CHOICE AND THE HAT GAME

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1. Introduction

Consider the following game: n people are sitting in a row, in such a way that the k-th person sees the $n-k$ people in front of him. Each person wears a hat that is either black or white and whose color is only visible to the people sitting behind him. Starting with the person who sees all the others, the people guess the color of their head, one after the other.

Surprisingly, there is a strategy that guarantees that at most one of the n people guesses the color of his hat wrongly. Namely, the first person is not concerned with guessing the correct color of his hat, which he cannot know anyway, but he states the parity of the number of white hats in front of him. He says "white" if he sees an odd number of white hats and "black", otherwise. Now the second person sees whether the parity of the number of white hats in front of him is the same as for the first person, in which case he knows that his hat is black. Otherwise the hat of the second person is white. From what the k people before him have said, the $(k+1)$-th person can conclude whether or not the parity of the number of white hats that the k-th person sees is odd or even. From the number of hats that the $(k+1)$-th person sees he can then conclude the color of his own hat.

We are interested in the countably infinite version of this game. If there are countably many people sitting in a row, indexed by natural numbers, so that the k-th person sees all other persons except for himself and the $k-1$ people behind him, is there a strategy so that if the people state a color one after the other according to the strategy, all but one person guess the color of their hat correctly?

This is closely connected to various other infinite *hat problems*, whose history is discussed in [1].

2. Strategies and parity function

We deviate from the usual terminology concerning infinite games and define strategies for the situation studied here as follows: a strategy takes as its input a sequence of zeroes and ones of some length $n \in \omega$, namely the guesses of the color of the hats of the previous n players, together with an infinite sequence of zeroes

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and ones, indexed by natural numbers $> n$, namely the colors of the hats that the n-th person sees ahead of him. The strategy then responds with 0 or 1, coding the guess of color of the hat of the n-th player.

Definition 1. Let

$$T = \{(s,t) : \text{for some } n \in \omega,\ s : n \to 2 \text{ and } t : \omega \setminus (n+1) \to 2\}.$$

A *strategy* is a map $f : T \to 2$. Given a strategy f and $s \in 2^\omega$ let $\sigma_s^f \in 2^\omega$ be defined recursively as follows:

$$\sigma_s^f(n) = f(\sigma_s^f \restriction n, s \restriction \omega \setminus (n+1))$$

Given a cardinal κ, a strategy f *guarantees* $\leq \kappa$ *mistakes* if for all $s \in 2^\omega$,

$$|\{n \in \omega : \sigma_s^f(n) \neq s(n)\}| \leq \kappa.$$

Lemma 2. *a) There is no strategy f that guarantees ≤ 0 mistakes.*

b) If a strategy f guarantees ≤ 1 mistakes, then for all $s \in 2^\omega$ and all $n > 0$, $\sigma_s^f(n) = s(n)$.

Proof. For a) let f be a strategy and choose $s \in 2^\omega$. Let $t \in 2^\omega$ be such that $s(0) \neq t(0)$ and for all $n > 0$, $s(n) = t(n)$. Then $\sigma_s^f(0) = \sigma_t^f(0)$ and hence either $\sigma_s^f(0) \neq s(0)$ or $\sigma_t^f(0) \neq t(0)$. It follows that f does guarantee ≤ 0 mistakes.

For b) let f be a strategy that guarantees ≤ 1 mistakes. Let $s \in 2^\omega$ and suppose that for some $n > 0$, $\sigma_s^f(n) \neq s(n)$. Since f guarantees ≤ 1 mistakes and $n > 0$, $\sigma_s^f(0) = s(0)$. Let $t \in 2^\omega$ be such that $t(0) \neq s(0)$ and for all $m > 0$, $s(m) = t(m)$.

Now $\sigma_t^f(0) = \sigma_s^f = s(0) \neq t(0)$. By induction we see that for all $m \in \omega$, $\sigma_s^f(m) = \sigma_t^f(m)$. It follows that for $m = 0$ and for $m = n$ we have $\sigma_t^f(m) \neq t(m)$, contradicting the fact that f guarantees ≤ 1 mistakes. $\square$

Strategies that guarantee ≤ 1 mistakes have been discovered by various people.

Theorem 3. *There is a strategy that guarantees ≤ 1 mistakes.*

We present three different proofs of this theorem. All of the proofs use what we call a *parity function* and we just give different constructions of such functions.

