On the Failure of BD-$\mathbb{N}$

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Introduction

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A subset A of $\mathbb{N}$ is *pseudo-bounded* if every sequence (a_n) of members of A is eventually bounded by the identity function: $\exists N \, \forall n > N \; a_n < n$ (equivalently, $a_n \leq n$).

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Question: *How could it fail?*

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A basic open set p is given by an unbounded sequence g_p of integers, with a designated integer $stem(p)$, beyond which g_p is non-decreasing.
$f \in p$ if $f(n) = g_p(n)$ for $n < stem(p)$ and $f(n) \leq g_p(n)$ otherwise.

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Conjecture: In the topological model over the space of unbounded sets of naturals, the generic is pseudo-bounded and unbounded.

Proof

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The proof that rng(G) is pseudo-bounded depends crucially on the following

Lemma

Let p be an open set forcing "$t \in rng(G)$", and I an integer such that $\max_{n<stem(p)} g_p(n) \leq I \leq g_p(stem(p))$. Then there is a q extending p with the same stem and $g_q(stem(q)) \geq I$ forcing "$t \leq I$".

Proof of the Main Lemma

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Notation:

For $i \leq I$, let $p_i \subseteq p$ be such that
a) $stem(p_i) = stem(p) + 1$,
b) $g_{p_i}(stem(p)) = i$, and
c) for $n \neq stem(p)$, $g_{p_i}(n) = g_p(n)$.
Notice that $\bigcup_{i \in I} p_i = p$.

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Proof.

If each p_i had a good extension q_i, then $\bigcup_i q_i$ is a good extension of p. So if p did not have a good extension, neither would some p_i. By the same argument, neither would some extension of p_i, say p_{ij}. Similarly, neither would some extension of p_{ij}, say p_{ijk}.

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Lemma

Let p be an open set forcing "$t \in rng(G)$", and l an integer such that $\max_{n<stem(p)} g_p(n) \leq l \leq g_p(stem(p))$. *Then there is a q extending p with the same stem and $g_q(stem(q)) \geq l$ forcing "$t \leq l$".*

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Anti-Specker Spaces

Definition

A metric space X satisfies the anti-Specker property if, for every metric space $Z \supseteq X$ and sequence $(z_n)(n \in \mathbb{N})$ through Z, if (z_n) is eventually bounded away from each point in X, then (z_n) is eventually bounded away from X.

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Question (Bridges): Does the converse implication hold?

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Question (Bridges): Does the converse implication hold?
Answer: No. In the topological model, the anti-Specker spaces are closed under products.

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Realizers are integers e, viewed as computable (a.k.a. recursive) functions $\{e\}$.

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So any countable set of naturals is either realized to be bounded or realized not to be pseudo-bounded.
Still, we would need a realizer for BD-$\mathbb{N}$.

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Conclusion: There is no realizer of BD-$\mathbb{N}$.

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Sketch of proof: We need to realize "if f enumerates a subset of $rng\{e\}$ then there is a bound beyond which $f(n) \leq n$." Suppose x provably realizes the antecedent. Let $N > x$ code such a proof.

fp-realizability

Beeson's formal-provable realizability:
$e \Vdash \phi \rightarrow \psi$ iff $\forall x\, (Pr(x \Vdash \phi) \rightarrow \{e\}(x) \Vdash \psi)$.

Let $\{e\}(n) = max\{k < n \mid \forall j, w, z < k$ if j codes a proof that w is total then $\{w\}(z) \downarrow < n\ \}$.

Clearly, the range of $\{e\}$ is countable and unbounded.

Claim: The range of $\{e\}$ is pseudo-bounded.

Sketch of proof: We need to realize "if f enumerates a subset of $rng\{e\}$ then there is a bound beyond which $f(n) \leq n$." Suppose x provably realizes the antecedent. Let $N > x$ code such a proof.

Then for $n > N$

$$\begin{aligned} f(n) &= \{e\}(\{x\}(n)_i) \\ &= max\{k < \{x\}(n)_i \mid \forall j, w, z < k \text{ if } j \text{ codes a proof} \\ &\qquad \text{that } w \text{ is total then } \{w\}(z) \downarrow < \{x\}(n)_i\ \}. \end{aligned}$$

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that w is total then $\{w\}(z) \downarrow< n$ steps$\}$.
Claim: The range of $\{e\}$ is pseudo-bounded.
Sketch: Let N be a proof that $x \Vdash$ "f enumerates a subset of $rng\{e\}$."
For $n > N$
$f(n) = \{e\}(\{x\}(n)_i)$
$= max\{k < \{x\}(n)_i \mid \forall j, w, z < k$ if j codes a proof
that w is total then $\{w\}(z) \downarrow< \{x\}(n)_i\ \}$.
Consider any $k > n$. Let j, w, z be N, x, n, respectively.

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Consider any $k > n$. Let j, w, z be N, x, n, respectively. We need to consider whether $\{x\}(n) \downarrow < \{x\}(n)_i$.

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For $n > N$

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$$= max\{k < \{x\}(n)_i \mid \forall j, w, z < k \text{ if } j \text{ codes a proof that } w \text{ is total then } \{w\}(z) \downarrow< \{x\}(n)_i \}.$$

Consider any $k > n$. Let j, w, z be N, x, n, respectively. We need to consider whether $\{x\}(n) \downarrow< \{x\}(n)_i$. Since $\{x\}(n) > \{x\}(n)_i$, $\{x\}(n) \downarrow> \{x\}(n)_i$. So $f(n)$ is the max of a set which includes nothing greater than n, hence $f(n) \leq n$.

Questions

Is there an example of A pseudo-bounded and yet unbounded?
Does the topological model over the unbounded sets of naturals suggested earlier work?
Is the topological model the right, or best, or simplest, or natural, or generic model of $\neg$BD-$\mathbb{N}$? What would that mean?
What other properties implied by BD-$\mathbb{N}$ could be shown not to imply BD-$\mathbb{N}$ by holding in the model given here?

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