Geometric Spaces with No Points

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Point-Free Mathematics

- Set Theory: category theory and topoi

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- Analysis: uniform vs. pointwise continuity
- (Banaschewski, Bishop, Coquand, Mulvey, Sambin, Spitters, Vickers, others)

Motivations

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- $z \in U \nVdash G - z$ has a square root.
- $\mathbb{C} \Vdash$ Not every polynomial has a root.

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- (Ruitenburg) FTA for Cauchy sequences (in particular, under Countable Choice)
- (Richman) FTA in the form of a correspondence between degree n polynomials and the completion of n-multisets of complex numbers
- (LR) For any given polynomial, it is contradictory that it doesn't have a root.
- $\mathbb{C}$ is not provably algebraically complete.

Distance

Definition

Distance: $d(z, X) = \inf_{x \in X} d(z, x)$.

Definition

Quasi-distance: $\delta(z, X) = glb_{x \in X} d(z, x)$.

Definition

X is *quasi-located* if $\delta(z, X)$ exists for all z.

Theorem

(LR) The root set of any monic non-constant polynomial over $\mathbb{C}$ is quasi-located.

Riesz Spaces

Definition

A Riesz space is a lattice-ordered vector space.

Canonical example: The set of continuous functions on a compact space into $\mathbb{R}$ ordered pointwise.
Stone-Yosida Representation Theorem (EM): Every Archimedean Riesz space can be embedded densely into the Riesz space of real-valued continuous functions on a compact Hausdorff space.

Theorem

(LR) Let S be a quasi-located subset of a closed disc D. Then the set of uniformly continuous functions on S that extend to D naturally forms a Riesz space. Moreover, this Riesz space is normable.

Example 2: The Topological Model over Subsets of $\mathbb{C}$

Let $F = \mathcal{P}_{fin}(\mathbb{C})$.
For $A \in F$ and $O \subseteq \mathbb{C}$ open, A satisfies O if $A \cap O \neq \emptyset$.
For $A \in F$ and $C \subseteq \mathbb{C}$ closed, A satisfies C if $A \cap C = \emptyset$.
$U \subseteq F$ is open if U is determined by finitely many open and closed subsets of $\mathbb{C}$.
Let H be such that $U \Vdash H \subseteq \overrightarrow{O}$, where $\overrightarrow{O}$ is the positive information in U.
$F \Vdash$ Nothing is in H.

Riesz Space

- In the ambient model, let R be the Riesz space generated by the constant function 1, the projection onto the real axis x, and the projection onto the imaginary axis y. The internal Riesz space in the topological model is the internalization of R, in which $U \Vdash r = s$ iff $r(z) = s(z)$ for all z outside of the closed sets determining U.

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- R is normable.

Distance

- L^1 metric: $d(0, X) = \inf_{(x,y)\in X}(|x| + |y|)$
  $d(z, X) = \inf_{(x,y)\in X}(|x - x_z| + |y - y_z|)$

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- So let $\delta(0, H)$ be $\inf(|x| + |y|)$
- and $d(z, X)$ be $\inf(|x - x_z| + |y - y_z|)$.
- Euclidean metric: Close the original Riesz space under squaring.

Questions

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- The defined distance functions are two-dimensional. What properties of the generators x and y allow this definition to go through? What other properties of the missing underlying space of a Riesz space can be read off from the Riesz space itself?

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- Let E be the space of compact subsets of $\mathbb{C}$. E is the completion of F. The above proof goes through for E. How do those two topological models differ?
- The defined distance functions are two-dimensional. What properties of the generators x and y allow this definition to go through? What other properties of the missing underlying space of a Riesz space can be read off from the Riesz space itself?
- Normability gives us that x and y values can be determined, just not simultaneously. Is there a 3-D model in which any two of the coordinates x, y and z can be determined, but not all three?

References

- Coquand/Spitter, Formal Topology and Constructive Mathematics: the Gelfand and Stone-Yosida Representation Theorems, Journal of Universal Computer Science, v. 11, 2005, p. 1932-1944
- Lubarsky/Richman, Zero Sets of Univariate Polynomials, TAMS, to appear
- Lubarsky/Richman, Geometric Spaces with No Points, in negotiation
- Richman, The Fundamental Theorem of Algebra: a Constructive Development without Choice, Pacific Journal of Mathematics, v. 196, 2000, p. 213-230