GEOMETRIC SPACES WITH NO POINTS

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Abstract. Some models of set theory are given which contain sets that have some of the important characteristics of being geometric, or spatial, yet do not have any points, in various ways. What's geometrical is that there are functions to these spaces defined on the ambient spaces which act much like distance functions, and they carry normable Riesz spaces which act like the Riesz spaces of real-valued functions. The first example, already sketched in [4], has a family of sets, each one of which cannot be empty, but not in a uniform manner, so that it is false that all of them are inhabited. In the second, we define one fixed set which does not have any points, while retaining all of these geometrical properties.

1. Introduction

There has been increasing interest over the years in the point-free approach to mathematics. Examples of this include toposes as a model of set theory, in which the objects play the role of sets and arrows functions, and the members of sets play no role in the axiomatization (see [11] for a good introduction and further references); locales (or frames) for the study of topology, where the opens of a topological space are the objects of the locale and the points of the space again are not accounted for in the axiomatization [8]; and point-free geometry, first developed by Whitehead almost a century ago [15, 16, 17]. This paper is a contribution to point-free geometry and analysis, albeit from a more modern perspective than Whitehead's.

Part of the motivation for this approach is increased uniformity of the results. A statement that applies to individual points obscures the uniformity or continuity of the outcome. Point-free mathematics also can provide stronger results and be more widely applicable, by proving theorems with weaker hypotheses. Sometimes assumptions are made in a theorem, such as Excluded Middle or Countable Choice, which serve only to build certain points, whereas the actual content of the construction lies elsewhere. By eliminating reference to points, such powerful axioms can themselves be eliminated, and attention focused on what's most important.

The framework of this paper is that of constructive mathematics, and so Excluded Middle is not assumed. Since constructive reasoning is a widely known and accepted paradigm, this is not what is novel about this work. Rather, what is a bit different is to work without Countable Choice. That is, we want to develop models, necessarily violating CC, in which interesting phenomena happen. This focus on models indicates a big difference from other efforts. Much of the literature on point-free mathematics develops theories and proves theorems in a point-free framework. The goal of this work is to present some models.

2. Polynomials without Uniform Roots

In [12], a plausibility argument is given for why the assertion that $X^2 - a$ has a root (a a complex number) is tantamount to accepting a certain choice principle. The argument comes down to the fact that there is no continuous square root function on any neighborhood of 0 in the complex numbers. Of course, if a is in a simply connected region excluding 0, then one of the two square roots can be chosen arbitrarily; if a equals 0, then 0 itself is a square root; it is when we don't know whether a is 0 or not that we're not able to define a square root. That choice plays a role here is indicated by the fact that polynomials over $\mathbb{C}$ have roots in $\mathbb{C}$, if the coefficients are limits of Cauchy sequences of complex rational numbers [13]; that every complex number (given as a pair of Dedekind cuts in the reals) is the limit of a Cauchy sequence is a typical application of Countable Choice. In [4], this plausibility argument is made more precise by casting it as a topological model. The main point of this section is to provide the details of this latter model, in which not all complex numbers have square roots.

Before turning to this model, let's clarify its significance by gathering some previously known results, working within IZF to be definite. It's clear that the set of roots of any polynomial over $\mathbb{C}$ is geometric, in that it is a subset of $\mathbb{C}$. Better than that, it was shown in [10] that the root set of a polynomial over $\mathbb{C}$ is *quasilocated*, meaning that it supports a *quasidistance* function δ, where $\delta(x, L) = \mathrm{glb}_{y \in L} d(x, y)$ (and d is the standard real-valued distance function on $\mathbb{C}$). Here we distinguish a greatest lower bound from the infinimum $\inf(S)$ of a set S: $\inf(S)$ is a lower bound of S such that $\forall \epsilon > 0 \; \exists x \in S \; x < \inf(S) + \epsilon$. If $\inf(S)$ exists, then in particular S is inhabited. For S perhaps not inhabited, the more useful concept is the glb, which is just what it says – a lower bound for S greater than or equal to all other lower bounds – without implying that S is inhabited. Furthermore, it is shown in the same paper that, for D a closed disc containing a quasilocated set S, the set of uniformly continuous functions on S that extend to uniformly continuous functions on D is a Riesz space V; the reader is reminded of the various Stone-like representation theorems, by which every Riesz space is isomorphic to a space of functions on a (non-empty) set, under certain assumptions, such as Excluded Middle or Dependent Choice [5, 3]. So while the root set of some polynomial might not be inhabited (i.e. have an element), it can be dealt with like an inhabited set in many ways, via its quasidistance function and its Riesz space. Finally, while the root set might not be inhabited, it is also not empty [10].

