ESSENTIAL SELF-ADJOINTNESS OF ORNSTEIN-UHLENBECK OPERATORS PERTURBED BY CERTAIN DRIFTS AND SINGULAR POTENTIALS

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ABSTRACT: In this paper, we establish the essential self-adjointness of Ornstein-Uhlenbeck operators perturbed by certain drifts and singular $L^2$-potentials.

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1. INTRODUCTION

In this paper, we are concerned with the essential self-adjointness of Ornstein-Uhlenbeck operators under certain drift and singular potential perturbations. In the finite dimensional settings, the essential self-adjointness of Schrödinger operators with singular potentials has been studied extensively and deeply by many authors (cf. Kato [13], Reed and Simon [21], Berezanskii and Samoilenko [5] [6] and some references therein). In the infinite dimensional settings, Segal [22] and Simon [24] established some abstract results on the essential self-adjointness of the self-adjoint generators of hypercontractive semigroups with potential perturbations over measure spaces. Some similar results were also obtained for self-adjoint generators of $L^p$-contractive semigroups with potential perturbations by Berezanskii and Samoilenko [5] via a parabolic evolution criterion. By applying these abstract criteria, Berezanskii and Samoilenko [5] [6] were able to prove the essential self-adjointness of infinite-dimensional elliptic operators with separable variables and Ornstein-Uhlenbeck operators with singular potentials, on domains of definition consisting of smooth functions, via the finite dimensional approximation method. The question of essential self-adjointness for infinite dimensional operators with potentials has also been discussed in [8], [11], [14], [12] and [19]. In [8], Daletskii showed that a Schrödinger operator with respect to
a Gaussian measure is self-adjoint if the potential is twice differentiable and semi-
bounded on some admissible set. In [11], Frolov proved the essential self-adjointness
of a Schrödinger type operator in a rigged Hilbert space under the assumption that
the potential is locally bounded. This result was further generalized to Dirichlet op-
erators with respect to certain smooth measures in the sense of Fomin, by Kondratiev
and TasycaLENko [14]. Moreover, in [14], Kondratiev and TasycaLENko also proved the
essential self-adjointness of Dirichlet operators (with smooth drift coefficients) under
$L^p$-potential perturbations ($p > 2$) by using an infinite dimensional version of Kato’s
inequality. In [12], Kazumi and Shigekawa established the essential self-adjointness
for the self-adjoint generator of a strongly continuous contraction semigroup with
certain $L^p$-potential perturbations ($p > 2$) by using the abstract version of Kato’s
inequality and the theory of Sobolev spaces. In [19], Long and Simão showed the es-
sential self-adjointness of Ornstein-Uhlenbeck operators perturbed by Lipschitz drifts
and potentials having polynomial growth. For the essential self-adjointness of Dirich-
let operators without potentials in infinite dimensional settings, we refer to [3], [10],
[14], [15], [16], [17], [18] and [23].

In this paper, we shall follow the method in [5] to deal with the essential self-
adjointness of Ornstein-Uhlenbeck operators under certain drift and singular $L^2$-
potential perturbations. We need to use a lemma due to Albeverio and Hoegh-Krohn
[2] and the domain characterization for Ornstein-Uhlenbeck operators under certain
Drift perturbations due to Da Prato [10].

This paper is organized as follows. In section 2, we shall prepare some preliminary
lemmas which are used in the proof of our main result in Section 3. We first introduce
some Sobolev spaces by following Da Prato [9] and prove a priori integral estimates in
certain Sobolev spaces. By using a lemma due to Albeverio and Hoegh-Krohn [2], we
obtain some properties (see Lemma 2.10) for the conditional expectation of functions
in certain Sobolev spaces. In Section 3, by adapting the method from Berezanskiii
and Samoilenko [5] and using those preliminary results shown in Section 2, we give a
proof for our main result.

2. PRELIMINARIES ON SOBOLEV SPACES

Let $H$ be a separable Hilbert space with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$, and
let $A : D(A) \subset H \rightarrow H$ be a negative self-adjoint operator such that $A^{-1}$ is of trace
class. We denote by $\gamma$ the Gaussian measure with mean 0 and covariance operator
$-\frac{1}{2}A^{-1}$. Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis in $H$, consisting of eigenvectors of $A$
such that $Ae_k = \lambda_k e_k$, $\lambda_k < 0$. We define the space $\mathcal{F}C^k_b$ for $k \in \mathbb{N} \cup \{\infty\}$ by

$$\mathcal{F}C^k_b = \{ f : H \to \mathbb{R} | f(x) = \varphi(<e_1, x>, \ldots, <e_n, x>), n \in \mathbb{N}, \varphi \in C^k_b(\mathbb{R}^n) \}.$$ 

We use the notation $<Df(x), h> = \partial_h f(x) := \frac{d}{ds}f(x + sh)|_{s=0}$ to denote the directional derivative of $f$ along the direction $h \in H$. Sometimes for simplicity, we also use $\partial_i f$ instead of $\partial_{e_i} f$. We first recall the following integration by parts formula with respect to the Gaussian measure $\gamma$ on the Hilbert space $H$ (cf. Lemma 2.1 in Berezanskii and Samoilenko [6]).

