ELEMENTARY EPIMORPHISMS BETWEEN MODELS OF SET THEORY

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Abstract. We show that every Π_1-elementary epimorphism between models of ZF is an isomorphism. On the other hand, nonisomorphic Σ_1-elementary epimorphisms between models of ZF can be constructed, as can fully elementary epimorphisms between models of ZFC^-. Elementary epimorphisms were introduced by Philipp Rothmaler in [Rot05]. A surjective homomorphism $f : M \to N$ between two model-theoretic structures is an elementary epimorphism if and only if every formula with parameters satisfied by N is satisfied in M using a preimage of those parameters.

1. Introduction

Philipp Rothmaler introduced the concept of the elementary epimorphism in [Rot05], weakening the related concept of an elementary surjection studied by George Sacerdote in [Sac74] and [Sac75]. In [Rot05], Rothmaler analyzed inverse limits of systems of elementary epimorphisms between various types of model-theoretic structures. An elementary epimorphism is a sort of backwards elementary embedding, which is surjective but need not be injective. Formally, elementary epimorphisms are defined as follows.

Definition 1. Let M and N be model-theoretic structures over a common language, $\mathcal{L}$, and let Γ be a collection of formulas. A homomorphism $f : M \to N$ is an **elementary epimorphism** if and only if for every formula $\varphi \in \Gamma$ and for every tuple of elements $y_0, \ldots y_n \in N$ such that $N \models \varphi(y_0, \ldots y_n)$, there is a tuple $x_0, \ldots x_n \in M$ such that $M \models \varphi(x_0, \ldots x_n)$ and such that $f(x_i) = y_i$. If Γ is the collection of all $\mathcal{L}$-formulas, then f is a (fully) elementary epimorphism.

To verify that a function is an elementary epimorphism, one must verify two properties: the homomorphism property and the elementarity property. Note that the elementarity property implies that every elementary epimorphism is surjective: to see that $x \in N$ is in the image of an elementary epimorphism f, use the formula $x = x$.

One of us (Perlmutter) analyzed inverse limits of systems of elementary embeddings between models of set theory in his dissertation

[Per13]. Rothmaler asked me whether I had also cosidered inverse-directed systems of elementary epimorphisms between models of set theory, motivating the present paper. The main results are as follows. It is easy to construct Σ_1-elementary epimorphisms between models of ZF that are not isomorphisms (theorem 2). However, every Π_1-elementary epimorphism between models of ZF is an isomorphism (theorem 3). It is possible to construct elementary epimorphisms between weaker models of set theory, in particular between models of ZFC$^-$, that is, ZFC without power set (theorem 7). Furthermore, we provide an example of an inverse-directed system of such elementary epimorphisms with an inverse limit (section 4).

2. Elementary epimorphisms between models of ZF

Theorem 2. *Let $N \subseteq M$ be transitive models of* ZF *with the same ordinals. Let $f : M \to N$ be defined as follows. For all $x \in N$, let $f(x) = x$. Otherwise, let $f(x) = V_\alpha \cap N$, where α is the $\in$-rank of x. Then the function f is a Σ_1-elementary epimorphism.*

Proof. To show that f is a homomorphism, let $x_0, x_1 \in M$ such that $x_0 \in x_1$. Break the proof into cases depending on whether each of x_0, x_1 is an element of N. The case $x_1 \in N, x_0 \notin N$ is impossible since N is transitive. In the other three cases, it is immediate to check that $f(x_0) \in f(x_1)$. That f satisfies the main defining property of a Σ_1-elementary epimorphism follows from the fact that Σ_1 formulas are upwards absolute between transitive models of ZF. □

The following corollaries are immediate.

- There is a Σ_1-elementary epimorphism $f : V \to L$.
- If $V[G]$ is a forcing extension of V, then there is a Σ_1-elementary epimorphism $f : V[G] \to V$.

Next, we prove that every Π_1-elementary epimorphism between models of ZF is an isomorphism.

