

Well-Founded Iterations of Infinite Time Turing Machines

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Applications

Useful for ordinal analysis

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Iteration and hyper-iteration/feedback

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Iteration and hyper-iteration/feedback

- ▶ Turing jump \mapsto hyperarithmetical sets
- ▶ Inductive definitions \mapsto the μ -calculus
- ▶ ITTMs \mapsto ???

First Definitions

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- ▶ the machine is in a dedicated state
- ▶ the head is on the 0^{th} cell
- ▶ the content of a cell is limsup of the previous contents (i.e. 0 if eventually 0, 1 if eventually 1, 1 if cofinally alternating)

Writable reals and ordinals

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Proposition

$R \subseteq \omega$ is writable iff $R \in L_\lambda$.

Eventually writable reals and ordinals

Definition

$R \subseteq \omega$ is **eventually writable**

if its characteristic function is on the output tape, never to change, of a computation.

An ordinal α is **eventually writable**

if some real coding α (via some standard representation) is eventually writable.

$\zeta := \sup \{ \alpha \mid \alpha \text{ is eventually writable} \}$

Proposition

$R \subseteq \omega$ is eventually writable iff $R \in L_\zeta$.

Accidentally writable reals and ordinals

Definition

$R \subseteq \omega$ is **accidentally writable**

if its characteristic function is on the output tape at any time during a computation.

An ordinal α is **accidentally writable**

if some real coding α (via some standard representation) is accidentally writable.

$\Sigma := \sup \{ \alpha \mid \alpha \text{ is accidentally writable} \}$

Proposition

$R \subseteq \omega$ is accidentally writable iff $R \in L_\Sigma$.

Summary and conclusions

λ is the supremum of the writable numbers.

ζ is the supremum of the eventually writable numbers.

Σ is the supremum of the accidentally writable numbers.

Clearly, $\lambda \leq \zeta \leq \Sigma$.

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Theorem

(Welch) ζ is the least ordinal α such that L_α has a Σ_2 -elementary extension. (ζ is the least Σ_2 -**extendible** ordinal.) The ordinal of that extension is Σ . L_λ is the least Σ_1 -elementary substructure of L_ζ .

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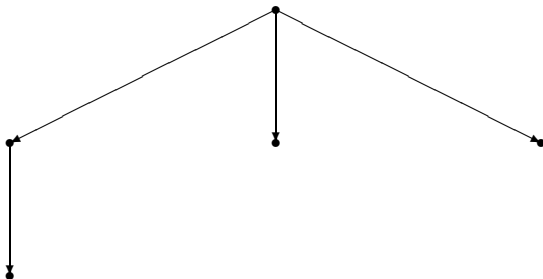
The definitions of λ , ζ , and Σ relativize (to λ^∇ , ζ^∇ , and Σ^∇) to computations from 0^∇ . Furthermore, ζ^∇ is the least Σ_2 -extendible limit of Σ_2 -extendibles, the ordinal of its Σ_2 extension is Σ^∇ , and λ^∇ is the ordinal of its least Σ_1 -elementary substructure.

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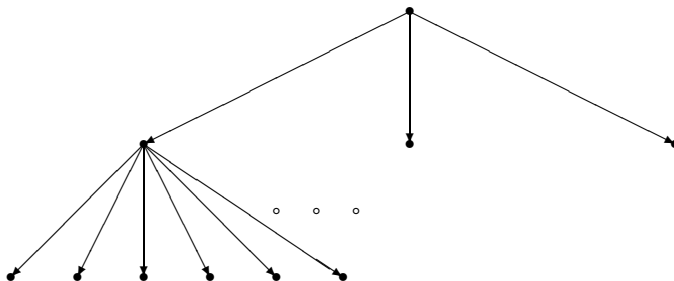
ITTMs with arbitrary iteration:

A computation may ask a convergence question about another computation. This can be considered calling a sub-computation. That sub-computation might do the same. This can continue, generating a *tree of sub-computations*. Eventually, perhaps, a computation is run which calls no sub-computation. This either converges or diverges. That answer is returned to its calling computation, which then continues.

Good examples



Good examples



Bad example



Bad example



One can naturally define the course of a computation if and only if the tree of sub-computations is well-founded. How is this to be dealt with?

Option I

When the main computation makes a sub-call, the call must be made with an ordinal. When a sub-call makes a sub-call itself, that must be done with a smaller ordinal. The definitions of λ , ζ , and Σ relativize (to λ^{it^∇} , ζ^{it^∇} , and Σ^{it^∇}).

Results

Definition

β is 0- (or 1-) extendible if its Σ_2 -extendible.

β is $(\alpha+1)$ -extendible if its a Σ_2 -extendible limit of α -extendibles.

β is κ -extendible if its a Σ_2 -extendible limit of α -extendibles for each $\alpha < \kappa$.

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Proposition

$\zeta^{it\nabla}$ is the least κ which is κ -extendible, $\Sigma^{it\nabla}$ is its Σ_2 extension, and $\lambda^{it\nabla}$ its least Σ_1 substructure.

Option II

Allow all possible sub-computation calls, even if the tree of sub-computations is ill-founded, and consider only those for which the tree of sub-computations just so happens to be well-founded.

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Proof.

Given e , run the computation of ϕ_e . Keep asking “if I continue running this computation until cell 0 changes, is that computation convergent or divergent?” Eventually you will get “divergent” as your answer. Then go on to cell 1, then cell 2, etc. After going through all the natural numbers, you know the real on your output tape is the eventually writable real you want. So halt. \square

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SECOND BIG FACT

If a real is accidentally writable in this fashion, then it's writable.

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QUESTION

Why isn't this a contradiction? Why can't you diagonalize?

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ANSWER

You can't run a universal machine.

As soon as a machine with code for a universal machine makes an ill-founded sub-computation call, it freezes.

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Claim In order to understand the writable reals in this context, one needs to understand the freezingly writable reals. One also needs to understand the tree of sub-computations for freezing computations.

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Proposition

Options a) and b) are incompatible: there cannot be one tree of sub-computations with more than Λ -much splitting beneath a node and another with the splittings beneath all the nodes cofinal in Λ .

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An oracle call may be the question “does one of these computations converge?” The computation asked about has an index e , parameter x , and free variable n . If one (natural number) value for n yields a convergent computation, the answer is “yes”, even if other values yield freezing computations. The answer “yes” means some natural number yields a convergent computation, even if other numbers yield freezing. The answer “no” means all parameter values yield non-freezing computations and all are divergent.

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to be continued ...

References

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