**Lemma 2.1.** Let $f, g \in \mathcal{F}C_{\infty}^\infty$. Then for $h \in D(A)$

$$\int_H <Df(x), h> g(x)\gamma(dx) = \int_H f(x)[- <Dg(x), h> - 2 <x, Ah> g(x)]\gamma(dx). \quad (2.1)$$

From Lemma 2.1, it follows that $\partial_k$ and $D$ are closable on $L^2(H, \gamma)$ for any $k \in \mathbb{N}$. For simplicity, we shall still denote by $\partial_k$ ($k \in \mathbb{N}$) and $D$ the closure. Now we introduce some Sobolev spaces. We denote by $W^{1,p}(H, \gamma)$ ($p > 1$) the linear space of all functions $f \in L^p(H, \gamma)$ such that $\|Df\| \in L^p(H, \gamma)$. When $p = 2$, $W^{1,2}(H, \gamma)$, endowed with the inner product

$$<f, g> = \int_H f(x)g(x)\gamma(dx) + \int_H <Df(x), Dg(x)> \gamma(dx),$$

is a Hilbert space. We define the Sobolev space $W^{2,2}(H, \gamma)$ consisting of all functions $f \in W^{1,2}(H, \gamma)$ such that $\partial_j \partial_k f \in L^2(H, \gamma)$ for all $j, k \in \mathbb{N}$ and

$$\int_H \|D^2f(x)\|^2_{L_2(H)}\gamma(dx) = \sum_{j,k=1}^\infty \int_H |\partial_j \partial_k f(x)|^2 \gamma(dx) < +\infty.$$ 

Here $L_2(H)$ denotes the space of all Hilbert-Schmidt operators on $H$. Moreover, $W^{2,2}(H, \gamma)$, endowed with the inner product

$$<f, g> = <f, g> + \sum_{j,k=1}^\infty \int_H \partial_j \partial_k f(x)\partial_j \partial_k g(x)\gamma(dx)$$

$$= <f, g> + \int_H <D^2f(x), D^2g(x)>_{L_2(H)} \gamma(dx),$$

is a Hilbert space.
We define
\[ W^{1,2}_A(H, \gamma) = \{ f \in W^{1,2}(H, \gamma) : \|(-A)^{1/2}Df\| \in L^2(H, \gamma) \}, \]
which, endowed with the norm
\[ \|f\|^2_{W^{1,2}_A(H, \gamma)} = \int_H (|f(x)|^2 + \|(-A)^{1/2}Df(x)\|^2 \gamma(dx)), \]
is a Hilbert space.

We consider a new measure \( \mu(dx) = ce^{-2U(x)} \gamma(dx) \), where \( U \) is a Borel measurable function such that \( \int_H e^{-2U(x)} \gamma(dx) < \infty \) and \( c = (\int_H e^{-2U(x)} \gamma(dx))^{-1} \). We have an integration by parts formula with respect to the measure \( \mu \) (cf. Da Prato [10] or Long and Simão [18]).

**Lemma 2.2.** If \( U \) is Gâteaux differentiable and such that \( e^{-U} \in W^{1,2}(H, \gamma) \), then, for any \( f, g \in FC^\infty_b \) and \( h \in D(A) \), we have
\[ \int_H [\partial_h f(x) \cdot g(x) + f(x)\partial_h g(x)] \mu(dx) = -2 \int_H [\langle x, Ah \rangle - \partial_h U(x)f(x)g(x)] \mu(dx). \]  
(2.2)

If we adopt the notion of logarithmic derivative of a measure along an admissible vector in the sense of Fomin (cf. Bogachev and Smolyanov [7] ), then the above integration by parts formula (2.2) means that the measure \( \mu \) is differentiable along any vector \( h \in D(A) \) and has logarithmic derivative
\[ \beta_h = 2(\langle x, Ah \rangle - \langle DU(x), h \rangle). \]  
(2.3)

From the integration by parts formula (2.2), we know that \( \partial_h \) as well as the gradient operator \( D \) is closable in \( L^2(H, \mu) \). We shall denote the closure of \( \partial_h \) and the closure of \( D \) by the same symbols. We can define Sobolev spaces \( W^{1,p}(H, \mu) \) (\( p > 1 \)), \( W^{2,2}(H, \mu) \) and \( W^{1,2}_A(H, \mu) \) in the same way as \( W^{1,p}(H, \gamma) \), \( W^{2,2}(H, \gamma) \) and \( W^{1,2}_A(H, \gamma) \).

We need the following assumption on \( U \):
(U.1) \( U \) is twice Gâteaux differentiable and such that \( e^{-U} \in W^{1,2}(H, \gamma) \) and \( \sup_{x \in H} \|D^2U(x)\| \leq K < \infty \).

As in Da Prato [9], we have the following property of \( W^{1,2}(H, \mu) \).
Proposition 2.3. Let (U.1) be satisfied. If \( h \in D(A) \) and \( \varphi \in W^{1,2}(H, \mu) \), then the function
\[
x \mapsto (\langle x, Ah \rangle - \partial_h U(x)) \varphi(x)
\]
belongs to \( L^2(H, \mu) \) and we have the following inequality
\[
\int_H | \langle x, Ah \rangle - \partial_h U(x) |^2 \varphi^2(x) \mu(dx) \leq \left( \| (-A)^{\frac{1}{2}} h \|^2 + \sup_{x \in H} \| D^2 U(x) \|_{L^2(H)} \| h \|^2 \right) \int_H \varphi^2(x) \mu(dx) + \int_H |\partial_h \varphi(x)|^2 \mu(dx).
\]

Proof. By Proposition 2.4 of Da Prato [9], we know that \( |\beta_h| \in L^2(H, \mu) \). From Lemma 2.2 it follows, by a limiting procedure, that for each \( f \in W^{1,2}(H, \mu) \), \( g \in FC_b^\infty \)
\[
\int_H (\partial_h f \cdot g + f \cdot \partial_h g) d\mu = - \int_H \beta_h f \cdot g d\mu.
\]
We only need to prove (2.4) for \( \varphi \in FC_b^\infty \). We apply the integration by parts formula (2.5) to \( f(x) = - \langle x, Ah \rangle + \partial_h U(x) \) and \( g(x) = \varphi^2(x) \). Since
\[
\partial_h f(x) = - \langle Ah, h \rangle + \partial^2 U(x)h, h >
\]
and
\[
\partial_h g(x) = 2\varphi(x) \partial_h \varphi(x),
\]
by Cauchy-Schwarz inequality, we have
\[
2 \int_H | \langle x, Ah \rangle - \partial_h U(x) |^2 \varphi^2(x) \mu(dx)
= \int_H -\langle Ah + D^2 U(x)h, h \rangle \varphi^2(x) \mu(dx)
+ \int_H (\langle x, Ah \rangle + \partial_h U(x)) \cdot 2\varphi(x) \partial_h \varphi(x) \mu(dx)
\leq \left( \| (-A)^{\frac{1}{2}} h \|^2 + \sup_{x \in H} \| D^2 U(x) \|_{L^2(H)} \cdot \| h \|^2 \right) \int_H \varphi^2(x) \mu(dx)
+ \int_H | \langle x, Ah \rangle - \partial_h U(x) |^2 \varphi^2(x) \mu(dx)
+ \int_H |\partial_h \varphi(x)|^2 \mu(dx),
\]
which immediately yields (2.4).
From Lemma 2.2 and Proposition 2.3, we easily get the following proposition.