Theorem 3. *Let M and N be (not necessarily transitive) models of* ZF*. Let the function $f : M \to N$ be a Π_1-elementary epimorphism. Then f is an isomorphism.*

Proof. As a preliminary result, we will show that f maps the ordinals of M isomorphically onto the ordinals of N. Since the statement "α is an ordinal" is Δ_0^{ZF}, it follows from the elementarity of f that for each ordinal α of N, there is an ordinal α_0 of M such that $f(\alpha_0) = \alpha$. Furthermore, by the homomorphism property of f, if α_0 and β_0 are ordinals of M such that $f(\beta_0) = f(\alpha_0)$, then $\alpha_0 = \beta_0$. So it remains to be shown that for every ordinal γ of M, the set $f(\gamma)$ is an ordinal of N. If there is an M-ordinal δ such that $\gamma <_M \delta$ and such that $f(\delta)$ is an ordinal of N, then $f(\gamma) <_N f(\delta)$, so that $f(\gamma)$ is an ordinal of N. The only other possibility is that for every N-ordinal α there is an M-ordinal α_0 such that $f(\alpha_0) = \alpha$ and such that $\alpha_0 <_M \gamma$. But this would imply by the homomorphism property of f that every ordinal of N is an N-element of $f(\gamma)$, which is impossible, because then $f(\gamma)$ would be a proper class in the sense of N.

With this preliminary complete, we present the main proof. To show that f is an isomorphism, it suffices to show that f is injective. Towards this end, let $y \in N$, and suppose that $x, w \in M$ are both preimages of y, that is, $f(x) = f(w) = y$. Let β be an ordinal of M such that M believes that both w and x are subsets of V_β. Consider the set S of disjoint two-part partitions of $V_{f(\beta)}$ in N. To be precise,

$$N \models S = \{\, \{\, a, b \,\} \in V_{f(\beta)+2} \mid a \cup b = V_{f(\beta)} \;\;\&\;\; a \cap b = \emptyset \,\}.$$

The formula above defining S in N is Π_1^{ZF}. (In particular, $\{\, a, b \,\} \in V_{\beta+2}$ is Δ_0^{ZF}, and $a \cup b = V_\beta$ is Π_1^{ZF}.) Therefore, by the elementarity of f, along with the fact that f is injective on the ordinals of M, it follows that there is a set S_0 in M such that $f(S_0) = S$, and such that S_0 is the set of disjoint two-part partitions of V_β in M. That is to say,

$$M \models S_0 = \{\, \{\, a, b \,\} \in V_{\beta+2} \mid a \cup b = V_\beta \;\;\&\;\; a \cap b = \emptyset \,\}.$$

In particular, M believes that the sets $\{\, x, V_\beta - x \,\}$ and $\{\, w, V_\beta - w \,\}$ are elements of S_0. Since the function f is an $\in$-homomorphism, it follows from the structure of the set S that

$$f\Big(\{\, x, V_\beta - x \,\}^M\Big) = \{\, y, V_{f(\beta)} - y \,\}^N = f\Big(\{\, w, V_\beta - w \,\}^M\Big)$$

In particular,

$$f\Big(\big(V_\beta - x\big)^M\Big) = \big(V_{f(\beta)} - y\big)^N = f\Big(\big(V_\beta - w\big)^M\Big).$$

Suppose towards a contradiction that $x \neq w$. Without loss of generality, there is $u \in_M x$ such that $u \notin_M w$. Since f is a homomorphism, it follows that $f(u) \in_N f(x) = y$, but also that $f(u) \in_N f\big((V_\beta - w)^M\big) = \big(V_{f(\beta)} - y\big)^N$. This contradiction completes the proof. □

The following corollary follows immediately, because any isomorphism between transitive models of ZF is the identity map.

