ITTMs with Feedback

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Abstract

Infinite time Turing machines are extended in several ways to allow for iterated oracle calls. The expressive power of these machines is discussed and in some cases determined.

1 Introduction

Infinite time Turing machines, or ITTMs, introduced in [2], are regular Turing machines that are allowed to run for transfinitely many steps. The only changes to the standard definition of a Turing machine that need making are what to do at limit stages: the head goes to the front of the tape(s), the state entered is a dedicated state for limits, and the value of each cell is the lim sup of the previous values.

That introductory paper also discussed various kinds of oracles computations and corresponding jump operators. One such jump operator encodes the information "does the ITTM with index e on input r converge?" If e is allowed to call an oracle A, this is called the **strong jump** $A^{\blacktriangledown}$ of A: $\{(e, x) \mid \{e\}^A(x) \downarrow\}$. The jump can of course be used as an oracle itself, and the process iterated: you can, for instance, ask whether $\{e\}(r)$ converges, where $\{e\}$ can itself ask oracle questions of simple (non-oracle) ITTMs.

We would like to investigate ultimate iterations of this jump, for several reasons. Iterations of a procedure can lead to new phenomena. A well-known example of that in a context similar to the current one is transfinite iterations of the regular Turing jump. If you iterate the Turing jump along any well-order that appears along the way, you get the least admissible set containing your starting set, admissible computability theory being a quantum leap beyond ordinary computability theory [1]. Arguably the next example right after this one would be iterations of inductive definitions. Admissible set theory is exactly what you need to develop a theory of positive inductive definitions, the least fixed point of such being Σ_1 definable over any admissible set containing the definition in question (e.g. its parameters) [1]. If the language of least fixed points of positive inductive definitions is closed in a straightforward manner, you end up with the μ-calculus. Determining the sets definable in the μ-calculus is however anything but a straightforward extension of admissibility, needing a generalization of the notion of reflection, gap reflection [4, 5, 3]. Something similar happens with ITTMs, as some of the extensions are quite different from the base case, as we will see.

A potential application of this work is in proof theory. The strongest fragment of second-order arithmetic for which an ordinal analysis has been done to date is Π^1_2 Comprehension [6]. Regular (i.e. non-iterated) ITTMs are already more powerful than that. Perhaps having descriptions of stronger subsystems of analysis other than the straightforward hierarchy of Π^1_n Comprehension principles will help the proof theorists make progress.

The goal of this line of inquiry is to examine what kind of iterations of ITTMs make sense, and to quantify how powerful those iterations are by characterizing the reals, or what amounts to the same thing ordinals or sets, that can be so written. This situation is different from that for regular Turing machines, because an ITTM computation can halt after infinitely many steps, and so ITTMs have the power to write reals. Hamkins and Lewis insightfully classified the reals that come up in this context as **writable** if they appear as the output of a halting computation, **eventually writable** if they are eventually the unchanging content of the output tape for a divergent computation, and **accidentally writable** if they appear anywhere on any tape during any ITTM computation, even if they are overwritten later. The same concepts apply to ordinals, where an ordinal is writable (resp. eventually, accidentally) if some real coding that order-type is writable (resp. eventually, accidentally). This distinction among these kinds of reals and ordinals turned out to be crucial to their characterization, as announced in [7] and detailed in [8], with improved proofs and other results in [9]. Let λ, ζ, and Σ respectively be the suprema of the writable, eventually writable, and accidentally writable ordinals.

Theorem 1.1. *(Welch) ζ is the least ordinal α such that L_α has a Σ_2 elementary extension, L_λ is the smallest Σ_1 substructure of L_ζ, and L_Σ is the unique Σ_2 extension of L_ζ.*

The relativization of this theorem to a real parameter holds straightforwardly.

In the next section, we give some notions of the syntax and semantics of these iterations fundamental to what follows. The three after that each gives a different kind of extension of ITTMs, and about as much as is currently know about them. Some are characterized pretty fully, others only to the point where it's clear that there's something very different going on. The final section offers a generalization of the semantics.