Definition 4. A function $p : 2^\omega \to 2$ is a *parity function* if for all $s, t \in 2^\omega$ such that $|\{n \in \omega : s(n) \neq t(n)\}| = 1$, $p(s) = 1 - p(t)$.

A parity function $p : 2^\omega \to 2$ extends naturally to partial functions: if $A \subseteq \omega$ and $s : A \to 2$, let $t : \omega \to 2$ be the function that agrees with s on A and is constantly 0 at all $n \in \omega \setminus A$ and define $p(s) = p(t)$.

Let us point out that parity functions are also called 2-flutters (see [6]).

Lemma 5. *There is a strategy that guarantees ≤ 1 mistakes iff there is a parity function $p : 2^\omega \to 2$.*

Proof. Given a parity function p we define f as follows: for all $n \in \omega$, $\sigma \in 2^n$, and $s : \omega \setminus (n+1) \to 2$ let $f(\sigma, s) = p(\sigma \cup s)$. Note that for all $s \in 2^\omega$,

$$f(\emptyset, s \restriction (\omega \setminus \{0\})) = p(s \restriction (\omega \setminus \{0\})).$$

It follows that $\sigma_s^f \restriction 1 \cup s \restriction (\omega \setminus \{0\})$ is even. By induction we see that for all $n \in \omega$, $\sigma_s^f \restriction (n+1) \cup s \restriction (\omega \setminus (n+1))$ is even. But since $\sigma_s^f \restriction 1 \cup s \restriction (\omega \setminus \{0\})$ is even, this implies, again by induction on n, that for all $n > 0$, $\sigma_s^f(n) = s$. Hence f is a strategy that guarantees ≤ 1 mistakes.

Now assume that f is a strategy that guarantees ≤ 1 mistakes. Given a function $s : \omega \to 2$, let $s' : \omega \setminus \{0\} \to 2; n \mapsto s(n-1)$. Let $p(s) = f(\emptyset, s')$. We show that p is a parity function.

Let $s, t \in 2^\omega$ be such that there is a unique $n \in \omega$ with $s(n) \neq t(n)$. We have to show that $p(s) \neq p(t)$. Suppose this is not the case.

Let s'' and t'' be the functions from ω to 2 that agree with s', respectively t', on $\omega \setminus \{0\}$ and are 0 at 0. Then $f(\emptyset, s') = f(\emptyset, t')$ and therefore $\sigma_{s''}^f(0) = \sigma_{t''}^f(0)$. By Lemma 2 b), for all $m > 0$ we have $\sigma_{s''}^f(m) = s'(m)$ and $\sigma_{t''} f(m) = t'(m)$. In particular, $\sigma_{s''}^f \restriction n = \sigma_{t''}^f \restriction n$. Since s'' and t'' agree after n,

$$\begin{aligned} s''(n) = \sigma_{s''}^f(n) = f(\sigma_{s''}^f \restriction n, s'' \restriction (\omega \setminus (n+1))) \\ = f(\sigma_{t''}^f \restriction n, t'' \restriction (\omega \setminus (n+1))) = \sigma_{t''}^f(n) = t''(n), \end{aligned}$$

contradicting the choice of n. It follows that $p(s) \neq p(t)$, finishing the proof of the lemma. □

By the previous lemma, in order to find strategies that guarantee ≤ 1 mistake, it is enough to find a parity function on 2^ω.

Lemma 6. *There is a parity function $p : 2^\omega \to 2$.*

We present three proofs of this fact. The first proof uses as little of the Axiom of Choice as possible and is due to Robert Lubarsky.

Minimal proof. Suppose $p : 2^\omega \to 2$ is a parity function. If $s, t \in 2^\omega$ differ at only finitely many coordinates, then from $p(t)$ we can compute $p(s)$. Namely, $p(s) = p(t)$ if s and t differ at an even number of coordinates, and $p(s) = 1 - p(t)$ if they differ at an odd number of coordinates.

This suggests the following equivalence relation E_{parity} on 2^ω: for $s, t \in 2^\omega$ let sE_0t iff s and t differ only at finitely many coordinates. E_0 is the well-known Vitali equivalence relation. Let $sE_{\text{parity}}t$ iff sE_0t and s and t differ at an even number of coordinates. It is easily checked that E_{parity} is an equivalence relation on 2^ω. Each E_0-equivalence class is the union of two E_{parity}-equivalence classes.