Corollary 4. *If M and N are transitive models of* ZF *and $f : M \to N$ is an elementary epimorphism, then $M = N$, and f is the identity map.*

Theorem 3 relies on the axiom of extensionality, so it is tempting to think that an elementary epimorphism could be obtained between two models of ZFA (ZF with atoms, sometimes also called ZF with ur-elements) by collapsing two atoms into one. However, this is not possible – it would run afoul of the same argument – if there are two atoms a_1, a_2, such that $f(a_1) = f(a_2)$ then consider the partition of the set of all atoms of the target model into two parts, one part consisting of just $f(a_1)$, the other consisting of all the other atoms. This partition would have to have a preimage which is a partition of all the atoms, but either both a_1 and a_2 are in the first part of the preimage partition, or neither is. So by the same argument as that of theorem 3, there is no elementary epimorphism between models of ZFA.

3. Elementary epimorphisms between models of ZFC$^-$

While there cannot be an elementary embedding between two models of ZF, it is possible to construct an elementary epimorphism between two models of ZFC$^-$, that is, ZFC without power set and with collection instead of replacement. (The paper [GHJ] explains collection should be used in the correct formulation for ZFC without power set. Using replacement instead of collection results in a theory which fails to prove several desirable statements.) This construction will make use of the following theorem of Caicedo.

Theorem 5 ([Cai06, lemma 2.1])**.** *Let $\mathbb{P} = \mathrm{Add}(\omega, \omega_1)$ denote the forcing with finite conditions that adds ω_1-many mutually generic Cohen reals. Let G_1 be $\mathbb{P}$-generic over V, and let G_2 be $\mathbb{P}$-generic over $V[G_1]$. Then there is an elementary embedding $j : L(\mathbb{R})^{V[G_1]} \to L(\mathbb{R})^{V[G_1][G_2]}$. This elementary embedding j has no critical point, and $j(r) = r$ for every real $r \in \mathbb{R}^{V[G_1]}$.*

The embedding of theorem 5 is defined by noting that $L(\mathbb{R}) = \mathrm{HOD}(\mathbb{R})$, and so every element of $L(\mathbb{R})^{V[G_1]}$ can be encoded by an ordinal and a real. This same ordinal and real encode an element of the $L(\mathbb{R})^{V[G_1][G_2]}$ as well, and so there is only one possible way to define the embedding.

Note that in theorem 5, the definition of $\mathbb{P}$ is absolute between models of ZFC with the same ω_1, so that in particular, $\mathbb{P} = \mathrm{Add}(\omega, \omega_1) = \mathrm{Add}(\omega, \omega_1)^{V[G_1]}$.

The following lemma will also be used in the elementary epimorphism construction. Note that in any model of ZFC, the hereditarily countable sets form a model of ZFC$^-$.

Lemma 6. *Let $G \subseteq \mathrm{Add}(\omega, \omega_1)$ be L-generic. Then the hereditarily countable sets (denoted HC) of $L[G]$ are the same as the hereditarily countable sets of $L(\mathbb{R})^{L[G]}$, and these sets are given by*

$$HC^{L[G]} = \bigcup_{\beta < \omega_1^L} L_{\omega_1^L}[G \restriction \beta].$$

Proof. Let $x \in L[G]$ be hereditarily countable in $L[G]$. The set x can be encoded by some real r using the standard technique of coding with sets of ordinals.[1] This encoding can be decoded in $L(\mathbb{R})^{L[G]}$, and the encoding witnesses the hereditary countability of x in $L(\mathbb{R})^{L[G]}$, so the first part of the lemma is proven.

Since the real r is determined by at most countably many forcing conditions as counted in $L[G]$, and since the forcing $\mathrm{Add}(\omega, \omega_1)$ does not collapse cardinals, it follows that there is an ordinal $\beta < \omega_1^L$ such that $r \in L[G \restriction \beta]$. Let X be a countable elementary substructure of $L[G \restriction \beta]$ such that $r \in X$. By the condensation lemma, the Mostowski collapse of X has the form $L_\alpha[G \restriction \beta]$ for some ordinal $\alpha < \omega_1^L$. The conclusion of the lemma follows. □

With these preliminaries established, we are ready to construct an elementary epimorphism between two models of ZFC^-.

