PARAMETER ESTIMATION FOR ORNSTEIN-UHLENBECK PROCESSES DRIVEN BY $\alpha$-STABLE LÉVY MOTIONS

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Abstract. The parameter estimation theory for stochastic differential equations driven by Brownian motions or general Lévy processes with finite second moments has been well developed. In this paper, we consider the parameter estimation problem for Ornstein-Uhlenbeck processes driven by $\alpha$-stable Lévy motions. The classical maximum likelihood method does not apply in this context because the likelihood ratio does not exist. We shall use the trajectory fitting method combined with the weighted least squares technique. We discuss the consistency and the asymptotic distributions of our estimators for general weights in both the ergodic and the non-ergodic cases. Some simulation results are also provided.

1. Introduction and Notation

We consider the Ornstein-Uhlenbeck processes $X = \{X_t, t \geq 0\}$ determined by the following linear stochastic differential equation

$$
\begin{align*}
\begin{cases}
\, dX_t = -\theta X_t dt + mt + \sigma Z_t, & t \geq 0 \\
\, X_0 = x
\end{cases}
\end{align*}
$$

(1.1)

where $\theta$, $m$ and $\sigma$ are given constants and $Z_t$ is a given standard $\alpha$-stable Lévy motion. The basic probability space is $(\Omega, \mathcal{F}, P)$ equipped with a right continuous and increasing family $\{\mathcal{F}_t, t \geq 0\}$ of $\sigma$-algebras. The expectation on this probability space is denoted by $E$. For some technical reason we also assume that $1 < \alpha < 2$.

Suppose we don’t know the parameter $\theta$ ($m$ and $\sigma$ known). We have observation of the process $X = \{X_t, 0 \leq t \leq T\}$ up to the time instant $T$. We are interested in estimating the unknown parameter $\theta$. When $Z_t$ is replaced by a standard Brownian motion, the parameter estimation for $\theta$ has been extensively studied by using classical maximum likelihood method or by using the least squares technique (see Liptser and Shiryaev [12]). However, the naive classical maximum likelihood estimator (MLE) is no longer valid in our setting because the explicit density functions are not available and the Girsanov measure transformation is not well defined for the $\alpha$-stable Lévy motions. In the next section we shall sketch how to...
use the least squares technique to study the estimator of $\theta$ when $Z_t$ is a Brownian motion and explain why the naive least squares technique can no longer apply in our case. To find a consistent estimator $\hat{\theta}$ of $\theta$ for (1.1) we shall use the trajectory fitting method combined with the weighted least squares technique. The trajectory fitting method was first proposed by Kutoyants [10] as a numerically attractive alternative to the well-developed maximum-likelihood estimators for continuous diffusion processes (see Dietz and Kutoyants [3], [4], Dietz [2], and Kutoyants [11]).

To obtain our estimator we introduce

$$A_t = \int_0^t X_s ds, \ t > 0.$$  

The equation (1.1) can be written as

$$X_t = x - \theta A_t + mt + \sigma Z_t.$$  

Let $w_t$ be a deterministic positive (weight) function. Multiply the above equation by $w_t$ we have

$$w_t X_t = w_t x - \theta w_t A_t + mt w_t + \sigma w_t Z_t.$$  

The weighted trajectory fitting estimate (TFE) of $\theta$ from the observation of $(X_t, 0 \leq t \leq T)$ is to minimize

$$\int_T^T |w_t X_t - (w_t x - \theta w_t A_t + mt w_t)|^2 dt.$$  

It is easy to see that the minimum is attained when $\theta$ is given by

$$\hat{\theta}_T = -\frac{\int_0^T w_t^2 (X_t - x - mt) A_t dt}{\int_0^T w_t^2 A_t^2 dt}. \quad (1.2)$$  

