Feedback ITTMs and Σ^0_3 Determinacy

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Introduction

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(Welch) The least model L_γ of Σ^0_3-Determinacy is between the least Σ_2-Admissible and the least Σ_2-Non-Projectible ordinals.
Actually, Welch showed, from above, if

- $\gamma_0 < \gamma_1 < \gamma_2$
- $L_{\gamma_0} \prec_{\Sigma_2} L_{\gamma_1}$
- $L_{\gamma_0} \prec_{\Sigma_1} L_{\gamma_2}$ and
- L_{γ_2} is a limit of admissibles,

then $\gamma < \gamma_0$.

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γ is greater than the least hyperextendible.

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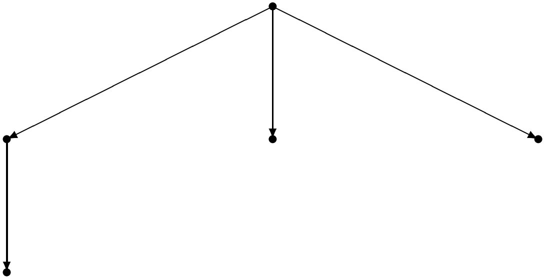
(Welch) The latest stage at which an ITTM can enter into a loop is at the least 0-extendible (i.e. the least Σ_2-extendible).
(L) The least hyperextendible can be characterized with iterated ITTMs, which are machines that are allowed certain oracle calls.

Feedback ITTMs

ITTMs with arbitrary iteration:
A computation may ask a convergence question about another computation. This can be considered calling a sub-computation. That sub-computation might do the same. This can continue, generating a *tree of sub-computations*. Eventually, perhaps, a computation is run which calls no sub-computation. This either converges or diverges. That answer is returned to its calling computation, which then continues.

Good examples



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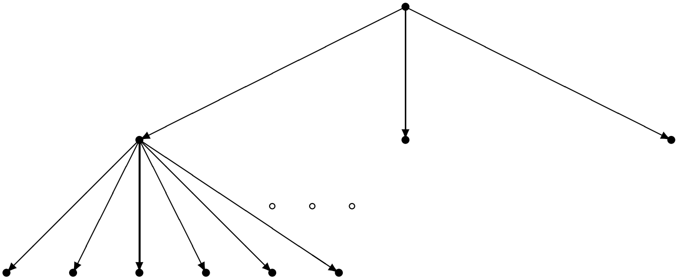

Bad example

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One can naturally define the course of a computation if and only if the tree of sub-computations is well-founded. How is this to be dealt with?

FITTMs

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Proof continued on next slide. □

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(continued)If I ever plays something false of L_α, then Welch showed how II can find an i.d.c. in that model, mod the following problem: in M, there could be ordinals such that

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$$\beta_0 < \beta_1 < \beta_2 < \ldots < \delta_2 < \delta_1 < \delta_0, \ \beta_n \text{ standard, and } L_{\beta_n} \prec_{\Sigma_2} L_{\delta_n}.$$

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Any tree of sub-computations can be adorned with ordinals in a natural way. In particular, the pair β_n, δ_n is assigned to a node which is a parent to the node of $\beta_{n+1}, \delta_{n+1}$.

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Any tree of sub-computations can be adorned with ordinals in a natural way. In particular, the pair β_n, δ_n is assigned to a node which is a parent to the node of $\beta_{n+1}, \delta_{n+1}$. Hence the β_n's would give an i.d.c. in $\{e\}$'s sub-computation tree, which was assumed to be well-founded. So that problem can't happen, giving II an opportunity to win, forcing I to play the truth.

Goals

Since we can't get the freezing computations themselves to be in L_γ, only initial segments of them, perhaps the ordinal of one of them is γ itself.

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It would also be nice to have a description of γ and of the FITTM-ordinals in terms of reflection/extendibility properties.

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