2 Feedback ITTMs and the Tree of Subcomputations

A **feedback ITTM** (**FITTM**) is an ITTM with two additional tapes, and an additional state, which is the oracle query "does the feedback ITTM with program the content of the first additional tape on input the content of the second converge?" Clearly, the additional tapes are merely an expository convenience, as they could be coded as dedicated parts of the original tape.

The semantics of feedback ITTMs is defined via the tree of subcomputations. The idea is that the tree keeps track of oracle calls by having each one be a child of the calling computation. This tree is in general not itself ITTM computable. Rather, it is defined within ZF, even if a fragment of ZF would suffice, inductively on the ordinals. At every ordinal stage, each extant node is labeled with some computation, and control is with one node.

At stage 0, control is with the root, which we think of as at the top of the downward growing subcomputation tree. The root is labeled with the index (and input, if any) of the main computation.

At a successor stage, if the node currently in control is in any state other than the oracle call, action is as with a regular Turing machine. If taking that action places that machine in a halting state, then, if there is a parent, the parent gets the answer "convergent" to its oracle call, and control passes to the parent. If there is no parent, then the current node is the root, and the computation halts. If the additional step does not place the machine in a halting state, then control stays with the current node. If the current node makes an oracle call, a new child is formed, after (to the right of) all of its siblings, labeled with the calling index and parameter; a new machine is established at that node, with program the given index and with the parameter written on the input tape; and control passes to that node.

At a limit stage, there are three possibilities. One is that on some final segment of the stages there were no oracle calls, and so control was always at one node. Then the rules for limit stages of regular ITTMs apply, and the snapshot of the computation at the node in question is determined (where the snapshot includes all of the current information about the computation – the state, the tape contents, and so on). If that snapshot repeats an earlier one, then that computation is divergent. (Here we are using the standard convention, first articulated in [2], that a snapshot qualifies as repeating only if it guarantees an infinite loop. In point of fact, a snapshot might be identical to an earlier one, which guarantees that it will recur ω-many times, but it is possible that at the limit of those snapshots, we escape the loop. So by convention, a repeating snapshot is taken to be one that guarantees that you're in a loop.) At that point, if there is a parent, then the parent gets the answer "divergent" to its oracle call, and control is passed to the parent. If there is no parent, then the node in question is the root, and the entire computation is divergent.

A second possibility is that cofinally many oracle calls were made, and there is a node ρ such that cofinally many of those calls were ρ's children. Note that such a node must be unique. Then ρ was active cofinally often, and again the rules for regular ITTMs at limit stages apply. If ρ is seen at that stage to be repeating, then control passes to ρ's parent, if any, which also gets the answer that ρ is divergent; if ρ is the root, then the main computation is divergent. If ρ is not seen to be repeating at this stage, then ρ retains control and the computation continues.

The final possibility is that, among the cofinally many oracle calls made, there is an infinite descending sequence, which is the right-most branch of the tree. This is bad. It is troublesome, at best, to define what to do at the next step. Various ways to avoid this last situation are the subject of the next sections.

3 Pre-Qualified Iterations

The problem cited above is that the subcomputation tree has an infinite descending sequence. The most obvious way around that is to ensure that that does not happen, that the tree is well-founded. That can be enforced by attaching an ordinal to each node of the tree and requiring that children of a node have smaller ordinals.

That is in essence what is done with the strong jump $\emptyset^{\blacktriangledown}$ of [2]. $\emptyset^{\blacktriangledown}$ is $\{(e, x) \mid e(x)\downarrow\}$, which is the same thing as labeling the root of the subcomputation tree with 1, so none of its children, the oracle calls, can themselves make oracle calls. In unpublished work, Phil Welch has show that $\zeta^{\emptyset^{\blacktriangledown}}$ is the smallest Σ_2-extendible limit of Σ_2-extendibles, and that $\lambda^{\emptyset^{\blacktriangledown}}$ and $\Sigma^{\emptyset^{\blacktriangledown}}$ are such that $L_{\lambda^{\emptyset^{\blacktriangledown}}}$ is the least Σ_1 substructure of $L_{\zeta^{\emptyset^{\blacktriangledown}}}$, which is itself a Σ_2 substructure of $L_{\Sigma^{\emptyset^{\blacktriangledown}}}$.