Theorem 7. *Let $\mathbb{P} = \mathrm{Add}(\omega, \omega_1)$, as defined in L. Let G_1 be $\mathbb{P}$-generic over L, and let G_2 be $\mathbb{P}$-generic over $L[G_1]$. Then there is an elementary epimorphism from the hereditarily countable sets,* HC_2*, of $L[G_1][G_2]$ to the hereditarily countable sets,* HC_1*, of $L[G_1]$.*

Proof. Let $\mathbb{R}_1$ denote the reals of $L[G_1]$, and let $\mathbb{R}_2$ denote the reals of $L[G_1][G_2]$. Let $j : L(\mathbb{R}_1) \to L(\mathbb{R}_2)$ be the elementary embedding given by theorem 5.

The restriction of j to HC_1 is an elementary embedding, $j \restriction \mathrm{HC}_1 : \mathrm{HC}_1 \to \mathrm{HC}_2$. Indeed, since j fixes every real and every hereditarily countable set is encoded by a real, it follows from lemma 6 that $j \restriction \mathrm{HC}_1$ is given by the inclusion map. Define an elementary epimorphism, $f : \mathrm{HC}_2 \to \mathrm{HC}_1$, as follows. For all x in HC_1, let $f(x) = j^{-1}(x) = x$. Since HC_1 is transitive, the rest of f can be defined using $\in$-recursion. The definition takes place in $L[G_1][G_2]$. Note that the forcing $\mathrm{Add}(\omega, \omega_1)$ does not collapse any cardinals, so the models $L, L[G_1]$, and $L[G_1][G_2]$ have the same countable ordinals.

[1] This standard technique works as follows. First well-order the transitive closure of $\{ x \}$, and then use this well-ordering along with Gödel pairing to let r encode the set-membership relation, $\in_x$, on this transitive closure. To decode x, note that the transitive closure of $\{ x \}$ is equal to the Mostowski collapse of $(\omega, \in_x)$.

Suppose that $x \in \mathrm{HC}_2 - \mathrm{HC}_1$ and that $f(w)$ is defined for all $w \in x$. There is an at most countable enumeration, $\langle w_n \rangle$, of the elements of x. For each $n \in \omega$, by lemma 6, there exist ordinals $\alpha_n, \beta_n < \omega_1^L$ such that $f(w_n) \in L_{\alpha_n}[G_1 \upharpoonright \beta_n]$. It follows that there is a countable ordinal γ such that for all natural numbers n, $f(w_n) \in L_\gamma[G_1 \upharpoonright \gamma]$. Without loss of generality, we can assume that γ is a limit ordinal and that if x has the form $L_\delta[G \upharpoonright \delta]$ for some ordinal δ, then $\gamma > \delta$. (These additional assumptions are not necessary for the current proof, but it will be useful for an application in section 4.) Let $f(x) = L_\gamma[G_1 \upharpoonright \gamma]$ for the least ordinal γ meeting all the conditions above.

The function f has been fully defined. The verification that f is an elementary epimorphism is routine: the construction of f ensures that f is an $\in$-homomorphism, and the elementarity property of f follows immediately from the fact that j is an elementary embedding.

□

4. An inverse limit of elementary epimorphisms between models of ZFC$^-$

In [Rot05, section 3], Rothmaler studied inverse limits of elementary epimorphisms between modules. We extend this work by producing an example of an inverse limit of elementary epimorphisms between models of ZFC$^-$. More precisely, we mean an inverse limit in the category where the objects are models of ZFC^- and the arrows are elementary epimorphisms. This differs from Rothmaler's approach, where he spoke of the inverse limit in a category where the arrows were only required to be homomorphisms and then checked which formulas were preserved from the inverse-directed system to the inverse limit. The existence of an inverse limit in our sense is related to the preservation of formulas in Rothmaler's sense.

It should be noted that in the category of model-theoretic structures and elementary epimorphisms, it is not always the case that all formulas are preserved to the thread class. Rothmaler gives an example of such a system where formulas are not preserved in [Rot05, example 5.3]. In this example, he does not explicitly say that the epimorphisms are fully elementary, but this full elementarity follows from his comment in the first paragraph of page 476.