While we're on the subject of Riesz spaces, we'd like to address one of their less discussed properties. In [3], a constructive Stone-Yosida representation theorem is proven: under DC, if R is a separable and normable Riesz space, then R is (isomorphic to) a set of continuous functions on a (Heine-Borel) compact (i.e. complete and totally bounded) metric space. R is *normable* if it contains a strong unit 1, inducing an embedding of $\mathbb{Q}$ into R, and, for each $f \in R$, $U(f) := \{q \in \mathbb{Q} | q > f\}$ is a real number (i.e. a located upper cut in the rationals), in which case U provides a pseudo-norm on R, which is a norm if R is Archimedean. It is not necessary for R to be normable in order to be a function space: just take R to be the set of functions on $\{0\} \cup \{1 \mid \mathrm{P}\}$, where P is some proposition; then the normability of

R is equivalent to the decidability of P. But for R to be a nice function space on a nice set, some such hypothesis is necessary. For instance:

Proposition 1. *If R is a set of uniformly continuous functions on a compact metric space, then R is normable.*

The very simple proof is left as an exercise, and also follows easily from results in [1] or [2]. The purpose of this discussion is to justify the assumption of normability in [3], and the effort spent proving normability in [10] and in the next section here.

At this point, we are ready for the construction. Take the standard topological model over $\mathbb{C}$. For those unfamiliar with topological models, this can be viewed as the standard sheaf model over $\mathbb{C}$, or as a Heyting-valued model over the Heyting algebra of the open subsets of $\mathbb{C}$. As is standard, $\mathbb{C} \Vdash$ IZF [4, 6, 9, 11].

Theorem 2. *(Fourman-Hyland* [4]*)* $\mathbb{C} \Vdash$ "*Not every polynomial has a root.*"

Proof. Let G be the generic complex number. G is characterized by the relation $O \Vdash G \in O$ for any open set O. We claim that no neighborhood of 0 forces that anything is a root of $X^2 - G$. Suppose to the contrary that $0 \in O \Vdash z^2 = G$. O contains all circles with center 0 of sufficiently small radius. Consider the circle in O centered at 0 of radius ϵ^2. Each point w of the circle is contained in a small open set O_w forcing z to be in an open set U_w of diameter less than ϵ. U_w must contain a square root of w, and each square root of w has absolute value ϵ, so they are a distance of 2ϵ apart. Hence U_w contains exactly one such square root. Furthermore, for any other such neighborhood O'_w of w (i.e. one determining z to be in some U'_w of diameter less than ϵ), U_w and U'_w contain the same square root of w. (If not, then $\emptyset \neq O_w \cap O'_w \Vdash z \in U_w \cap U'_w = \emptyset$.) Notice that the square root of w so determined is a continuous function of w: by choosing O_w to be sufficiently small, U_w can be made arbitrarily small, and the values of the square root function on the elements of O_w can be limited to an arbitrarily small arc. This, however, is a contradiction, as there is no continuous single-valued square root function on a circle around the origin.

This argument shows that $\mathbb{C} \not\Vdash$ "$X^2 - G$ has a root." In contrast, it is not hard to see that $\mathbb{C} - \{0\} \Vdash$ "$X^2 - G$ has a root." However, given any open set, it can be translated to contain the origin, to come up with a similar example. That is, for any $v \in \mathbb{C}$, arguments similar to the ones above will show that no neighborhood v can force "$X^2 - (G - v)$ has a root." So no non-empty open set forces "every polynomial has a root." So $\mathbb{C} \Vdash$ "Not every polynomial has a root." $\square$

3. A Set of Complex Numbers with No Members

In the previous section, we identified a family of subsets of $\mathbb{C}$ which are all nonempty but not all inhabited. In this section, we present a subset of $\mathbb{C}$ which is not inhabited but still has a nontrivial distance function and nontrivial Riesz space.