Choose a function $f : 2^\omega/E_0 \to 2^\omega/E_{\text{parity}}$ that assigns to each E_0-equivalence class A one of the two E_{parity}-equivalence classes that union up to A. For each $s \in 2^\omega$ let $[s]_{E_0}$ denote the E_0-equivalence class of s. Let $p(s) = 1$ if $s \in f([s]_{E_0})$ and $p(s) = 0$ otherwise. It is easily checked that p is a parity function. $\square$

Our next proof is a variation of the first proof that (seems) to use a bit more of AC. This is essentially the proof of Lenstra's theorem presented in [1].

The E_0-transversal proof. Let A be a system of representatives for the E_0-equivalence classes. Given $s \in 2^\omega$, let t be the unique element of A with sE_0t. Let $p(s) = 0$ if s and t differ at an even number of coordinates and $p(s) = 1$, otherwise. It is easily checked that p is a parity function. $\square$

The last proof uses an ultrafilter limit to extend the obvious parity functions for finite sequences to infinite sequences. This is essentially Wagon's proof of Lenstra's theorem as presented in [1].

The ultrafilter proof. Let $\mathcal{U}$ be a non-principal ultrafilter on ω. For $A \subseteq \omega$, a function $s : A \to 2$, and $n \in \omega$ let

$$g(s, n) = |n \cap s^{-1}(1)| \mod 2.$$

For A and s as before let

$$p_{\mathcal{U}}(s) = \begin{cases} 1, & \text{if } \{n \in \omega : g(s, n) = 1\} \in \mathcal{U}, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Claim 7. Let $A \subseteq \omega$, $s : A \to 2$, and $n \in A$. Let $t : A \to 2$ be the function that agrees with s on all coordinates except for n and has $t(n) = 1 - s(n)$. Then $p_{\mathcal{U}}(t) = 1 - p_{\mathcal{U}}(s)$.

For the proof of this claim observe that the set

$$B = \{n \in \omega : g(s, n) = 1\}$$

agrees with the complement of $C = \{n \in \omega : g(t, n) = 1\}$ up to a finite set. It follows that B is in $\mathcal{U}$ if and only if C is not. This shows the claim. From the claim it follows that $p_{\mathcal{U}}$ is a parity function. $\square$

3. Use of the Axiom of Choice

The three proofs of the existence of a parity function presented in the last section all use the Axiom of Choice. We will see that this cannot be avoided. Also, we can compare the proofs by the amount of choice that each proof uses.

3.1. **Various incarnations of E_0.** Two of the three proofs of Lemma 7 directly use transversals for certain equivalence relations. The third proof uses a free ultrafilter on ω, which also happens to be a transversal for an equivalence relation, namely the relation E_{comp} that identifies a subset of ω with its complement. Choosing an E_{comp}-transversal does not require the Axiom of choice, since each E_{comp}-equivalence class has exactly two elements and we can always choose the representative that contains 0. However, the ultrafilter gives an E_{comp}-transversal that is closed under finite changes.

We will now investigate the equivalence relations that appear in our context more systematically, without using choice.

By identifying each subset of ω with its characteristic function, we can consider E_{comp} as an equivalence relation on 2^ω, just as E_0 and E_{parity}. Given two equivalence relations E and F on the same set X, by $E \vee F$ we denote the smallest equivalence relation that includes both E and F.

Let A be the set of characteristic functions of the elements of a non-principal ultrafilter $\mathcal{U}$ on ω. The set A is closed under finite modifications, i.e., it is the union of a family of E_0-equivalence classes. Now consider the relation $E_0 \vee E_{\text{comp}}$. We have $(x, y) \in E_0 \vee E_{\text{comp}}$ iff either xE_0y or $xE_0(1-y)$, where $1-y$ denotes the function $n \mapsto 1 - y(n)$.

Each $E_0 \vee E_{\text{comp}}$-equivalence class is the union of two E_0-equivalence classes. The intersection of A with an $E_0 \vee E_{\text{comp}}$-equivalence class is exactly one E_0-equivalence class. In other words, the set A/E_0 of E_0-equivalence classes contained in A is a transversal for the equivalence relation $(E_0 \vee E_{\text{comp}})/E_0$ induced by $E_0 \vee E_{\text{comp}}$ on the quotient $2^\omega/E_0$.