We would like to generalize this to ordinals as large as possible, certainly to ordinals greater than 1. An **ordinal oracle ITTM** is an FITTM with not two but three additional tapes. On the third tape is written a real coding an ordinal α. The oracle calls allowed are about other ordinal oracle ITTMs, and on the third tape must be written some ordinal $\beta < \alpha$. Since one of the other tapes is for parameter passing, it is unimportant just how the ordinals are written on the latest tape. With this restriction, the third outcome above can never happen, and all computations are well-defined (as either convergent or divergent).

An **iterated ITTM**, or **IITTM**, is an FITTM that may make an oracle call about any ordinal oracle ITTM writing on the third tape any ordinal at all. So an IITTM is like an ordinal oracle ITTM only the length of the ordinal iteration is not fixed in advance. Rather, it is limited only by what the machine figures out to write down.

Definition 3.1. *λ^{it}, ζ^{it}, and Σ^{it} are the respective suprema of the ordinals writable, eventually writable, and accidentally writable by IITTMs.*

Definition 3.2. *An ordinal α is*

- *0-extendible if it is Σ_2 extendible,*
- *$\beta+1$-extendible if it is a Σ_2 extendible limit of β-extendibles, and*
- *γ-extendible (γ a limit) if it is Σ_2 extendible and a limit of β-extendibles for each $\beta < \gamma$.*

As pointed out by the referee, the limit clause actually works perfectly well for all three clauses.

The definition above relativizes to any parameter x. The corresponding notation is for α to be $\beta[x]$-extendible. Notice that, in the limit case, when $\gamma < \alpha$, α is also the limit of ordinals which are themselves limits of β-extendibles for each $\beta < \gamma$.

Theorem 3.3. *For ordinal oracle ITTMs with ordinal α coded by the input real x_α and parameter y, the supremum ζ of the eventually writable ordinals is the least $\alpha[x_\alpha, y]$-extendible. Moreover, the supremum Σ of the accidentally writable ordinals is such that $L_\Sigma[x_\alpha, y]$ is the (unique) Σ_2 extension of $L_\zeta[x_\alpha, y]$, and the supremum λ of the writable ordinals is such that $L_\lambda[x_\alpha, y]$ is the smallest Σ_1 substructure of $L_\zeta[x_\alpha, y]$. Finally, the writable (resp. eventually, accidentally) reals are those in the corresponding segment of $L[x_\alpha, y]$.*

Proof. By induction on α.

$\alpha = 0$: This is the relativized version of Welch's theorem cited above.

$\alpha = \beta + 1$: Let γ be any ordinal less than ζ. Run some machine which eventually writes γ. Dovetail that computation with the following. Simulate running all ordinal oracle ITTMs with input β and as parameters the output of the first machine, which is eventually γ, and y. This is essentially running a universal machine: clear infinitely many cells on the scratch tape, split them up into countably many infinite sequences, and on the i^{th} sequence run a copy of the i^{th} machine. For each of those simulations, keep asking whether the current output will ever change. (That is, ask whether the computation that continues that simulation until the output tape changes, at which point it halts, is convergent.) This is a legitimate question for the oracle, as $\beta < \alpha$. Whenever you get the answer "no," indicate as much on a dedicated part of the output tape. Eventually you will get all and only the indices of the eventually stable computations. So the least $\beta[x_\alpha, y]$-extendible ordinal is less than ζ, and so ζ is the limit of such.

Because of this closure under β-extendibility, $\mathrm{L}_\zeta[x_\alpha, y]$ can run correctly the computation of the ordinal oracle ITTMs with input β. So the rest of the proof – that the computations of eventually writable reals stabilize by ζ, and that the eventually writable reals form a Σ_2 substructure of the accidentally writables and a Σ_1 extension of the writables – follows by the same arguments pioneered in [8] and improved upon in [9]. In order to keep this paper self-contained, and to verify that the new context here really makes no difference, we present these arguments here.