Our example comes from an iteration of the elementary epimorphism of theorem 7. We begin by repeating the construction of Caicedo's elementary embedding to obtain a directed system of elementary embeddings of order type ω, as follows. Beginning in L, force ω many times to add ω_1 many Cohen reals, obtaining an ω-sequence of models: $L[H_1], \quad L[H_1][H_2], \quad L[H_1][H_2][H_3], \ \ldots$

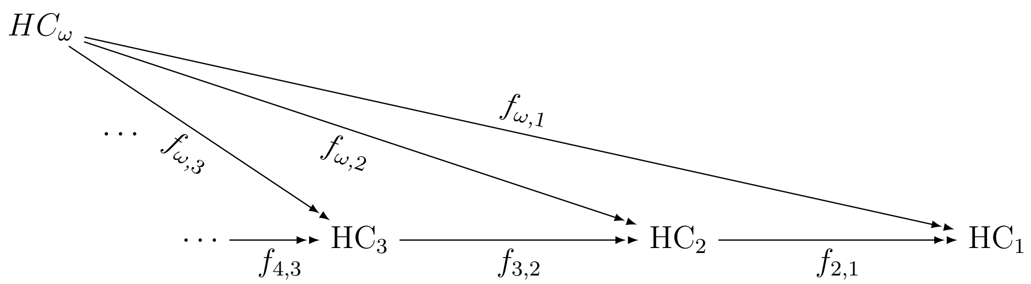
For each n, identify the model $L[H_1]\cdots[H_n]$ with the model $L[G_n]$, where G_n adds an ω_1-sequence of Cohen reals, and for each $k \in \omega$, the kth real of G_n encodes the kth reals of H_1 through H_n. This ensures that whenever $n > m$, for all countable limit ordinals γ, the set $L_\gamma[G_n \restriction \gamma]$ properly contains $L_\gamma[G_m \restriction \gamma]$.

$$L(\mathbb{R})^{L[G_1]} \xhookrightarrow{j_{1,2}} L(\mathbb{R})^{L[G_2]} \xhookrightarrow{j_{2,3}} L(\mathbb{R})^{L[G_3]} \xhookrightarrow{j_{3,4}} \cdots$$

As noted in the proof of theorem 7, each embedding $j_{n,n+1}$ fixes every hereditarily countable set. Therefore, by restricting this directed system to the hereditarily countable sets HC_n of each model $L(\mathbb{R})^{L[G_n]}$, we obtain an elementary chain. We will denote the union of this chain by HC_ω.

$$HC_1 \prec HC_2 \prec HC_3 \prec \cdots HC_\omega$$

Using the technique of theorem 7, we construct elementary epimorphisms $f_{n+1,n} : HC_{n+1} \to HC_n$, and we extend these epimorphisms to an inverse-directed system by composition. Next, for each n, we define a map $f_{\omega,n} : \mathrm{HC}_\omega \to HC_n$ as follows. Given $x \in \mathrm{HC}_\omega$, to determine $f_{\omega,n}(x)$, first let $m \geq n$ be minimal such that $x \in HC_m$. Then let $f_{\omega,n}(x) = f_{m,n}(x)$. It is easy to check that these maps f, illustrated below, commute.