First we shall prove the strong consistency of $\hat{\theta}_T$:

$$\lim_{T \to \infty} \hat{\theta}_T = \theta \quad P - a.s.$$  

[In the rest of the paper, when we don’t specify the convergence we always mean the almost sure convergence.] Once we have established the above result we can study the asymptotic distributions of $\hat{\theta}_T$. Namely, we will find a functional $\kappa(w, T)$ such that

$$\kappa(w, T) \left( \hat{\theta}_T - \theta \right)$$  

converges in distribution to a $\alpha$-stable random variable $\Xi$ (independent of $T$ and $w$). This means that

$$\hat{\theta}_T - \theta \approx \frac{1}{\kappa(w, T)} \Xi \quad \text{as} \ T \to \infty.$$  

If $\frac{1}{\kappa(w, T)}$ is of the order of $T^{-\delta}$ for some positive $\delta$, namely, $\frac{T^\delta}{\kappa(w, T)}$ converges to a functional $F(w)$ of $w$, then we have

$$\hat{\theta}_T - \theta \approx F(w) \frac{1}{T^\delta} \Xi \quad \text{as} \ T \to \infty.$$
\( F(w) \) is called the leading coefficient. We prefer to have a smaller value of \( F(w) \) for a given class of weight functions.

In the following we demonstrate how to view the stable process from general context of Lévy process. Namely, we consider

\[
\begin{cases}
    dX_t = -\theta X_t dt + dL_t, & t \geq 0 \\
    X_0 = x
\end{cases}
\]

where \( \theta \) is an unknown parameter, \( \{L_t, t \geq 0\} \) is a one-dimensional Lévy process. Lévy processes are closely related to stable distributions. A random variable \( Z \) use the notation \( Z \) for the random variable.

\( \text{Itô-Lévy decomposition, we have} \)

\[
\begin{align*}
    N &= \sum_{s \leq t} 1_A(\Delta L_s) \\
    \Phi(u) &= \exp \left\{ i u X_0 + \frac{1}{2} \int_0^t \frac{1}{\alpha} (1 - i \beta \text{sgn}(u) \tan \frac{\alpha \pi}{2}) + |u| \right\} \\
    \phi(u) &= \exp \left\{ i u X_0 + \frac{1}{2} \int_0^t \frac{1}{\alpha} (1 + i \beta \text{sgn}(u) \log |u|) + |u| \right\}
\end{align*}
\]

For the random variable \( Z \) distributed according to the rule described above we use the notation \( Z \sim S_\alpha(\sigma, \beta, \mu) \). When \( \mu = 0 \), we say \( Z \) is strictly stable. If in addition \( \beta = 0 \), we call \( Z \) symmetric \( \alpha \)-stable. We refer to Samorodnitsky and Taqqu [15], Janicki and Weron [8] and Sato [16] for more details on stable distributions.

Suppose that \( \{L_t, t \geq 0\} \) is a Lévy process generated by the triplet \((0, \rho, \lambda)\), i.e. the distribution of \( L_t \) has characteristic function

\[
\Phi_L(u) = E[e^{iuL_t}] = \exp \left\{ i t \lambda u + t \int_{\mathbb{R} \setminus \{0\}} (e^{iuX} - 1 - iux1_D(x)) \rho(dx) \right\}, \quad u \in \mathbb{R},
\]

where \( D = \{x : |x| \leq 1\} \) and \( \rho \) is the Lévy measure given by

\[
\rho(dx) = \frac{c_1}{x^{1+\alpha}} 1_{(0, \infty)}(x) dx + \frac{c_2}{|x|^{1+\alpha}} 1_{(-\infty, 0)}(x) dx,
\]

where \( 1 < \alpha < 2, \ c_1 \geq 0, \ c_2 \geq 0, \ \text{and} \ c_1 + c_2 > 0 \). It is easy to see that (1.4) can be written as

\[
\Phi_L(u) = \exp \left\{ i t \left( \lambda + \int_{|x|>1} x \rho(dx) \right) u - t \sigma^\alpha |u|^\alpha \left[ 1 - \beta \text{sgn}(u) \tan \left( \frac{\pi \alpha}{2} \right) \right] \right\},
\]

