in the form of statement 1, and apply Lemma 4. By Lemma 3, KS_ω follows from statement 2. ■

Note that in computable analysis [14] an open subset of a (separable) metric space can be characterised as a countable union of open balls.

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