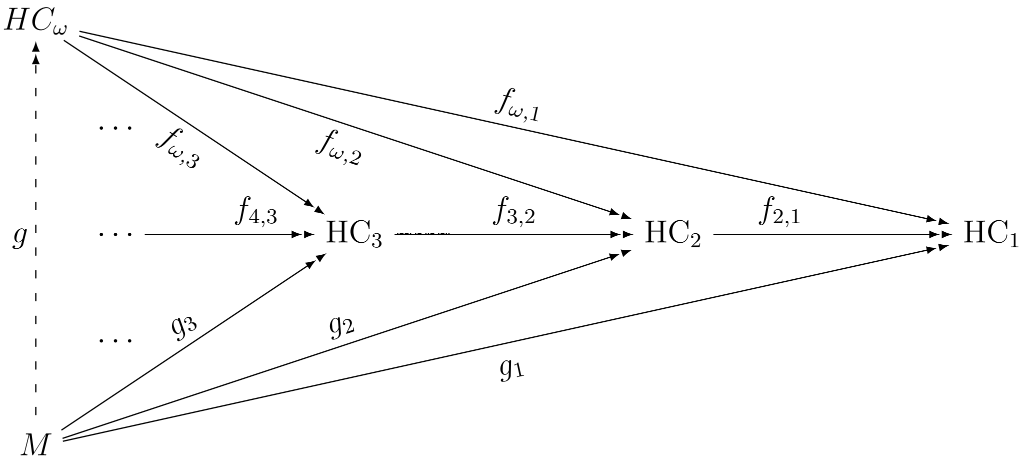


Proposition 8. *The model HC_ω along with the corresponding maps $f_{\omega,n}$ comprise the inverse limit of the inverse directed system illustrated immediately above.*

Proof. We must show that the maps $f_{\omega,n}$ are elementary epimorphisms and that the purported inverse limit satisfies the universal property.

To see that the map $f_{\omega,n}$ is an elementary epimorphism, first note that this map is surjective because $f_{\omega,n}(x) = x$ whenever $x \in \mathrm{HC}_n$. Using this fact, the elementarity property of $f_{\omega,n}$ follows immediately from the fact that $HC_n \prec HC_\omega$. Finally, that $f_{\omega,n}$ is an $\in$-homomorphism can be verified directly from its definition, since the $f_{m,n}$ are $\in$-homomorphisms.

To check the universal property, suppose that $(M, \langle g_n\rangle_{n\in\omega})$ is another natural source for the system. We must exhibit an elementary epimorphism g so that the diagram below commutes.



Towards this end, we consider the **threads** through the system, that is, the sequences of the form $\langle x_n\rangle_{n\in\omega}$ such that for each n, we have $x_n \in \mathrm{HC}_n$ and $f_{n+1,n}(x_{n+1}) = x_n$. A thread is **induced** by $x \in HC_\omega$ if it has the form $\langle f_{\omega,n}(x)\rangle$, and similarly, it is induced by $y \in M$ if it has the form $\langle g_n(y)\rangle$.[2]

We claim that every thread is $\langle x_n\rangle$ is eventually constant as $n \to \omega$. To verify this claim, first note that once $x_{n+1} = x_n$, the thread must remain constant for all larger n. Additionally, for any $m \in \omega$, the only way to have $x_{m+2} \neq x_{m+1} \neq x_m$ is if $x_{m+1} = L_\delta[G_n \restriction \delta]$ and $x_m = L_\gamma[G_n \restriction \gamma]$ where $\delta < \gamma$. Thus, if the thread were not eventually contant, there would be an infinite decreasing sequence of ordinals.

Define the map g as follows. Suppose that some $x \in M$ induces the thread with eventual constant value y. Then let $g(x) = y$. To see that g is an $\in$-homomorphism, note that given sets $a, b \in M$, there exists some n such that $g(a) = g_n(a)$ and $g(b) = g_n(b)$. Since g_n is an $\in$-homomorphism, then so is g. To check the elementarity property of g, suppose that $HC_\omega \models \varphi(a_1, \ldots a_k)$. Choose some n sufficiently large such that $a_1, \ldots a_k \in \mathrm{HC}_n$ and such that for each $i \leq n$, the set a_i does not have the form $L_\gamma[G_n \restriction \gamma]$. The map g_n is an elementary epimorphism, and $HC_n \prec HC_\omega$, so there are sets $b_1, \ldots b_n \in M$ such that $M \models \varphi(b_1, \ldots b_n)$ and such that $g_n(b_i) = a_i$. Since each a_i does not

[2]It is possible that M and HC_ω have nonempty intersection, but the meaning of *induce* will always be clear from the context.

have the form $L_\gamma[G_n \upharpoonright \gamma]$, the thread induced by b_i must have constant value a_i at all positions with index $\geq n$. Therefore, $g(b_i) = g_n(b_i) = a_i$, so the proof is complete. □

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