On the other hand, each E_0-equivalence class is the union of two E_{parity}-equivalence classes. A parity function $p : 2^\omega \to 2$ chooses one E_{parity}-equivalence class from the two that make up an E_0-equivalence class, just as in the minimal proof of Lemma 7. The similarity of $(E_0 \vee E_{\text{comp}})/E_0$ and E_0/E_{parity} is no accident.

Recall that 2^ω carries a natural metric such that a bijection $f : 2^\omega \to 2^\omega$ is an isometry if for all $x \in 2^\omega$ and all $n \in \omega$, $f(x) \restriction n$ only depends on $x \restriction n$ and vice versa.

Theorem 8. *a) The structures $(2^\omega, E_0, E_{\text{parity}})$ and $(2^\omega, E_0 \vee E_{\text{comp}}, E_0)$ are isomorphic by an isometry $g : 2^\omega \to 2^\omega$.*

b) The structures $(2^\omega, E_0)$, $(2^\omega, E_{\text{parity}})$, and $(2^\omega, E_0 \vee E_{\text{comp}})$ are pairwise isomorphic by isometries.

Proof. a) Let $+_2$ denote addition modulo 2 on $\{0, 1\}$. For $x \in 2^\omega$ and $n \in \omega$ let $g(x)(n) = x(0) +_2 \cdots +_2 x(n)$. The map g is a permutation of 2^ω as we can define its inverse as follows: for $y \in 2^\omega$ let $h(y)(0) = y(0)$ and $h(y)(n+1) = y(n) +_2 y(n+1)$.

For all $x \in 2^\omega$ and $n \in \omega$ we have

$$(h \circ g)(x)(0) = g(x)(0) = x(0) = h(x)(0) = (g \circ h)(x)(0)$$

and

$$(h \circ g)(x)(n+1) = x(0) +_2 \cdots +_2 x(n) +_2 x(0) +_2 \cdots +_2 x(n+1) = x(n+1).$$

Also,

$$\begin{aligned}(g \circ h)(x)(n+1)& \\ &= x(0) +_2 (x(0) +_2 x(1)) +_2 \cdots +_2 (x(n) +_2 x(n+1)) = x(n+1).\end{aligned}$$

It is clear from the definitions of g and h that $g(x) \restriction n$ depends only on $x \restriction n$ and vice versa. It follows that g is an isometry.

Let $x, y \in 2^\omega$ be such that $(x, y) \in E_0$. Let $n \in \omega$ be such that for all $m > n$ we have $x(m) = y(m)$. Then $g(x)(n) = g(y)(n)$ iff $(x, y) \in E_{\text{parity}}$ iff for all $m \geq n$, $g(x)(m) = g(y)(m)$. On the other hand, $g(x)(n) = 1 +_2 g(y)(n)$ iff $(x, y) \notin E_{\text{parity}}$ iff for all $m \geq n$, $g(x)(m) = 1 +_2 g(y)(m)$. Hence $g(x)(n) = g(y)(n)$ iff $(g(x), g(y)) \in E_0$ and $g(x)(n) = 1 +_2 g(y)(n)$ iff $(g(x), g(y)) \in E_{\text{comp}}$ but $(g(x), g(y)) \notin E_0$.

It remains to show that for $x, y \in 2^\omega$ with $(g(x), g(y)) \in E_0 \vee E_{\text{comp}}$ we have $(x, y) \in E_0$. There are two cases:

Either for some $n \in \omega$ and all $m \geq n$ we have $g(x)(m) = 1 +_2 g(y)(m)$ or for some $n \in \omega$ and all $m \geq n$ we have $g(x)(m) = g(y)(m)$. In the first case, for all $m \geq n$ we have

$$\begin{aligned}x(m+1) &= h(g(x))(m+1) = g(x)(m) +_2 g(x)(m+1) \\ &= 1 +_2 g(y)(m) +_2 1 +_2 g(y)(m+1) = g(y)(m) +_2 g(y)(m+1) \\ &= h(g(y))(m+1) = y(m+1).\end{aligned}$$

In the second case, for all $m \geq n$ we have

$$\begin{aligned}x(m+1) &= h(g(x))(m+1) = g(x)(m) +_2 g(x)(m+1) \\ &= g(y)(m) +_2 g(y)(m+1) = h(g(y))(m+1) = y(m+1).\end{aligned}$$

It follows that for all $m > n$ we have $x(m) = y(m)$ and hence $(x, y) \in E_0$.

b) follows from a). $\square$

Corollary 9. *a) There is an E_0-transversal iff there is an E_{parity}-transversal iff there is an $(E_0 \vee E_{\text{comp}})$-transversal.*

b) There is a parity function iff there is an E_0/E_{parity}-transversal iff there is an $(E_0 \vee E_{\text{comp}})/E_0$-transversal.

c) The existence of any of the transversals in a) implies the existence of all the transversals in b).