**Proposition 2.4.** Let (U.1) be satisfied. Let $f, g \in W^{1,2}(H, \mu)$ and $h \in D(A)$. Then we have

$$
\int_H \left[ \partial_h f(x) \cdot g(x) + f(x) \partial_h g(x) \right] \mu(dx) \\
= -2 \int_H [\langle x, Ah \rangle - \partial_h U(x)] f(x) g(x) \mu(dx).
$$

(2.6)

Next, we consider the following perturbed Ornstein-Uhlenbeck operator

$$Lf(x) = \frac{1}{2} \text{Tr} D^2 f(x) + \langle x, Af(x) \rangle - \langle DU(x), f \rangle, f \in \mathcal{F}_{b} \mathcal{C}^\infty,$$

where $U$ is a real-valued Gâteaux differentiable function. Then, we know that if $U$ is Gâteaux differentiable and $e^{-U} \in W^{1,2}(H, \gamma)$, then $(L, \mathcal{F}_{b} \mathcal{C}^\infty)$ is symmetric on $L^2(H, \mu)$. If we further assume that $DU$ is a Lipschitz mapping from $H$ to $H$, then $(L, \mathcal{F}_{b} \mathcal{C}^\infty)$ is essentially self-adjoint on $L^2(H, \mu)$ (cf. Long and Simão [18]). We remark that Da Prato [10] proved the essential self-adjointness of $(L, \mathcal{F}_{b} \mathcal{C}^\infty)$ under different conditions on $U$. As in Da Prato [10], we have the following energy identity formula and domain characterization for the closure of $L$.

**Lemma 2.5.** (1) Let (U.1) be satisfied. Then, we have for $f, g \in \mathcal{F}_{b} \mathcal{C}^\infty$

$$
\int_H Lf \cdot g d\mu = -\frac{1}{2} \int_H <Df, Dg> d\mu,
$$

(2.7) and

$$
2 \int_H (Lf)^2 d\mu = \frac{1}{2} \int_H \text{Tr}[(D^2 f)^2] d\mu + \int_H |(-A)^{1/2} \text{D} f|^2 d\mu \\
+ \int_H <D^2 U(x) Df, Df> d\mu.
$$

(2.8)

(2) $D(L) = W^{2,2}(H, \mu) \cap W^{1,2}_{A}(H, \mu)$ and the formula (2.8) is valid for all $f \in D(L)$.

**Proof.** (1) For the proof of (2.7), we refer to the proof of Theorem 3.4 of Long and Simão [18]. For the proof of (2.8), we refer to the proof of Proposition 3.3 (ii) in Da Prato [10].

(2) Note that $(L, \mathcal{F}_{b} \mathcal{C}^\infty)$ is essentially selfadjoint on $L^2(H, \mu)$ (see Long and Simão [18]). Therefore, for any $\phi \in D(L)$ we can choose a sequence of functions $\{\phi_n\}$ in $\mathcal{F}_{b} \mathcal{C}^\infty$ converging in $L^2(H, \mu)$ to $\phi$ and such that $\{L\phi_n\}$ is a Cauchy sequence in $L^2(H, \mu)$. We can then use formula (2.7) to show that the sequence $\{D\phi_n\}$ is Cauchy.
in $L^2(H, \mu)$. Since $D$ is a closed operator, this implies that $\phi$ is in the domain of $D$ and that $\phi_n$ converges to $\phi$ in $W^{1,2}(H, \mu)$. This proves that $D(\mathcal{L})$ is contained in $W^{1,2}(H, \mu)$. Now, applying the formula (2.8) to the difference $\phi_m - \phi_n$, it follows that

$$
0 < \frac{1}{2} \int_H \text{Tr}[(D^2(\phi_m - \phi_n))^2]d\mu + \int_H |(-A) \frac{1}{2} D(\phi_m - \phi_n)|^2 d\mu
= \frac{2}{2} \int_H (\mathcal{L}(\phi_m - \phi_n))^2 d\mu - \int_H <D^2U(x)D(\phi_m - \phi_n), D(\phi_m - \phi_n)> d\mu.
$$

Using the following facts:
(i) $|<D^2U D(\phi_m - \phi_n), D(\phi_m - \phi_n)>| \leq K \|D(\phi_m - \phi_n)\|^2$;
(ii) $\{D\phi_n\}$ is a Cauchy sequence in $L^2(H, \mu)$;
(iii) $\{\mathcal{L}\phi_n\}$ is a Cauchy sequence in $L^2(H, \mu)$,
we can conclude that $\{\phi_n\}$ is Cauchy in $W^{2,2}(H, \mu)$ and in $W^{1,2}_A(H, \mu)$ and therefore $\phi_n$ converges to $\phi$ in $W^{2,2}(H, \mu)$ and in $W^{1,2}_A(H, \mu)$. This shows that $D(\mathcal{L})$ is contained in $W^{2,2}(H, \mu) \cap W^{1,2}_A(H, \mu)$. Similarly, using the formula (2.8) and the above fact (i), we can easily show that $W^{2,2}(H, \mu) \cap W^{1,2}_A(H, \mu) \subseteq D(\mathcal{L})$. Now using approximation, it is easy to show that formulae (2.7) and (2.8) hold for functions in $D(\mathcal{L})$. This completes the proof.