where \( \sigma^\alpha = -(c_1 + c_2) \Gamma(-\alpha) \cos(\pi \alpha/2) \) and \( \beta = (c_1 - c_2)/(c_1 + c_2) \). Then, by the Itô-Lévy decomposition, we have

\[
L_t = \lambda t + \int_0^t \int_{|x|<1} x \tilde{N}(ds, dx) + \int_0^t \int_{|x|\geq1} x N(ds, dx),
\]

where \( N(dt, dx) \) is a Poisson random measure defined by

\[
N((0,t], A) = \sum_{s \leq t} 1_A(\Delta L_s)
\]
for \( A \in B(\mathbb{R} \setminus \{0\}) \) and \( \Delta L_s = L_s - L_s^- \) denoting the jump of \( L_s \) at time \( s \), and the compensated Poisson random measure \( \tilde{N}(dt, dx) \) is given by

\[
\tilde{N}((0,t], A) = N((0,t], A) - t\rho(A).
\]

with

\[
\rho(A) = \int_A \rho(dx).
\]

The Itô-Lévy decomposition can be rewritten as

\[
L_t = \lambda t + \int_0^t \int_{\mathbb{R} \setminus \{0\}} x\tilde{N}(ds, dx) + t\int_{|x|\geq 1} x\rho(dx)
\]

\[
= \left( \lambda + \int_{|x|\geq 1} x\rho(dx) \right) t + \int_0^t \int_{\mathbb{R} \setminus \{0\}} x\tilde{N}(ds, dx). \tag{1.7}
\]

Let

\[
m = \lambda + \int_{|x|>1} x\rho(dx).
\]

Then

\[
m = \lambda + \frac{c_1 - c_2}{\alpha - 1}.
\]

Denote

\[
\tilde{Z}_t = \int_0^t \int_{\mathbb{R} \setminus \{0\}} x\tilde{N}(ds, dx).
\]

Then \( \tilde{Z}_t \) is a \( \alpha \)-stable Lévy motion and \( \tilde{Z}_t - \tilde{Z}_s \sim S_\alpha(\sigma(t-s)^{1/\alpha}, \beta, 0) \) for any \( 0 \leq s < t < \infty \). We can renormalize \( \tilde{Z}_t \) and define \( Z_t = \tilde{Z}_t / \sigma \). Then we can easily see that \( \{Z_t, t \geq 0\} \) is a standard \( \alpha \)-stable Lévy motion (see Janicki and Weron [8]) so that \( Z_1 \) has a stable distribution \( S_\alpha(1, \beta, 0) \). It is clear that \( L_t = mt + \sigma Z_t \) and \( E[L_t] = mt \).

The paper is organized as follows. In Section 2, we sketch the least squares method in classical setting. In Section 3, we prove the strong consistency and discuss the asymptotic distribution of the weighted TFE in the ergodic case. In Section 4, we shall establish some results for the weighted TFE in the non-ergodic case. In Section 5, we present some numerical simulations.

## 2. Classical Least Squares Technique

Consider the Langevin equation

\[
dX_t = -\theta X_t dt + dB_t \tag{2.1}
\]

where \((B_t, t \geq 0)\) is a standard Brownian motion and \(\theta\) is the unknown parameter to be estimated from the observation \((X_t, 0 \leq t \leq T)\). To explain the least squares technique we write formally

\[
\dot{X}_t = -\theta X_t + \dot{B}_t
\]

and we minimize

\[
\int_0^T |\dot{X}_t + \theta X_t|^2 dt = \int_0^T \dot{X}_t^2 dt + 2\theta \int_0^T X_t \dot{X}_t dt + \theta^2 \int_0^T X_t^2 dt.
\]
The minimizer is given by

\[ \hat{\theta}_T = -\frac{\int_0^T X_t dX_t}{\int_0^T X_t^2 dt} \]  

(2.2)