Proof. a) and b) immediately follow from the isomorphisms exhibited in Theorem 8. The minimal proof of the existence of a parity function shows that an E_0/E_{parity}-transversal exists iff there is a parity function. The E_0-transversal proof of the existence of a parity function shows that the existence of an E_0-transversal implies the existence of a parity function. $\square$

A $(E_0 \vee E_{\text{comp}})/E_0$-transversal is essentially a 2-chameleon in the language of [6]. In [6] it is shown that the existence of a 2-flutter is equivalent to the existence of a 2-chameleon. Since 2-flutters are just parity functions, this is just part b) of Corollary 9.

Let us consider 2^ω as a group for a moment. The addition $+$ on 2^ω is pointwise addition modulo 2. Every element of this group is of order 2 and hence $-$ is the same as $+$. It is easily checked that the bijections $g, h : 2^\omega \to 2^\omega$ in the proof of Theorem 8 are actually group homomorphisms.

Let $\mathtt{fin}$ denote the subgroup of elements of 2^ω that are equal to 1 only on finitely many coordinates. The equivalence relation E_0 is just the equivalence relation induced by this (normal) subgroup, i.e.,

$$xE_0y \quad \Leftrightarrow \quad x + y \in \mathtt{fin}.$$

Similarly, E_{parity} is the equivalence relation induced by the subgroup $h[\mathtt{fin}]$ and $E_0 \vee E_{\text{comp}}$ is the equivalence relation induced by $g[\mathtt{fin}] = \mathtt{h}^{-1}[\mathtt{fin}]$. For each $n \in \mathbb{Z}$ let $G_n = h^n[\mathtt{fin}]$ and let $E(G_n)$ be the corresponding equivalence relation. Since $E_{\text{parity}} \subseteq E_0$ and h is an isomorphism from $(2^\omega, E_0)$ to $(2^\omega, E_{\text{parity}})$, for all $n, m \in \mathbb{Z}$ with $n < m$ we have $E(G_m) \subseteq E(G_n)$ and hence $G_m \subseteq G_n$. Also, the structures $(2^\omega, E(G_n))$, $n \in \mathbb{Z}$, are all isomorphic, and this is witnessed by isometries of 2^ω.

Now observe the following: if $x \in \mathtt{fin}$ has its last 1 at coordinate $n \in \omega$, then $h(x)$ has its last 1 at coordinate $n+1$. It follows that for every $x \in h^n[\mathtt{fin}]$ that is not constantly 0 there is $m \geq n$ such that $x(m) = 1$. This shows that $\bigcap_{n\in\omega} h^n[\mathtt{fin}]$ only consists of the sequence that is constantly 0. It follows that $\bigcap_{n\in\mathbb{Z}} E(G_n)$ is the identity on 2^ω.

3.2. **Transversals, the Baire property and non-measurable sets.** We now observe that some fragment of the Axiom of Choice is necessary to prove Lemma 7. Recall the a set $A \subseteq 2^\omega$ has the Baire property if it has a meager symmetric difference with an open set. In Solovay's model every subset of 2^ω has the Baire property. But even though Solovay constructed his model using an inaccessible cardinal, Shelah proved that if ZF is consistent, then so is ZF together with the statement "every subset of 2^ω has the Baire property".

It is well known that if $\mathcal{U}$ is a non-principal ultrafilter on ω, then the set A of all characteristic functions of elements of $\mathcal{U}$ is non-measurable and does not have the property of Baire. The usual proof of this only uses the fact that the set A/E_0 is an $(E_0 \vee E_{\mathrm{comp}})/E_0$-transversal. Using the isomorphism from Theorem 8 we can transfer this non-measurability result to E_0/E_{parity}-transversals, i.e., to parity functions. We give a direct proof that from a parity function we can obtain a non-measurable set without the Baire property.