Suppose, toward a contradiction, that $\mathrm{L}_\zeta[x_\alpha, y]$ satisfies some Π_2 sentence ϕ, but $\mathrm{L}_\Sigma[x_\alpha, y]$ does not. By the nature of Π_2 sentences, the set of ordinals $\xi \leq \Sigma$ such that $\mathrm{L}_\xi[x_\alpha, y] \models \phi$ is closed, and so contains its maximum. By hypothesis, that maximum is strictly less than Σ. Take some machine that accidentally writes each of the ordinals less than Σ. A universal machine will do, for instance, so we will call this machine u. We also need a machine, say p, which eventually writes the ϕ's parameter. It is safe to assume that there is only one parameter, as finitely many can be combined into one set by pairing. If no parameter is necessary, then $\emptyset$ as a dummy parameter can be used. Our final machine, call it e, runs p and u simultaneously. It takes the output of u and uses it to generate the various $\mathrm{L}_\xi[x_\alpha, y]$s. When it finds such a set modeling ϕ, with parameter the current output of p, it compares ξ to the current content of the output tape. If the current content is an ordinal greater than or equal to ξ, nothing is written and the computation continues. Else ξ is written on the output. Eventually the output of p settles down. Once that happens, when the largest such ξ ever appears, it will be so written, after which point it will never be overwritten, making ξ eventually writable. This is a contradiction.

Regarding λ, suppose $\mathrm{L}_\zeta[x_\alpha, y]$ satisfies some Σ_1 formula ψ with parameters from $\mathrm{L}_\lambda[x_\alpha, y]$. Consider the computation which first computes the parameters using a halting computation, then runs a machine which eventually writes a witness to ψ and halts when it finds one. This is a halting computation for such a witness.

By the foregoing, ζ is $\alpha[x_\alpha, y]$-extendible. That it is the least such is ultimately because the assertion that any particular cell in a computation stabilizes is Σ_2. In detail, let ζ_α be the least $\alpha[x_\alpha, y]$-extendible ordinal and Σ_α its Σ_2 extension. Since stabilization is a Σ_2 assertion, any computation has the same eventually stable cells at ζ_α as at Σ_α. Moreover, if δ is a stage beyond which a certain cell is stable in ζ_α, the assertion that that cell beyond δ is stable is Π_1, so that same δ is also a stabilization point in Σ_α. So the snapshot of a computation at ζ_α is that same at Σ_α, and all looping has occurred by then.

α a limit: Since ordinal oracle ITTMs with input α subsume those with input $\beta < \alpha$, ζ is $\beta[x_\alpha, y]$-extendible for each $\beta < \alpha$, and hence, considering successor βs, a limit of $\beta[x_\alpha, y]$-extendibles. The rest follows as above. □

Theorem 3.4. *ζ^{it} is the least κ which is κ-extendible, λ^{it} its smallest Σ_1 substructure, and Σ^{it} its (unique) Σ_2 extension.*

Proof. For every $\alpha < \zeta^{it}$, the ordinal oracle ITTMs with input α are also IITTMs. Hence the least α-extendible is $\leq \zeta^{it}$, and ζ^{it} is a limit of α-extendibles. The rest, again, follows as above. □

4 Freezing Computations

Another way to deal with the possible ill-foundedness of the subcomputation tree is not to worry about it. That is, while no steps are taken to rule out such computations, there will be some with perfectly well-founded subcomputation trees, even if only by accident. We remain positive, and focus our attention on those, where we have a well-defined semantics, including whether a computation converges or diverges. So we can define the reals writable, eventually writable, and accidentally writable by FITTMs.