(The meaningless term \( \int_0^T \dot{X}_t^2 dt \) does not appear). To show the least squares estimator \( \hat{\theta}_T \) converges to \( \theta \), we have

\[ \hat{\theta}_T - \theta = -\frac{\int_0^T X_t dX_t}{\int_0^T X_t^2 dt} - \theta = \frac{\theta}{\int_0^T X_t^2 dt} - \theta = -\frac{\int_0^T X_t dB_t}{\int_0^T X_t^2 dt} \]  

(2.3)

Now \( \int_0^T X_t dB_t \) is a martingale with the bracket \( \int_0^T X_t^2 dt \). Roughly speaking when \( T \to \infty \), \( \int_0^T X_t dB_t \) is the order of \( \sqrt{\int_0^T X_t^2 dt} \). But when \( T \to \infty \), \( \frac{1}{T} \int_0^T X_t^2 dt \to E(X_\infty^2) \). So \( \frac{\int_0^T X_t dB_t}{\int_0^T X_t^2 dt} \) converges to 0 with the order \( \frac{1}{\sqrt{T}} \). This argument also works if \( B_t \) is replaced by a square integrable martingale. Unfortunately the stable process \( Z_t \) is not square integrable. So this naive least squares technique does not apply directly to our case.

Another approach is the maximum likelihood ratio. In the Brownian motion case, both the maximum likelihood ratio method and the least squares method give the same estimator. But in our stable process case there is no direct extension of the Girsanov theorem because of the infinite variance property. So the maximum likelihood ratio method cannot be directly applied here neither.

### 3. Ergodic Case

In this section we consider the consistency and the asymptotic distribution of the weighted TFE in ergodic case. That means we assume \( \theta > 0 \). Then, the solution of the SDE (1.3) can be written in the following way:

\[ X_t = e^{-\theta t} x + \int_0^t e^{-\theta (t-s)} dL_s \]

\[ = e^{-\theta t} x + m \int_0^t e^{-\theta (t-s)} ds + \sigma \int_0^t e^{-\theta (t-s)} dZ_s. \]

(3.1)

The general properties of generalized Ornstein-Uhlenbeck processes driven by Lévy processes have been comprehensively studied in the monograph of Sato [16]. We shall use some important results in Sato [16] freely in this paper.

**Lemma 3.1.** The generalized Ornstein-Uhlenbeck processes \( \{X_t, t \geq 0\} \) (generated by the triplet \( (0, \rho, \lambda) \)) has a unique invariant distribution \( \mu_\infty \) which is self-decomposable and generated by the triplet \( (0, \nu, \gamma) \), where

\[ \nu(B) = \frac{1}{\theta} \int \rho(dy) \int_0^\infty 1_B(e^{-s}y) ds, B \in \mathcal{B}(R) \]

and

\[ \gamma = \frac{\lambda}{\theta} + \frac{1}{\theta} \int \frac{\nu}{|y|} \rho(dy). \]
Proof. By Theorem 17.5 of Sato [16], we only need to verify that the Lévy measure \( \rho \) satisfies the following condition

\[
\int_{|x| > 2} |x| \log |x| \rho(dx) < \infty. \tag{3.2}
\]

In fact, we have

\[
\int_{|x| > 2} |x| \log |x| \rho(dx) = \int_{|x| > 2} |x| \left( \frac{c_1}{x^{1+\alpha}} 1_{(0,\infty)}(x) + \frac{c_2}{|x|^{1+\alpha}} 1_{(-\infty,0)}(x) \right) dx
\]

\[
= c_1 \int_2^\infty \frac{\log x}{x^{1+\alpha}} dx + c_2 \int_{-\infty}^{-2} \frac{\log(-x)}{(-x)^{1+\alpha}} dx
\]

\[
= (c_1 + c_2) \int_2^\infty \log x \cdot x^{-1-\alpha} dx
\]

\[
= \frac{c_1 + c_2}{\alpha 2^\alpha} \left( \log 2 + \frac{1}{\alpha} \right) < \infty.
\]

This completes the proof. \qed

We can easily find that \( X_t \) converges weakly to a random variable

\[
X_\infty = \frac{m}{\theta} + \sigma \int_0^\infty e^{-\theta s} dZ_s.
\]

Note that \( \{Z_t\} \) is a \( \mathcal{F}_t \)-martingale and the random variable \( \int_0^\infty e^{-\theta s} dZ_s \) has mean zero. Hence \( X_\infty \) is a \( \alpha \)-stable random variable with mean \( E[X_\infty] = \frac{m}{\theta} \). By Lemma 3.1 and ergodic theorem, we have

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T X_t dt = E[X_\infty] = \frac{m}{\theta}. \tag{3.3}
\]

Remark: When we don’t specify the way of convergence we always mean the almost sure convergence.