Theorem 10. *If $p : 2^\omega \to 2$ is a parity function, then the set $A = p^{-1}(1) \subseteq 2^\omega$ is non-measurable and does not have the Baire property. In particular, the existence of a parity function cannot be proved in ZF alone.*

Proof. The proofs of non-measurability and failure of the Baire property are almost the same. For $n \in \omega$ and $s \in 2^n$ let $[s] = \{x \in 2^\omega : s \subseteq x\}$. The map that flips the n-th bit of every element of $[s]$ is measure and category preverving and maps $A \cap [s]$ onto the set $[s] \setminus A$. It follows that in the measurable case, the measure of $A \cap [s]$ is exactly half the measure of $[s]$. Similarly, if A has the Baire property, then $A \cap [s]$ cannot be meager or co-meager in $[s]$.

However, in the measurable case, by the Lebesgue density theorem, there is $s \in 2^{<\omega}$ such that the measure of $A \cap [s]$ is either less that $1/4$ of the measure of $[s]$ or more than $3/4$ of the measure of $[s]$, a contradiction. In the Baire property case, there is $s \in 2^{<\omega}$ such that $A \cap [s]$ is either meager or comeager in $[s]$, also a contradiction. □

3.3. **Models where those choice principles fail.** Theorem 10 brings up the question how much choice is needed for our three proofs of Lemma 6. It follows from results of Di Prisco and Todorcevic [3] that after forcing with $\mathcal{P}(\omega)/\mathtt{fin}$ over a model $L(\mathbb{R})$ satisfying the partition relation $\omega \to (\omega)^\omega$ we obtain a model of set theory in which there is a non-principal ultrafilter on ω while there is no selection of representatives of all E_0-equivalence classes (see [5]).

Hence there are models of set theory (at least assuming the existence of certain large cardinals) in which the ultrafilter proof of Lemma 6 goes through while the E_0-transversal proof fails.

On the other hand, Paul Larson [7] showed that assuming the consistency of a proper class of Woodin cardinals implies that there is a model of ZF+"there is an E_0-transversal"+ "there is no free ultrafilter on ω".

So modulo large cardinals we see that the existence of a free ultrafilter on ω and the existence of an E_0-transversal are independent of each other (over ZF).

Question 11. Are there more elementary constructions of models of set theory witnessing that the existence of a free ultrafilter does not imply the existence of an

E_0-transversal and vice versa, in particular, constructions that do not require large cardinals?

It seem likely that there is a symmetric model construction of a model of ZF without a free ultrafilter but with an E_0-transversal. A natural construction would be to first build a model without a free ultrafilter and then add the E_0-transversal by forcing with countable approximations. Similarly, one could try to first construct a model without an E_0-transversal and then add an ultrafilter. However, in both cases it seems to be difficult to show that the last forcing extension does not add the unwanted object. Apparently, this is where the large cardinals come in.

We have already seen that E_0-transversals and free ultrafilters give parity functions. However, we do not know the answer to the following question:

Question 12. Does ZF+"there is a parity function" imply the existence of a free ultrafilter on ω or of an E_0-transversal?

In the previous section we argued that ZF does not prove the existence of a parity function since from a parity function one can construct a set of reals without the Baire property, while there are models of ZF where every set of reals has the Baire property. We finish by showing that it is possible to have no parity function, but not for the reason mentioned above.

Theorem 13. *If ZF is consistent, then so is ZF+"there is no parity function"+ "there is a set of reals without the Baire property".*

Proof. We use a standard symmetric model without a free ultrafilter on ω (see Example 15.59 in [4]). Let M be the ground model and let $\mathbb{P}$ be the forcing for adding a countable sequence of Cohen reals. That is, a condition is a function p from a finite subset of $\omega \times \omega$ to 2. For $n \in \omega$ and a $\mathbb{P}$-generic filter G over M, the n-th Cohen real added by G is the function $c_n : \omega \to 2$ defined by $c_n(m) = p(n, m)$ for some $p \in G$ with $(n, m) \in \mathrm{dom}(p)$.

Now consider $\mathcal{P}(\omega \times \omega)$ as a group with the binary operation symmetric difference defined by $x \triangle y = (x \setminus y) \cup (y \setminus x)$. Let S be the subgroup of $\mathcal{P}(\omega \times \omega)$ consisting of finite sets. With each $x \in S$ we associate an automorphism σ_x of $\mathbb{P}$ in the natural way: For each $p \in \mathbb{P}$ let

$$\sigma_x(p)(i,j) = \begin{cases} p(i,j), & x(i,j) = 0 \text{ and} \\ 1 - p(i,j), & x(i,j) = 1. \end{cases}$$

This action of S on $\mathbb{P}$ induces an action on the class of $\mathbb{P}$-names that we also denote by σ.