Let the topological space F consist of all finite subsets of $\mathbb{C}$. (Since we are dealing only with the topological properties here, you can just as well think of taking $\mathbb{R}^2$ instead.) The topology is that induced by the Hausdorff metric, which is also known as the Vietoris topology. We give a self-contained description. A basic open set U is determined by finitely many pieces of information. Information is either positive or negative. A positive piece of information is an open set O of $\mathbb{C}$, and holds of $A \in F$ iff A contains a point of O. A negative piece of information is an open set N of $\mathbb{C}$, and holds of $A \in F$ if and only if $A \subseteq N$. Notice that the finitely many pieces of negative information can be combined into one piece, by taking the intersection of the determining open sets, so we will assume that any basic open U has only one piece of negative information. A point $A \in F$ is in an open set U if and only if A satisfies all of the information determining U. Since the information determining U can be of any finite size, such sets U are a basis, and so determine a topology on F. The model desired is the full topological model built on F.

An open set U is in *normal form* if its determining closed set N is the union of the determining open sets $\overrightarrow{O}$. Observe that the open sets in normal form are a basis for the topology on F.

In this model, the obvious choice for the canonical subset of $\mathbb{C}$ is the set H defined via the relation $U \Vdash H \subseteq \bigcup \overrightarrow{O}$, where $\overrightarrow{O}$ is the set of positive information in (i.e. the finitely many open sets determining) U and U is in normal form. (More generally, $U \Vdash H \subseteq N$, where N is the negative information in U, U any basic open set.) H is indeed the set we want.

Theorem 3. *H does not contain any points.*

Proof. Suppose to the contrary $U \Vdash X \in H$. Let $A \in U$. Say that A has n elements. Shrink U to an open V in normal form containing A. Let $\epsilon > 0$ be such that the positive open sets determining V are more than ϵ apart from each other. Shrink V to $W \ni A$, again in normal form, forcing X to be within $\epsilon/2$ of some point with rational coordinates. So W forces X to be within a fixed choice among the n open sets determining W. Without loss of generality, we can assume that that open set is a rectangle, $I \times J$ say, by shrinking it if necessary. So $W \Vdash X \in I \times J$.

Now choose $B \in W$ to agree with A except that from $I \times J$ B has not one point but two, one from the upper-left quadrant and the other from the lower-right. Extend W to a basic neighborhood W_1 of B forcing X to be in either the upper-left or lower-right quadrant; without loss of generality, say it's the lower-right that's forced. For each y in the upper half of $I \times J$, let B_y be B with the point from the upper-left quadrant replaced by y. Consider $Y = \{y \mid$some neighborhood of B_y forces X to be in the lower-right quadrant of $I \times J\}$. This is open, because if z is close to $y \in W'$ then $B_z \in W'$. It's also closed, because if z is on the boundary of Y, some neighborhood of B_z either forces X to be close to z or forces X to be in the lower-right quadrant; if the former, then the same would hold for those y's in a neighborhood of z, which would be a contradiction when considering such a $y \in Y$. Because rectangles in $\mathbb{C}$ are connected, Y is the entire upper half-rectangle. Let y be in the upper-right quadrant of $I \times J$, and W_y such a neighborhood of B_y.

Now do the same on the bottom half of $I \times J$, moving the point in the lower right quadrant to the lower left. Then move the point in the upper right down to the lower right, and the point in the lower left down to the upper left. In fact, those last two steps could be chosen so that we end back at B. But the location of X was dragged all the way around, and we now have a neighborhood of B forcing X to be in the upper-left quadrant. This is a contradiction, so there is no such X. $\square$

In a very real sense, H is just the empty set. But it's really not. Consider distance. Let U be in normal form, with each determining positive open set of diameter less than ϵ. Intuitively, the distance from $z \in \mathbb{C}$ to H is between $\rho = \mathrm{dist}(z, \overrightarrow{O})$, $\overrightarrow{O}$ the positive information in U, and $\rho + \epsilon$. We could just give this as the definition of distance and be done with it. Instead, we would rather put this in a broader context. For that, we will first need to develop the Riesz space aspect, and then return to the discussion of distance.