Proposition 4.1. *Every feedback eventually writable real is feedback writable.*

Proof. Let e be a computation which writes a feedback eventually writable real. Consider an alternative computation which runs e on a dedicated part of the tapes. Every time e's output tape changes, the main computation asks the oracle: "Consider the computation which begins at the current snapshot of e, and continues e's computation until the output tape changes once more, and then halts. Does that converge or diverge?" Since e's tree of subcomputations is well-founded, so is that of the oracle call, and the oracle call will return a definite answer. If that answer is "converge," then the construction continues; if "diverge", then the construction halts. By hypothesis, this computation eventually halts, at which point e's output is written on the output tape. □

Even worse:

Proposition 4.2. *Every feedback accidentally writable real is feedback writable.*

Proof. Suppose e is a divergent computation. As in [2], e then has to loop, and does so already at some countable stage. The sledgehammer way to see that is that there are only set-many possible snapshots, so if a computation never halts then it has to repeat itself. As to why that would happen at some countable stage, that follows from Levy absoluteness. More concretely, the argument in [2] for regular ITTMs applies unchanged in the current setting. There are only countably many cells. So only countably many stop changing beneath $\aleph_1$. Moreover, there is some countable bound α by which those have all stopped changing. List the remaining cells in an ω-sequence $c_0, c_1, \ldots$.

Let α_0 be the least stage beyond α at which c_0 changes. Inductively, let α_n be the least stage beyond α_{n-1} by which all of $c_0, c_1, ..., c_n$ have changed since stage α_{n-1}. The configuration at stage $\alpha_\omega = \lim_n \alpha_n$ repeats unboundedly beneath $\aleph_1$, and so is a looping stage.

Let α be such that e has already started to loop by α many steps. Suppose we could write (a real coding) α via a halting computation. Then any real written at any time during e's computation would be writable, via the program "write α, then compute e for the number of steps given by the integer n in the coding of α, then output whatever's on e's tapes then" (with the desired choice of n, of course). So it suffices to write the looping time of a computation.

First we determine the first looping snapshot of the machine. At every stage of the computation in a simulation of e, the oracle is asked: "Consider the computation that begins with the current snapshot of e, saves it on a dedicated part of the tape, and continues with a simulation of e on a different part of the tape, halting whenever the original snapshot is reached again; does this computation halt?" If the answer is "no," the simulation continues. Eventually the answer will be "yes." That is the first looping snapshot. (Actually, as pointed out in [2], that's not quite right. A snapshot can repeat itself, which would then force it to repeat ω-many times, but the limit could be unequal to that repeating snapshot, and so this loop could be escaped. The constructions here could be modified easily enough to avoid this problem.)

The next thing to do would be to write the ordinal number of steps it took to get to that looping snapshot, and the ordinal number of steps it would take to make one loop, and then to add them. Since those ordinals are constructed the same way, we will describe only how to do the second.

During the construction, we will assign integers to ordinals in such a way that the $<$-relation will be immediate. The construction will take ω-many stages, during each of which we will use up countably (or finitely) many integers, so beforehand assign to each $n \in \omega$ countably many integers disjointly to be available at stage n. Furthermore, each integer has its own infinite part of the tape for its scratchwork.

Let C_i ($i \in \omega$) be the (simulated) i^{th} cell of the tape on which we're running (the simulation of) e. We will need to know which cells change value cofinally in the stage of interest (the return of the looping stage) and which don't. So simulate the run of e from the looping stage until its reappearance. Every time C_i changes value, toggle the i^{th} cell on another dedicated tape from 0 to 1 to 0. At the end of the computation, the i^{th} cell on the dedicated tape will be 0 iff C_i changed value boundedly often; so it will be 1 iff C_i changed value cofinally often.

Stage 0 starts in the looping snapshot, and is itself split into ω-many steps. Those steps interleave consideration of the cells that changed boundedly often and those that change cofinally. At step $2i$ continue the computation until the i^{th} cell with bounded change stops changing. That can be determined by asking the oracle whether the cell in question changes before the looping snapshot reappears. While this is not a converges-or-diverges question on the face of it, since the computation converges in any case (either when the cell changes or when the looping snapshot is reached, whichever happens first), one of those outcomes can be changed to a trivial loop, so that the question is a standard oracle call. If the answer is "yes," then continue the computation until the answer becomes "no," which is guaranteed to happen. At that point, use an available integer to mark that ordinal stage, which integer is then larger in the ordinal ordering than all other integers used so far. Also write the current snapshot in that integer's scratchwork part of the tape. Then proceed to the next step, $2i+1$.