For each $E \subseteq \omega$ let

$$\mathrm{fix}(E) = \{x \in S : x \cap (E \times \omega) = \emptyset\}.$$

Let $\mathcal{G}$ be the filter on S generated by the sets of the form $\mathrm{fix}(E)$, $E \subseteq \omega$ a finite set. A $\mathbb{P}$-name τ is *symmetric* if the set of all $x \in S$ with $\sigma_x(\tau) = \tau$ is an element of $\mathcal{G}$. In other words, τ is symmetric if there is a finite set $E \subseteq \omega$ such that for all $x \in \mathrm{fix}(E)$ we have $\sigma_x(\tau) = \tau$. Now let G be $\mathbb{P}$-generic over M and let N be the symmetric submodel of $M[G]$ consisting of all evaluations of symmetric $\mathbb{P}$-names.

We show that there is no parity function in N. Suppose there is. Let τ be a symmetric $\mathbb{P}$-name whose evaluation τ_G is a parity function in N. Let $p \in G$ be a condition that forces "τ is a parity function in N". Since τ is symmetric, there is a finite set E such that for all $x \in \mathrm{fix}(E)$, $\sigma_x(\tau) = \tau$. After enlarging E if necessary, we may assume that for all $(i,j) \in \mathrm{dom}\, p$, $i \in E$.

Let $n \in \omega \setminus E$. Let $\dot{c}_n$ be a $\mathbb{P}$-name for the n-th Cohen real and let $q \in G$ decide $\tau(\dot{c}_n)$ to be some $i \in 2$. We may assume $q \leq p$. Choose $m \in \omega$ such that $(n,m) \notin \mathrm{dom}(q)$ Let $x = \{(n,m)\}$. Now $\sigma_x(q) = q$, $\sigma_x(\tau) = \tau$, and $\sigma_x(\dot{c}_n) = \dot{c}_n$. Since

$$q \Vdash \tau(\dot{c}_n) = \check{i}$$

and σ_x is an automorphism of $\mathbb{P}$,

$$\sigma_x(q) \Vdash \sigma_x(\tau)(\sigma_x(\dot{c}_n))) = \sigma_x(\check{i}).$$

This implies

$$q \Vdash \tau(\sigma_x(\dot{c}_n)) = \check{i}.$$

But for every $\mathbb{P}$-generic filter G over M, $\sigma_x(\dot{c}_n)_G$ and $(\dot{c}_n)_G$ differ in exactly one coordinate. Hence

$$q \Vdash \tau(\sigma_x(\dot{c}_n)) = 1 - \check{i},$$

a contradiction.

It is well known that for every nonempty interval $(a,b) \subseteq \mathbb{R}$ with $a, b \in M$, the set $(a,b) \cap M$ is non-meager in $M[G]$. It follows that the sets of the form $(a,b) \cap M$ are non-meager in N as well. Hence $\mathbb{R} \cap M$ is nowhere meager in N. Clearly, in N every translate of $\mathbb{R} \cap M$ by a real not in M is disjoint from M and also nowhere meager. It follows that in N, $\mathbb{R} \cap M$ is nowhere meager and has a nowhere meager complement. But this implies that $\mathbb{R} \cap M$ does not have the Baire property. $\square$

References

[1] Christopher S. Hardin, Alan D. Taylor, *An introduction to infinite hat problems*, Math. Intelligencer 30 (2008), no. 4, 20–25

[2] Christopher S. Hardin, Alan D. Taylor, *Minimal predictors in hat problems*, Fund. Math. 208 (2010), no. 3, 273–285

[3] C.A. Di Prisco, S. Todorcevic, *Souslin partitions of products of finite sets*, Advances in Math 176 (2003), 145–173

[4] T. Jech, **Set Theory, The Third Millenium Edition, Revised and Expanded**, Springer Verlag, 2003

[5] R. Ketchersid, P. Larson, J. Zapletal, *Borel cardinals and Ramsey ultrafilters*, in preparation

[6] A.R.D. Mathias, slides from a presentation at the 4th European Set Theory Conference in Mon St. Benet (2013)

[7] P. Larson, private communication

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