We impose the following hypothesis for $U$:
(U.2) There exist constants $0 < K_0 < 1$ and $0 < K_1 < \infty$ such that $\|A^{-1}DU(x)\|^2 \leq K_1 + K_0 \|x\|^2$.

Then we have the following a priori estimate.

**Lemma 2.6.** Let (U.1) and (U.2) be satisfied. Then, $\forall \varphi \in W^{1,2}(H, \mu)$, the function $x \mapsto \|x\|\varphi(x)$ belongs to $L^2(H, \mu)$ and there exists $K_2 > 0$ such that

$$
\int_H \|x\|^2 \varphi^2(x)\mu(dx) \leq K_2 \|\varphi\|_{W^{1,2}(H, \mu)}^2. \quad (2.9)
$$

**Proof.** From Proposition 2.3, it follows that

$$
\int_H |\lambda_i x_i - \partial_i U(x)|^2 \varphi^2 \mu(dx) \leq (|\lambda_i| + K + 1) \|\varphi\|_{W^{1,2}(H, \mu)}^2, \quad (2.10)
$$

where $K$ is the constant in Hypothesis (U.1). On the other hand, we have for all $\varepsilon > 0$

$$
\int_H \lambda_i x_i^2 \varphi^2 \mu(dx) \leq \left(1 + \frac{1}{\varepsilon}\right) \int_H |\lambda_i x_i - \partial_i U(x)|^2 \varphi^2 \mu(dx)
+ (1 + \varepsilon) \int_H |\partial_i U(x)|^2 \varphi^2 \mu(dx). \quad (2.11)
$$
From (2.10) and (2.11), we get
\[
\int_H x_i^2 \varphi^2 \mu(dx) \leq \left( 1 + \frac{1}{\varepsilon} \right) \lambda_i^{-2} (|\lambda_i| + K + 1) \| \varphi \|_{W^{1,2}(H,\mu)}^2 + (1 + \varepsilon) \int_H |\lambda_i^{-1} \partial_i U(x)|^2 \varphi^2 \mu(dx).
\]

Summing up in \(i\), we get
\[
\int_H \|x\|_2^2 \varphi^2 \mu(dx) \leq \left( 1 + \frac{1}{\varepsilon} \right) C_0 \| \varphi \|_{W^{1,2}(H,\mu)}^2 + (1 + \varepsilon) \int_H \| A^{-1} DU(x) \|^2 \varphi^2 \mu(dx),
\]
where \(C_0\) is a positive constant. This together with hypothesis (U.2) gives
\[
\int_H \|x\|_2^2 \varphi^2 \mu(dx) \leq \left( 1 + \frac{1}{\varepsilon} \right) C_0 \| \varphi \|_{W^{1,2}(H,\mu)}^2 + (1 + \varepsilon) \int_H (K_1 + K_0 \|x\|^2) \varphi^2 \mu(dx).
\] (2.12)

Since \(K_0 < 1\), we can choose \(\varepsilon > 0\) such that \(K_0 (1 + \varepsilon) < 1\). Therefore, it follows from (2.12) that
\[
[1 - K_0 (1 + \varepsilon)] \int_H \|x\|_2^2 \varphi^2 \mu(dx) \leq \left( 1 + \frac{1}{\varepsilon} \right) C_0 \| \varphi \|_{W^{1,2}(H,\mu)}^2 + (1 + \varepsilon) K_1 \int_H \varphi^2 \mu(dx).
\]
which gives the result.

**Corollary 2.7.** Let the conditions on \(U\) in Lemma 2.6 be satisfied. Let \(\varphi \in W^{r,2}(H,\mu)\). Then, the function \(x \mapsto \|x\| \|D\varphi\|\) belongs to \(L^2(H,\mu)\) and there exists a constant \(K_3 > 0\) such that
\[
\int_H \|x\|_2^2 \|D\varphi\|_2^2 \mu(dx) \leq K_3 \| \varphi \|_{W^{r,2}(H,\mu)}^2.
\] (2.13)

**Proof.** This follows from Lemma 2.6. We omit the details.

Now, we consider the conditional differential calculus for functions in certain Sobolev spaces. Let \(H_m = \text{span}\{e_1, \ldots, e_m\}\), \(P_m x := \sum_{i=1}^m x_i e_i\) and \(\Pi_m(x) = \ldots\).
(x_1, \ldots, x_m) with x_i =< x, e_i > (i = 1, \ldots, m) for each x \in H. We define a projection system \( S_m : L^2(H, \mu) \to L^2(H, \mu) \) by
\[
S_m f(x) = f_m(x) = \tilde{f}_m \circ \Pi_m(x) := \mathbb{E}_\mu[f|\sigma_m](x),
\]
where \( \mathbb{E}_\mu[\cdot|\sigma_m] \) denotes the conditional expectation with respect to the measure \( \mu \) given the sub-\( \sigma \)-algebra \( \sigma_m = \sigma(<e_1, \cdot, \cdot, e_m, \cdot>) \). We denote by \( \mu^{(m)} = \mu \circ \Pi^{-1}_m \) the induced image measure of \( \mu \) under the projection \( \Pi_m \) on \( \mathbb{R}^m \).

