A Constructive View of Continuity Principles

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CCA 2012
Cambridge, UK
June 24-27, 2012

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Theorem

(Ishihara) (Countable Choice)
a) iff $\neg$WLPO (Weak Limited Principle of Omniscience)
b) iff WMP (Weak Markov's Principle)
c) iff BD (Boundedness Principle)

BD and BD-N

Definition

A subset A of $\mathbb{N}$ is *pseudo-bounded* if every sequence (a_n) of members of A is eventually bounded by the identity function: $\exists N \, \forall n > N \; a_n < n$ (equivalently, $\lim_n a_n/n = 0$).

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BD: Every inhabited pseudo-bounded set (of natural numbers) is bounded.
BD-N: Every countable pseudo-bounded set is bounded.
(Ishihara) BD-N iff every sequentially continuous function from a separable metric space to a metric space is continuous.

The Truth of BD-N

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The topological model: Put the right topology on the space of (pseudo-)bounded sequences. This is effectively taking a generic pseudo-bounded sequence, which will not be bounded.

Anti-Specker Spaces

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$p \Vdash G(n) = x$ iff $n < stem(p)$ and $g_p(n) = x$.

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Theorem

$T \Vdash rng(G)$ is countable, pseudo-bounded, but not bounded. Also, $T \Vdash DC$.

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Theorem

$T \Vdash rng(G)$ is a counter-example to the RPT.

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Theorem

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Theorem

$T \Vdash X$ and Y are A-S spaces, whereas G is a counter-example to $X \times Y$ being an A-S space.

Questions

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- How can they be reformulated to look more like BD-N?
- Are they independent of each other?
- What other non-provable statements are strictly weaker than BD-N?

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The contrapositive: "If a set of nodes is not uniform, then it's not a bar." Classically, "if a set of nodes does not cover a whole level, then there's a path avoiding it," that is, (Weak) König's Lemma.

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What do these have to do with continuity?

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Question: Are any arrows reversible? Provable outright?

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Most $K(T)$ satisfy full FAN.

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(Longley) Under mild restrictions on a pca A, the realizability model over A either satisfies full FAN or falsifies D-FAN. Furthermore, the same holds for all known extensional realizability models.

A Kripke Model of $\neg$ D-FAN

Force (in classical ZF) to get a binary tree with labels IN (the bar), OUT (of the bar), and ∞ (really out of the bar).

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Successor nodes are based on an ultrapower of V[G].
In all possible ways, change hyper-finitely many non-standard nodes by moving them from out of the generic to in the generic.

A Kripke Model of D-FAN + $\neg$ c-FAN

Hide a tree like the previous one so that it's at best c-definable.

A Kripke Model of D-FAN $+ \neg$ c-FAN

Hide a tree like the previous one so that it's at best c-definable.
At the bottom node, the decidable tree contains everything.

A Kripke Model of D-FAN + $\neg$ c-FAN

Hide a tree like the previous one so that it's at best c-definable. At the bottom node, the decidable tree contains everything. Successor nodes are based on an ultrapower of V[G], and omit from the decidable tree a non-standard point labeled ∞. So the induced c-set at the bottom node looks like the generic.

A Kripke Model of D-FAN + $\neg$ c-FAN

Hide a tree like the previous one so that it's at best c-definable. At the bottom node, the decidable tree contains everything. Successor nodes are based on an ultrapower of V[G], and omit from the decidable tree a non-standard point labeled ∞. So the induced c-set at the bottom node looks like the generic. Include only those terms definable from the decidable tree.

A Kripke Model of c-FAN + $\neg\Pi^0_1$-FAN

Hide a tree like the previous one so that it's at best Π^0_1-definable.

A Kripke Model of c-FAN + $\neg\Pi^0_1$-FAN

Hide a tree like the previous one so that it's at best Π^0_1-definable. At the bottom node, the decidable sequence of trees contains everything.

A Kripke Model of c-FAN + $\neg\Pi^0_1$-FAN

Hide a tree like the previous one so that it's at best Π^0_1-definable. At the bottom node, the decidable sequence of trees contains everything.

Successor nodes are based on an ultrapower of V[G], and omit from a tree with non-standard index a binary sequence either if it's labeled ∞ or has non-standard length. So the induced Π^0_1-set at the bottom node looks like the generic.

A Kripke Model of c-FAN + $\neg\Pi^0_1$-FAN

Hide a tree like the previous one so that it's at best Π^0_1-definable.
At the bottom node, the decidable sequence of trees contains everything.
Successor nodes are based on an ultrapower of V[G], and omit from a tree with non-standard index a binary sequence either if it's labeled ∞ or has non-standard length. So the induced Π^0_1-set at the bottom node looks like the generic.
Include only those terms definable from the decidable sequence.

A Kripke Model of Π^0_1-FAN + $\neg$ full FAN

The easiest of all, because the tree does not have to be decidable.

A Kripke Model of Π^0_1-FAN $+ \neg$ full FAN

The easiest of all, because the tree does not have to be decidable.
At the bottom node, the tree looks like the generic (the IN nodes).

A Kripke Model of Π^0_1-FAN $+ \neg$ full FAN

The easiest of all, because the tree does not have to be decidable. At the bottom node, the tree looks like the generic (the IN nodes). Successor nodes need no ultrapower. For each binary sequence labeled ∞, there is some successor node at which that binary sequence and its predecessors are the only nodes not in the tree. Include only those terms definable from this tree.

Goals

- To determine the computational content of these principles. Find computational/realizability models separating them. Perhaps there are complexity issues involved.

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- Find the canonical models, if any.
- Study the weak versions of these principles, by which the bar is concluded not to be uniform but rather to take up (at least) half of a level.

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