We need the following well-known integral version of Toeplitz’s Lemma (see Dietz and Kutoyants [3]):

**Lemma 3.2.** If \( \varphi_T \) is a probability measure defined on \([0, \infty)\) such that \( \varphi_T([0,T]) = 1 \) and \( \varphi_T([0,K]) \to 0 \) as \( T \to \infty \) for each \( K > 0 \), then

\[
\lim_{T \to \infty} \int_0^\infty f \varphi_T(dt) = f_\infty
\]

for every bounded and measurable function \( f : [0, \infty) \to \mathbb{R} \) for which the limit \( f_\infty := \lim_{t \to \infty} f_t \) exists.
Denote
\[ \gamma(T) = \int_0^T t^2 w_t^2 dt. \]

We assume that the weight function \( w_t \) is given so that \( \gamma(T) \) is well defined for all \( T > 0 \) and \( \gamma(T) \to \infty \) as \( T \to \infty \).

**Theorem 3.3.** Assume that \( \theta > 0 \) and \( m \neq 0 \). Then the weighted TFE is strongly consistent in the following sense:
\[ \lim_{T \to \infty} \hat{\theta}_T = \theta \quad \mathbb{P} \text{-a.s.} \]

**Proof.** Since the observed process \( \{X_t\} \) is the solution of (1.1), we find
\[ \hat{\theta}_T = \theta - \frac{\sigma \int_0^T w_t^2 A_t Z_t dt}{\int_0^T w_t^2 A_t^2 dt}. \tag{3.4} \]

By the strong law of large numbers, Lemma 3.1 and the ergodic theorem, we have
\[ \lim_{T \to \infty} T^{-1} Z_T = 0 \tag{3.5} \]
(since \( E[Z_1] = 0 \)) and
\[ \lim_{T \to \infty} T^{-1} A_T = \lim_{T \to \infty} \frac{1}{T} \int_0^T X_t dt = E[X_\infty] = \frac{m}{\theta}. \]

By the Toeplitz’s Lemma 3.2, we find
\[ \lim_{T \to \infty} \frac{\sigma \int_0^T w_t^2 A_t Z_t dt}{\int_0^T w_t^2 A_t^2 dt} = \lim_{T \to \infty} \frac{\sigma \int_0^T \left( \frac{A_t}{\sqrt{T}} \right) \left( \frac{Z_t}{\sqrt{T}} \right) w_t^2 dt}{\int_0^T \left( \frac{A_t}{\sqrt{T}} \right)^2 w_t^2 dt} = 0. \tag{3.6} \]

The last identity follows from the fact that \( \int_0^T w_t^2 \tau_t \frac{dt}{\tau(T)} = 1 \), \( \lim_{T \to \infty} T^{-1} Z_T = 0 \), and \( \lim_{T \to \infty} T^{-1} A_T = \frac{m}{\theta} \). This proves the theorem. \( \Box \)

It is well known that the least squares estimator or the maximum likelihood estimator for diffusion processes driven by Brownian motion are asymptotically normal with the order of convergence \( T^{-\frac{1}{2}} \). Here we shall prove that the weighted TFE is asymptotically \( \alpha \)-stable with the order of convergence \( T^{-(1 - \frac{1}{\alpha})} \). When \( \alpha \) is formally set to 2, our result coincides with the classical one.