To phrase it imprecisely but hopefully suggestively, the Riesz space desired is the one generated by what would be the projections of H onto the x (real) and y (imaginary) axes, if H did have any points. To phrase it precisely, working in the ambient (i.e. external) model, let R be the Riesz space of functions from $\mathbb{C}$ to $\mathbb{R}$ generated by the three functions 1 (the constant function with value 1), x (projection onto the real part), and y (projection onto the imaginary part). Each element of R can be thought of as a word in the language of Riesz spaces, an expression built from the generators using vector addition, scalar multiplication, and the lattice operations; or better: an equivalence class of such words, mod functional equality. The internal Riesz space in the topological model, ambiguously also called R, consists of those same functions, or words. Given any $r, s \in R$, $U \Vdash r = s$ iff, for N the negative information determining U and $z \in N$, $r(z) = s(z)$. The Riesz space structure is inherited from the external R, and the equality axioms are clearly satisfied. There remains only one fact to check.

Claim 4. $F \Vdash R$ is normable.

Proof. Let $r \in R$ and $A \in F$. Notice that r as a function externally is continuous. Cover A with disjoint open sets $\overrightarrow{O}$ so that r varies less than ϵ on each of them. Let U be the open set of F in normal form the positive information of which is given by $\overrightarrow{O}$. Then U forces $\sup(r)$ to be between $\sup_{z \in O \in \overrightarrow{O}} r(z)$ on the upper end and $\sup_{O \in \overrightarrow{O}} \inf_{z \in O} r(z)$ on the lower, a distance less than ϵ. □

In [3], Coquand and Spitters asked whether Choice was necessary to construct a Riesz space homomorphism into $\mathbb{R}$ of a discrete, countable Riesz space, where the Riesz space is a vector space over $\mathbb{Q}$, or of a separable Riesz space. In this example, if there were such a homomorphism σ, then $(\sigma(x), \sigma(y))$ would be a point in H, so there is no such homomorphism. Furthermore, R is separable, because it is externally; if taken to be only over the rationals instead of the reals, then R is countable. R is, however, not discrete: if z is on the boundary of $\{z \mid r(z) = s(z)\}$ then no neighborhood of z will decide whether $r = s$.

Using R, we can now deal with distance. There is first the choice of which metric to use. While the Euclidean metric is the most common, it is easier to do the taxicab (L^1) metric, which we will do first. For a warm-up, let's consider what should be the distance $d(0, H)$ from the origin 0 to the set H. Classically, the distance would be calculated between 0 and each point $z \in H$ — namely, $|x| + |y|$, where x and y are z's coordinates — and the minimum would be taken over all such values. Observe that this is exactly what the Riesz space is set up to do: $d(0, H) = \inf(|x| + |y|)$. More generally, $d(z, H) = \inf(|x - x_z| + |y - y_z|)$.

To define Euclidean distance, we need to have squares available. This calls for an expanded Riesz space. When generating R, close not just under the Riesz space operations, but also squaring. Everything else remains the same. Given that squares are available, Euclidean distance can be defined as $d(z, H) = \sqrt{\inf(|x - x_z|^2 + |y - y_z|^2)}$; notice that there is no problem taking the square root, since it is the non-negative root of a non-negative real number, and so always exists.

Remark 5. An interesting question is based on the observation that the finiteness of the members $A \in F$ is not essential. The proof given above is unchanged if we allow A to be an arbitrary compact subset of $\mathbb{C}$ (making allowances for the fact that there are no open sets in normal form anymore – in general, overlaps of the open sets used as positive information must be allowed). Then we're dealing with a different topological space E, which is the completion of F. How do the models built over F and E differ? Are they elementarily equivalent?

Remark 6. This construction brings up questions regarding dimension. Distance was defined as in a two-dimensional space, and we knew to do that because the original construction was based on a 2-D space. How could the dimensionality of the missing underlying space be recovered from the Riesz space alone? More generally, under what circumstances can which properties of the invisible underlying space be inferred from just the Riesz space?

Also, this construction can be viewed as producing a set, the points of which can have their x-coordinates determined, and their y-coordinates too, just not both together. Is it possible to build a 3-D model with no points in which you can determine any two coordinates from among x, y, and z simultaneously, just not all three?

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