Set \( \varphi_{j,m} = \partial_j U - \mathbb{E}_\mu[\partial_j U|\sigma_m] \). We need the following condition on \( U \):

(U.3) There exists a constant \( C > 0 \) such that
\[
\sum_{j=1}^m \varphi_{j,m}^2(x) \leq C(1 + \|x\|^2), \quad \forall m \in \mathbb{N}.
\]

**Lemma 2.8.** Assume that \( \|x\| \in L^4(H, \mu) \) and that (U.1), (U.2) and (U.3) hold. If \( f \in W^{1,2}(H, \mu) \), then \( f_m \in W^{1,2}(H, \mu) \), and we have that for any \( k = 1, \ldots, m \)
\[
\partial_k \mathbb{E}_\mu[f|\sigma_m] = \mathbb{E}_\mu[\partial_k f|\sigma_m] + \mathbb{E}_\mu[f(\beta_k - \mathbb{E}_\mu[\beta_k|\sigma_m])|\sigma_m]
= \mathbb{E}_\mu[\partial_k f|\sigma_m] - 2\mathbb{E}_\mu[f\varphi_{k,m}|\sigma_m].
\] (2.14)

**Proof.** We can use the same arguments as in Takeda [26] to prove that (2.14) is valid for \( f \in \mathcal{F}C_b^\infty \). We only need to point out that the projection operator \( L^{(m)} \), of \( L \) under the projection mapping \( \Pi_m \), given by
\[
L^{(m)} \phi(x^{(m)}) = \mathbb{E}_\mu[L(\phi \circ \Pi_m)|\sigma_m](x), x^{(m)} = \Pi_m(x), \phi \in \mathcal{C}_b^\infty(\mathbb{R}^m),
\]
is essentially self-adjoint on \( L^2(\mathbb{R}^m, \mu^{(m)}) \). This fact follows easily from Liskevich and Semenov [16] since \( \|DU\| \in L^4(H, \mu) \).

**Remark 2.9.** Obviously, if \( f \in L^\infty(H, \mu) \), then \( f_m \in L^\infty(H, \mu) \). The formula (2.14) is due to Albeverio and Hoegh-Krohn [2].

For convenience, we shall denote \( \|f\|_p \) the \( L^p \)-norm of \( f \) in \( L^p(H, \mu) \) for \( 1 \leq p \leq +\infty \).

**Lemma 2.10.** Let \( U \in W^{2,2}(H, \mu) \) and \( \|x\| \in L^4(H, \mu) \). Assume that conditions (U.1), (U.2) and (U.3) are satisfied. If \( f \in W^{2,2}(H, \mu) \cap W^{1,2}_A(H, \mu) \cap L^\infty(H, \mu) \), then \( f_m \in W^{2,2}(H, \mu) \cap W^{1,2}_A(H, \mu) \cap L^\infty(H, \mu) \) and we have for \( 1 \leq j, k \leq m \),
\[
\partial_j \partial_k f_m = \mathbb{E}_\mu[\partial_j \partial_k f|\sigma_m] - 2\mathbb{E}_\mu[g_{j,k,m}|\sigma_m].
\] (2.15)
where
\[ g_{j,k,m} = \partial_k f \cdot \varphi_{j,m} + \partial_j f \cdot \varphi_{k,m} + 2f \mathbb{E}_\mu[\partial_k U \cdot \varphi_{j,m}] \sigma_m \]
\[ -2f \cdot \varphi_{k,m} \varphi_{j,m} + f (\partial_j \partial_k U - \mathbb{E}_\mu[\partial_j \partial_k U] \sigma_m). \]  \hspace{1cm} (2.16)

**Proof.** From Lemma 2.8 and Remark 2.9, it follows that \( f_m \in W^{1,2}(H, \mu) \cap W^{1,2}_A(H, \mu) \cap L^\infty(H, \mu) \) and
\[ \partial_k f_m = \mathbb{E}_\mu[\partial_k f] \sigma_m + \mathbb{E}_\mu[f \varphi_{k,m}] \sigma_m. \]
Since \( \partial_k f \in W^{1,2}(H, \mu) \), we can apply Lemma 2.8 again to calculate \( \partial_j \mathbb{E}_\mu[\partial_k f] \sigma_m \).
On the other hand, we have \( f \varphi_{k,m} \in L^2(H, \mu) \), and by (U.2)
\[
\int_H \| D(f \varphi_{k,m}) \|^2 d\mu \leq 2 \int_H \| Df \|_2 \| \varphi_{k,m} \|^2 d\mu + 2 \| f \|^2_\infty \int_H \| D \varphi_{k,m} \|^2 d\mu
\]
\[ \leq 2C \int_H \| Df \|_2 d\mu + 2C \int_H \| x \|^2 \cdot \| Df \|^2 d\mu
\]
\[ + 4 \| f \|^2_\infty \int_H \| D(\partial_k U) \|^2 d\mu + 4 \| f \|^2_\infty \int_H \| \mathbb{E}_\mu[\partial_k U] \sigma_m \|^2 d\mu. \]
This together with Corollary 2.7 gives
\[
\int_H \| D(f \varphi_{k,m}) \|^2 d\mu \leq C_1 \left( \| f \|^2_{W^{2,2}(H, \mu)} + \| \partial_k U \|^2_{W^{2,2}(H, \mu)} + \| \mathbb{E}_\mu[\partial_k U] \sigma_m \|^2_{W^{2,2}(H, \mu)} \right).
\]
Hence, \( f \varphi_{k,m} \in W^{1,2}(H, \mu) \) and we can apply Lemma 2.8 to calculate \( \partial_j \mathbb{E}_\mu[f \varphi_{k,m}] \sigma_m \).
Thus, we easily get (2.15) and (2.16) from (2.14). By using Lemma 2.6 and Corollary 2.7, it is easy to check that \( \partial_j \partial_k f_m \) given by (2.15) and (2.16) is in \( L^2(H, \mu) \).
Since \( f_m \) only depends on the first \( m \) coordinates of \( x \), this is enough to show that \( f_m \in W^{2,2}(H, \mu) \).

### 3. Essential Self-Adjointness

In this section, we aim to establish the essential self-adjointness of \( L \) with certain singular potentials, i.e., the following operator
\[ \mathcal{F}C^\infty_b \ni f \mapsto L_V f = -Lf + Vf, \]
where \( V \) is a real-valued function in \( L^2(H, \mu) \). Here, we need to impose a restriction on the underlying measure \( \mu \) for a technical reason. We denote by \( \mathcal{M}^{(0)}(H) \) the class of quasi-invariant measures \( \nu \) on \( (H, \mathcal{B}(H)) \) with the property that their induced image measures \( \nu_n = \nu \circ \Pi_n^{-1} \) \((n \in \mathbb{N})\) admit strictly positive densities with respect
to Lebesgue measure on $\mathbb{R}^n$, of class $C(\mathbb{R}^n)$. We also impose our last condition on $U$, the so-called Hoegh-Krohn condition (see Remark 4.4 of Albeverio, Bogachev and Röckner [1]):

\[(U.4) \|(-A)^{1/2}(P_mD U - \mathbb{E}_\mu[P_mD U|\sigma_m])\| \to 0 \text{ in } L^2(H,\mu) \text{ as } m \to \infty.\]

Now we state our main result as follows.