For notational simplicity, we denote
\[ \xi(T) = \int_0^T tw_t^2 dt \quad \tau(T) = \int_0^T \left| \int_t^T s w_s^2 ds \right|^\alpha dt. \]

We assume that the weight function \( w_t \) satisfies the following condition:

\[ \text{(C1)} \quad \xi(T) \to \infty \text{ as } T \to \infty, \quad \xi(K)/\xi(T) \to 0 \text{ as } T \to \infty \text{ for each } K > 0, \quad \text{and} \]
\[ \frac{\xi(T)^{1/\alpha}}{\tau(T)^{1/\alpha}} = O(1). \tag{3.7} \]
Sufficient conditions on $w_t$ satisfying (C1) will be provided later on.

**Theorem 3.4.** If the generalized Ornstein-Uhlenbeck process is ergodic with $\theta > 0$ and $m \neq 0$, then the weighted TFE is asymptotically $\alpha$-stable under the condition (C1), i.e.

$$\frac{\gamma(T)}{\tau(T)^{1/\alpha}} \left( \tilde{\theta}_T - \theta \right) \Rightarrow \frac{-\sigma \theta}{m} \kappa$$

as $T \to \infty$, where $\kappa$ is a $\alpha$-stable random variable with distribution $S_{\alpha}(1, \beta, 0)$.

**Proof.** By (3.4), it follows that

$$\tilde{\theta}_T - \theta = -\frac{\sigma}{\tau(T)} \int_0^T w_t^2 A_t Z_t dt = \left[ -\sigma \int_0^T \left( \frac{A_t}{T} - \frac{m}{\theta} \right) w_t^2 Z_t dt - \frac{m\sigma}{\theta} \int_0^T w_t^2 Z_t dt \right] \int_0^T (A_t)^2 \cdot w_t^2 t^2 dt$$

$$= \frac{\phi_1(T) + \phi_2(T)}{\phi_3(T)}.$$

Then, we have

$$\frac{\gamma(T)}{\gamma(T)^{1/\alpha}} \left( \tilde{\theta}_T - \theta \right) = \frac{\tau(T)^{-1/\alpha} \phi_1(T) + \tau(T)^{-1/\alpha} \phi_2(T)}{\gamma^{-1}(T) \phi_3(T)}.$$  (3.8)

By the ergodic property and Toeplitz’s Lemma 3.2, we find

$$\lim_{T \to \infty} \frac{\phi_3(T)}{\gamma(T)} = \lim_{T \to \infty} \frac{\tau(T)}{\gamma(T)} \int_0^T \left( \frac{A_t}{T} - \frac{m}{\theta} \right) w_t^2 t^2 dt = \left( \frac{m}{\theta} \right)^2 \mathbb{P} - \text{a.s.}$$  (3.10)

Now, we consider $\tau(T)^{-1/\alpha} \phi_1(T)$. Note that

$$|\tau(T)^{-1/\alpha} \phi_1(T)| \leq \sigma \tau(T)^{-1/\alpha} Z_T^* \left( \int_0^T \left| \frac{A_t}{T} - \frac{m}{\theta} \right| tw_t^2 dt, \right.$$  (3.11)

where

$$Z_T^* = \sup_{0 \leq t \leq T} |Z_t|.$$

Denote $\tilde{\phi}_1(T) = \int_0^T \left| \frac{A_t}{T} - \frac{m}{\theta} \right| tw_t^2 dt$. By ergodic property and Toeplitz’s Lemma 3.2, it follows that

$$R_T := \left| \frac{\tilde{\phi}_1(T)}{\xi(T)} \right| = \int_0^T \left| \frac{A_t}{T} - \frac{m}{\theta} \right| tw_t^2 \frac{dt}{\xi(T)} \to 0 \quad \mathbb{P} - \text{a.s.}$$