At step $2i+1$, we will consider not just the i^{th} cofinally changing cell, but also the j^{th} such for all $j \leq i$, for purposes of dovetailing. Sequentially for each j from 0 to i, go to the next stage at which the j^{th} cofinally changing cell changes value again. After doing so for i, use an available integer to mark that ordinal stage, which integer is then larger in the ordinal ordering than all other integers used so far. Also write the current snapshot in that integer's scratchwork part of the tape. Then proceed to step $2(i+1)$.

Because stage 0 consists of ω-many steps, each of which picks out only one integer in an increasing sequence, it picks out a strictly increasing ω-sequence of ordinals. The limit of that ordinal sequence is the ordinal in the computation at which its looping snapshot reappears. That's because by then we're beyond the ordinal at which any cell with boundedly many changes will change again, thanks to the even steps, and those cells with cofinal changing change cofinally in that ordinal, thanks to the dovetailing in the odd steps.

To summarize, we have produced an ω-sequence cofinal in the ordinal at which the looping snapshot reappears. Inductively, suppose at stage $i > 0$ we have an integer assignment, with $<$, to a subset of e's ordinal stage, as well as a picture of the snapshot of the computation each at such stage of the computation. Then for each integer which is a successor in this partial assignment, replicate the construction above with the starting snapshot being the snapshot of e at the predecessor and the ending snapshot being the snapshot of e at the integer under consideration. By the well-foundedness of the ordinals, this process ends after ω-many stages.

□

It is easy to see that the feedback writable reals are those contained in the initial segment of L given by the feedback writable ordinals, which are also the FITTM clockable ordinals. We call the set of these ordinals Λ.

This result removes the basis of the analysis used in weaker forms of ITTM computation. It comes about because the divergence of a computation in this paradigm can be determined convergently by a computation of the same type. Why doesn't this run afoul of some kind of diagonalization result? The answer is that there's no universal machine! That is, the computations and oracle calls used in the proofs above were sometimes convergent and sometimes divergent, but conveniently they were in any case all well-defined: the tree of subcomputations was well-founded. If it is not, we have no semantical notion of how the computation should continue or what the outcome should be. This notion is captured in the following.

Definition 4.3. *A computation is* **freezing** *if its tree of subcomputations is ill-founded.*

Proposition 4.4. *There is no FITTM computation which decides on an input e whether the e^{th} FITTM is freezing.*

Proof. If there were, you could diagonalize against the non-freezing computations, for a contradiction. □

We expect that as with most models of computation, the key to understanding what's computable will be an analysis of the uncomputable. While the freezing computations do not have an output or even a divergent computation, they are perfectly well-defined up until the point when an oracle call is made about a freezing subcomputation. For that matter, on the tree of subcomputations, that freezing subcomputation generates a good tree underneath it, until it calls its own freezing subcomputation. More generally, even for a freezing computation, its subcomputation tree, albeit ill-founded, is well-defined. Hence the following definition makes sense.

Definition 4.5. *A real is* **freezingly writable** *if it appears anywhere on a tape during a freezing computation or any of its subcomputations.*

We expect that the role that the eventually and accidentally writable reals played in the understanding of the writable reals for basic ITTMs will be played here by the freezingly writable reals. In any case, it should be of interest to understand better the freezing computations. Centrally, what does the subcomputation tree of a freezing computation look like? Since the computation cannot continue once an infinite path through the tree develops, that infinite path is unique, and is the right-most path. So each of the ω-many levels on the tree has width some successor ordinal. For each freezing computation e, let λ_n^e be the width of level n of e's subcomputation tree. For a fixed e, there are three possibilities for the λ_n^es:

a) λ_n^e is bounded beneath Λ.

b) λ_n^e is cofinal in Λ.

c) Some λ_n^e is greater than Λ.

Option a) is simply unavoidable: it is a simple task to write a machine which immediately makes an oracle call about itself, producing a subcomputation tree of order-type ω^* (ω backwards).