**Theorem 3.1.** Assume that $\mu \in \mathcal{M}^{(0)}(H)$, $\|x\| \in L^4(H,\mu)$ and $U \in W^{2,2}(H,\mu)$. Assume also that (U.1), (U.2), (U.3) and (U.4) are satisfied. If $V$ is a real-valued function bounded from below in $L^2(H,\mu)$, then the operator

\[FC^\infty_b \ni f(x) \mapsto (LVf)(x) = -(Lf)(x) + V(x)f(x)\]

is essentially self-adjoint on $L^2(H,\mu)$.

We need the following abstract result on the essential self-adjointness of a non-negative self-adjoint operator with potential perturbations (cf. Theorem 1.3 in Berezanskii and Samoilenko [5] and Theorem 3.1 in Simon [25])

**Theorem 3.2.** Assume that $(E,\nu)$ is a $\sigma$-finite measure space. Let $S_0$ be a non-negative self-adjoint operator defined on $L^2(E,\nu)$ for which $D = D(S_0) \cap L^p(E,\nu)$ for fixed $p \in (2,\infty]$ is a core and $e^{-tS_0}$ is a contractive semigroup in $L^p(E,\nu)$. We suppose that the real-valued function $q \in L^r(E,\nu)$, where $r = 2p/(p - 2)$, is bounded below, i.e. $\inf_{x \in E} q(x) > -\infty$. Then the operator on $L^2(E,\nu)$ given by

\[D \ni f \mapsto (Sf)(x) = (S_0f)(x) + q(x)f(x)\]

is essentially self-adjoint.

Now we give the proof of Theorem 3.1.

**Proof of Theorem 3.1.** Under the conditions on $U$, from Long and Simão [18], we know that $(L,FC^\infty_b)$ is essentially self-adjoint on $L^2(H,\mu)$. This obviously implies that the closure of $(L,FC^\infty_b)$, denoted by $(\overline{L},D(\overline{L}))$, is a negative self-adjoint operator on $L^2(H,\mu)$. Moreover, by Lemma 2.5, we have that $D(\overline{L}) = W^{2,2}(H,\mu) \cap W^{1,2}_A(H,\mu)$. The operator $\overline{L}$ generates a strongly continuous contraction semigroup $e^{t\overline{L}}$ on $L^2(H,\mu)$. This semigroup is sub-Markovian, therefore it is a semigroup of contractions on any space $L^p(H,\mu), p \in [1,\infty]$. Set $D = D(\overline{L}) \cap L^\infty(H,\mu)$. By Theorem 3.2 with $r = 2,$
the operator

\[ D \ni f \mapsto \tilde{L}_Vf(x) = -(\tilde{L}_f)(x) + V(x)f(x) \]

is essentially self-adjoint on \( L^2(H, \mu) \). In order to prove the desired result, we only need to verify that the essential self-adjointness of \((\tilde{L}_V, D)\) implies that of \((L_V, \mathcal{F}C_b^\infty)\). This follows from

\[ D \subseteq D((\tilde{L}_V|\mathcal{F}C_b^\infty)). \tag{3.1} \]

We shall prove (3.1) by several steps.

**Step 1.** We establish the convergence of \( f_m \) to \( f \) in the graph norm of the operator \( \tilde{L} \) on \( L^2(H, \mu) \), i.e. \( \|\tilde{L}f_m - \tilde{L}f\|_2^2 \to 0 \) as \( m \to \infty \).

By using the energy identity formula (2.8) in Lemma 2.5, we have

\[ 2\|\tilde{L}f - \tilde{L}f_m\|_2^2 = \frac{1}{2} \int_H \text{Tr}[(D^2(f - f_m))^2]d\mu + \int_H \|(-A)^{\frac{1}{2}}D(f - f_m)\|^2d\mu \]
\[ + \int_H <D^2U, D(f - f_m), D(f - f_m)> d\mu. \tag{3.2} \]

We note that for any mapping \( F : H \to H \), in \( L^p(H, H; \mu) \) the sequence \( F_m = \mathbb{E}_m[F|\sigma_m] \) converges to \( F \) in \( L^p(H, \mu) \) \((p \geq 1)\) (cf. Theorem 4.1 of Chapter II in Vakhania, Tarieladze and Chobanyan [27]). Moreover, it is easy to see that \( P_mF_m \) also converges to \( F \) in \( L^p(H, \mu) \). From (2.14), it follows that

\[ Df_m = \mathbb{E}[P_mDf|\sigma_m] - 2\mathbb{E}_\mu[f(P_mDU - \mathbb{E}_\mu[P_mDU|\sigma_m])|\sigma_m]. \]

Therefore,

\[ \int_H \|(-A)^{\frac{1}{2}}D(f - f_m)\|^2d\mu \]
\[ \leq 2\int_H \|(-A)^{\frac{1}{2}}Df - \mathbb{E}_\mu[(-A)^{\frac{1}{2}}P_mDf|\sigma_m]\|^2d\mu \]
\[ + 4\int_H \|\mathbb{E}_\mu[f(-A)^{\frac{1}{2}}(P_mDU - \mathbb{E}_\mu[P_mDU|\sigma_m])|\sigma_m]\|^2d\mu \]
\[ \leq 2\int_H \|(-A)^{\frac{1}{2}}Df - \mathbb{E}_\mu[(-A)^{\frac{1}{2}}Df|\sigma_m]\|^2d\mu \]
\[ + 4\|f\|^2_2 \int_H \|(A)^{\frac{1}{2}}P_mDU - \mathbb{E}_\mu[(-A)^{\frac{1}{2}}P_mDU|\sigma_m]\|^2d\mu, \tag{3.3} \]

which tends to zero as \( m \to \infty \) by using the fact that \( f \in W^{1,2}_A(H, \mu) \), condition \((U.4)\) and Theorem 4.1 in Chapter II of Vakhania, Tarieladze and Chobanyan [27]. Therefore, the second term on the right hand side of (3.2) converges to zero as \( m \to \infty \).
From this and the boundedness of $\|D^2U\|_{\mathcal{L}(H)}$, it follows that the third term on the right hand side of (3.2) also converges to zero as $m \to \infty$. It only remains to prove that \( \int_H \text{Tr}[(D^2(f - f_m))^2]d\mu \to 0 \) as $m \to \infty$. We have

\[
\int_H \text{Tr}[(D^2(f - f_m))^2]d\mu \\
= \sum_{j,k=1}^{\infty} \int_H (\partial_j \partial_k (f - f_m))^2 d\mu \\
= \sum_{j,k=1}^{m} \int_H (\partial_j \partial_k (f - f_m))^2 d\mu + \sum_{k=m+1}^{\infty} \sum_{j=1}^{\infty} \int_H (\partial_j \partial_k f)^2 d\mu \\
+ \sum_{j=m+1}^{\infty} \sum_{k=1}^{\infty} \int_H (\partial_j \partial_k f)^2 d\mu.
\] (3.4)

Since $f \in W^{2,2}(H, \mu)$, the second and third terms on the right hand side of (3.4) converge to zero as $m \to \infty$. It remains to show that the first term on the right hand side of (3.4) converges to zero as $m \to \infty$. By (2.15) and (2.16), it follows that

\[
\sum_{j,k=1}^{m} \int_H (\partial_j \partial_k (f - f_m))^2 d\mu \\
\leq 2 \sum_{j,k=1}^{m} \int_H (\partial_j \partial_k f - \mathbb{E}_\mu[\partial_j \partial_k f|\sigma_m])^2 d\mu + 8 \sum_{j,k=1}^{m} \int_H |\mathbb{E}_\mu[g_{j,k,m}|\sigma_m]|^2 d\mu \\
\leq 2 \sum_{j,k=1}^{m} \int_H (\partial_j \partial_k f - \mathbb{E}_\mu[\partial_j \partial_k f|\sigma_m])^2 d\mu + 8 \sum_{j,k=1}^{m} \int_H |g_{j,k,m}|^2 d\mu \\
\leq 2 \sum_{j,k=1}^{m} \int_H (\partial_j \partial_k f - \mathbb{E}_\mu[\partial_j \partial_k f|\sigma_m])^2 d\mu \\
+ 40 \sum_{j,k=1}^{m} \int_H ((\partial_k f)^2 \varphi_{j,m}^2 + (\partial_j f)^2 \varphi_{k,m}^2 + 4f^2 \mathbb{E}_\mu[(\partial_k U)^2 \varphi_{j,m}^2]) \, d\mu \\
+ 160 \sum_{j,k=1}^{m} \int_H f^2 \varphi_{k,m}^2 \varphi_{j,m}^2 \, d\mu + 40 \sum_{j,k=1}^{m} \int_H f^2 (\partial_k \partial_k U - \mathbb{E}_\mu[\partial_k \partial_k U|\sigma_m])^2 d\mu \\
\leq 2 \sum_{j,k=1}^{\infty} \int_H (\partial_j \partial_k f - \mathbb{E}_\mu[\partial_j \partial_k f|\sigma_m])^2 d\mu \\
+ 40(2 + 4\|f\|_\infty^2) \int_H (\|DF\|^2 + \|DU\|^2) \|P_m DU - \mathbb{E}_\mu[P_m DU|\sigma_m]\|^2 d\mu \\
+ 160\|f\|_\infty^2 \int_H \|P_m DU - \mathbb{E}_\mu[P_m DU|\sigma_m]\|^4 d\mu \\
+ 40\|f\|_\infty^2 \sum_{j,k=1}^{\infty} \int_H (\partial_j \partial_k U - \mathbb{E}_\mu[\partial_j \partial_k U|\sigma_m])^2 d\mu.
\]
such that $f$ converges to zero as $m \to \infty$, we can then apply the dominated convergence to show that the integrals formula (2.8) in Lemma 2.5, to show that

$$
\forall m \in \mathbb{N}, \| P_m DU - \mathbb{E}_\mu[P_m DU | \sigma_m]\|^2 \leq C(1 + \|x\|^2).
$$

Therefore, the integrands are bounded above by $C(Df)^2 + \|DU\|^2(1 + \|x\|^2)$ and $C^2(1 + \|x\|^2)^2$ respectively, which are integrable functions by Corollary 2.7 and our assumption. We can then apply the dominated convergence to show that the integrals converge to zero as $m \to \infty$.

**Step 2.** For fixed $m$, we are going to construct a sequence of functions $\{f_{m,R}\}$ such that $f_{m,R} \in D(\overline{L})$, $f_{m,R} \to f_m$, $\overline{L}f_{m,R} \to \overline{L}f_m$ in $L^2(H, \mu)$ and

(i) $\forall R$, $f_{m,R}(x)$ depends only on $x_1, \ldots, x_m$;
(ii) $f_{m,R}(x) = 0$ if $x \notin \{x : x_i \in [-(R + 1), R + 1], i = 1, \ldots, m\}$;
(iii) $\|f_{m,R}(x)\| \leq \|f\|_\infty$.