Options b) and c), as it turns out, are incompatible with each other. To see this, first note that if c) holds for some computation, then n can be chosen to be 1 (level 0 consisting of the root alone). After all, if this is not the case for some given e, let e_1 be some computation that halts at a stage larger than $\max_{m<n} \lambda_m^e$. Use e_1 to write e_1's run-time (using methods like those in the main proposition above). Use that ordinal to run e substituting for the oracle calls an explicit computation until the right-most node on level $n-1$ (of e's original subcomputation tree) becomes active. That is the node which has more than Λ-many children, and which is now the root node of the tree of this modified computation.

Now assume we have indices e_b and e_c of types b and c respectively (and $\lambda_1^{e_c} > \Lambda$). Simulate e_c. Whenever an oracle call is made, write the new length of the top level in the subcomputation tree (using techniques as above). Use that ordinal to simulate the computation of e_b substituting explicit computation for oracle calls and building explicitly the subcomputation tree. Whenever the run of e_b demands an ordinal greater than that provided by e_c yet, break off the former computation and return to the latter. By hypothesis, at a certain point you will be able see that e_b's subcomputation tree is ill-founded. Then write $\sup_{n\in\omega} \lambda_n^{e_b}$, and halt. This would then be a halting computation of Λ, contradiction.

Unfortunately, we do not know which of b) or c) is excluded. For that matter, there could be no examples of either! Possibly all freezing computations are of type a), where those bounds over all freezing es are cofinal in Λ.

5 Parallel Oracle Calls

With sequential computation, as defined above, once an ill-founded oracle call is made, the entire computation is freezing. Parallel computation provides an alternative. In its essence, this is the same as with finite computation. In that setting, what should be the semantics of "A or B"? That both converge and one is true, or that one is true regardless of whether the other even converges? Similarly here, a machine could make a parametrized oracle call. This is perhaps most easily modeled by having another tape as part of the oracle call. The called computation asks for the convergence of a computation with index given on the first tape and inputs the second and third tapes. When making a call, the third tape is blank, but in generating the answer, the oracle substitutes all possible finite strings (equivalently: all integers) on the blank tape. If any return a convergent computation, the oracle answers "yes." If none of them freeze and all return a divergent computation, the oracle answers "no." If at least one of the parallel calls freezes and all those that do not diverge, then the oracle gives no answer and the current computation freezes.

Notice that the roles of convergence and divergence could be interchanged here, as convergent and divergent computations can be interchanged with each other: given e, ask the oracle whether e converges; if yes, diverge, if no, halt. Of course, if e freezes, so does this.

Arguments similar to those above show that the parallel writable, parallel eventually writable, and parallel accidentally writable reals are all the same.

Although it seems likely, we do not have a proof that the parallel writable reals include strictly more than the feedback writables do.

6 Extending Convergence and Divergence Consistently

For both (sequential) feedback and parallel computation above, the semantics was given conservatively. That is, the convergence/divergence answers to oracle calls were forced on us. Evidence for such was an explicit computation in which some tree was well-founded, as so is absolute. Once well-foundedness is brought into the picture, induction cannot be too far behind. In fact, the process can be described via an inductive definition.

Let $\downarrow$ and $\uparrow$ be a disjoint pair of sets of computation calls, where a computation call is a pair consisting of (an index for) a program and a parameter. Given $(\downarrow, \uparrow)$, computations can be defined as convergent or divergent relative to that pair. For the sake of concreteness we will restrict attention to feedback computation; analogous considerations apply to parallel computation. When making oracle calls, the given pair $(\downarrow, \uparrow)$ is used as the oracle. This is deterministic, as $\downarrow$ and $\uparrow$ are disjoint. It is also monotonic: any computation that asks only oracle calls already in $\downarrow$ or $\uparrow$ will be unaffected by increasing either or both of those; all other computations are freezing, and so can only thaw by increasing those. As a monotonic operator, it has a least fixed point. This is the semantics given for feedback computation, that is the sense in which the semantics was conservative. This description of the matter does allow for considering other fixed points as possible semantics for these computational languages.

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