**Proof of Step 2:** Let $\theta_1 : \mathbb{R} \to [0, 1]$ be a function in $C^\infty(\mathbb{R})$ such that $\theta_1(t) = 1$, $t \in [-1, 1]$ and $\theta_1(t) = 0$, $t \in (-\infty, -2] \cup [2, +\infty)$. For all $R \in \mathbb{N}$ ($R > 1$), define $\theta_R : \mathbb{R} \to [0, 1]$ by $\theta_R(t) = 1$, $t \in [-R, R]$, $\theta_R(t) = 0$, $t \in (-\infty, -(R + 1)] \cup [R + 1, +\infty)$, $\theta_R(t) = \theta_1(t + R)$, $t \in (-R - 1, -R)$ and $\theta_R(t) = \theta_1(t - R)$, $t \in (R, R + 1)$. Define $f_{m,R}(x) = \theta_R(x_1) \cdots \theta_R(x_m) f_m(x)$. It is easy to check that $|f_{m,R}(x)| \leq \|f\|_\infty$, for all $R \in \mathbb{N}$, $f_{m,R} \to f_m$ and $\overline{L}f_{m,R} \to \overline{L}f_m$ in $L^2(H, \mu)$ as $R \to \infty$. Note that both $f_{m,R}(x)$ and $f_m(x)$ depend only on the first $m$ coordinates of $x$, so by using the energy identity formula (2.8) in Lemma 2.5, to show that $\overline{L}f_{m,R} \to \overline{L}f_m$, it is enough to show that $\forall j, k = 1, \ldots, m$, $\partial_j(f_{m,R} - f_m) \to 0$ and $\partial_k(f_{m,R} - f_m) \to 0$ in $L^2(H, \mu)$. This follows easily by using the chain rule, and the fact that $f_m \in W^{2,2}(H, \mu)$.

**Step 3:** Fix $m, R \in \mathbb{N}$. We are going to construct a family of functions $\{f_{m,R,\varepsilon}\}$ such that

(i) $f_{m,R,\varepsilon} \in \mathcal{F}C^\infty_c$, $\forall \varepsilon > 0$;
(ii) $f_{m,R,\varepsilon} \to f_{m,R}$ and $\overline{L}f_{m,R,\varepsilon} \to \overline{L}f_{m,R}$ in $L^2(H, \mu)$ as $\varepsilon \to 0$;
(iii) $|f_{m,R,\varepsilon}(x)| \leq \|f\|_\infty$, $\forall \varepsilon > 0$. 

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Proof of Step 3: Let $\tilde{f}_{m,R}(y) = \theta_R(y_1) \cdots \theta_R(y_m)\tilde{f}_m(y)$, $y \in \mathbb{R}^m$, where $\tilde{f}_m \circ \Pi_m = E_\mu[f|\sigma_m] = f_m$. Define
\[
\tilde{f}_{m,R,\varepsilon}(y) = \int_{\mathbb{R}^m} \rho_\varepsilon(|y - x|)\tilde{f}_{m,R}(x)dx, \ y \in \mathbb{R}^m,
\]
where
\[
\rho_\varepsilon(|x - y|) = \varepsilon^{-m}\rho\left(\frac{|x - y|}{\varepsilon}\right)
\]
and $\rho$ is a non-negative function in $C^\infty(\mathbb{R}^m)$ vanishing outside the unit ball and satisfying $\int_{\mathbb{R}^m} \rho(x)dx = 1$. We have $\tilde{f}_{m,R,\varepsilon} \in C_0^\infty(\mathbb{R}^m)$. If we show that $\tilde{f}_{m,R,\varepsilon} \in W^{2,2}(\mathbb{R}^m)$ (the Sobolev space with respect to Lebesgue measure), then we can conclude that $\tilde{f}_{m,R,\varepsilon} \to \tilde{f}_{m,R}$ in $W^{2,2}(\mathbb{R}^m)$. To prove that $\tilde{f}_{m,R} \in W^{2,2}(\mathbb{R}^m)$, we use the fact that for $\varphi : \mathbb{R}^m \to \mathbb{R}_+$ with compact support $G$, to get
\[
\int_{\mathbb{R}^m} \varphi(y)dy = \int_{\mathbb{R}^m} \varphi(y)\frac{1}{p^{(m)}(y)}d\mu^{(m)}(y)
\leq \sup_{y \in G} \left(\frac{1}{p^{(m)}(y)}\right)\int_{\mathbb{R}^m} \varphi(y)d\mu^{(m)}(y)
= \sup_{y \in G} \left(\frac{1}{p^{(m)}(y)}\right)\int_{\mathbb{R}^m} \varphi(y)d\mu^{(m)}(y),
\]
where $p^{(m)}$ denotes the density of $\mu^{(m)}$ with respect to the Lebesgue measure. Here, we have used the fact that $p^{(m)}$ is continuous and strictly positive. By taking
\[
\varphi(y) = |\tilde{f}_{m,R}(y)| + \|D\tilde{f}_{m,R}(y)\|^2 + \sum_{j,k=1}^m |\partial_j \partial_k \tilde{f}_{m,R}(y)|^2,
\]
we see that $f_{m,R} \in W^{2,2}(H, \mu)$ implies that $\tilde{f}_{m,R} \in W^{2,2}(\mathbb{R}^m)$. Therefore, we can conclude that $\tilde{f}_{m,R,\varepsilon} \to \tilde{f}_{m,R}$ in $W^{2,2}(\mathbb{R}^m)$. Theorem.

Step 4: From steps 1-3, it follows that there exists a sequence of functions $\{\psi_n\}$ such that $\forall n \in \mathbb{N}$, $\psi_n \in FC^\infty$, $|\psi_n(x)| \leq \|f\|_\infty$, and both $\psi_n \to f$ and $\tilde{L}\psi_n \to \tilde{L}f$ in $L^2(H, \mu)$ as $n \to \infty$. Since the functions $\psi_n$ are uniformly bounded and $V \in L^2(H, \mu)$, we have $V\psi_n \to Vf$ in $L^2(H, \mu)$, by Lebesgue dominated convergence theorem. Therefore, $\tilde{L}V\psi_n \to \tilde{L}Vf$ in $L^2(H, \mu)$ as $n \to \infty$. This completes the proof.
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