as $T \to \infty$. Then, for $\varepsilon > 0$ and $\delta > 0$, we have

$$\mathbb{P}(|\tau(T)^{-1/\alpha} \phi_1(T)| \geq \varepsilon) \leq \mathbb{P} \left( \sigma \tau(T)^{-1/\alpha} \xi(T) Z_T^* R_T \geq \varepsilon \right) \leq \mathbb{P}(R_T \geq \delta) + \mathbb{P} \left( Z_T^* \geq \frac{\varepsilon \tau(T)^{1/2}}{\sigma \delta \xi(T)} \right).$$  (3.12)
By Markov inequality, we find
\[ P\left(Z^*_T \geq \varepsilon \frac{\tau(T)^{1/2}}{\sigma \delta(T)}\right) \leq \frac{\sigma \delta(T) \mathbb{E}[Z^*_T]}{\tau(T)^{1/2} \varepsilon} \leq \frac{\sigma \delta(T)}{\tau(T)^{1/2} \varepsilon} \cdot C \left(\int_0^T dt\right)^{1/2} = \frac{C \sigma \delta(T) T^{1/2} \varepsilon}{\varepsilon \tau(T)^{1/2}}, \quad (3.13) \]
where \( C \) is a positive constant depending only on \( \alpha \). By the condition (C1), for any fixed \( \varepsilon > 0 \), we can choose \( \delta \) arbitrarily small so that
\[ \lim_{T \to \infty} P\left(Z^*_T \geq \varepsilon \frac{\tau(T)^{1/2}}{\sigma \delta(T)}\right) = 0. \quad (3.14) \]
Obviously, \( P(R_T \geq \delta) \to 0 \) as \( T \to \infty \). Therefore, we have
\[ \lim_{T \to \infty} P(|\tau(T)^{-1/2} \phi_1(T)| \geq \varepsilon) = 0. \quad (3.15) \]
Next, we turn to consider the limiting behavior of \( \tau(T)^{-1/2} \phi_2(T) \). By integration by parts, it follows that
\[ \int_0^T Z_t \tau(T)^{1/2} dt = \int_0^T \left[ \int_t^T s \tau(T)^{1/2} ds \right] dZ_t. \]
By the inner clock property for the \( \alpha \)-stable stochastic integral (see Rosinski and Woyczynski [14], Kallenberg [9], and Zanzotto [17]), we know that
\[ \int_0^T \left[ \int_t^T s \tau(T)^{1/2} ds \right] dZ_t \]
has the same distribution as \( Z'_T \), where \( Z' := (Z'_T)' \) has the same law as \( Z \). Thus, \( Z'_T \) has an \( \alpha \)-stable distribution \( S_\alpha((\tau(T))^{1/2}, \beta, 0) \).

By basic properties of \( \alpha \)-stable random variable (see Janicki and Weron [8]), we find that
\[ \frac{\phi_2(T)}{\tau(T)^{1/2}} \]
converges weakly to a stable distribution \( S_\alpha\left(\frac{|m|}{\theta}, \text{sgn}(-m)\beta, 0\right) \).

Summarizing we have that
\[ \gamma(T) \left( \theta - \hat{\theta} \right) \]
converges weakly to \(-\frac{\alpha \theta}{m} \kappa\), where \( \kappa \) is a random variable with \( \alpha \)-stable distribution \( S_\alpha(1, \beta, 0) \). \( \square \)

**Remark 3.5.** We shall specify some conditions on \( w_t \) so that the condition (C1) is satisfied. Of course, we always assume that \( w_t \) is continuous on \([0, \infty)\). We further assume that there exists a nonnegative function \( g(s) \) such that
\[ \lim_{T \to \infty} \frac{w^2_t}{w^2_T} = g(s), \forall s \in [0, 1]. \quad (3.16) \]
and \( s^2 g(s) \leq 1. \)
Now, we can verify that (C1) holds if (3.16) is satisfied. Indeed, we have
\[
\lim_{T \to \infty} \frac{\xi(T)^{\alpha T}}{\tau(T)} = \lim_{T \to \infty} \frac{\xi(T)^{\alpha T}}{T} \int_0^T (\xi(T) - \xi(t))^{\alpha} \, dt \\
= \lim_{T \to \infty} \frac{1}{T} \int_0^T \left(1 - \frac{\xi(t)}{\xi(T)}\right)^{\alpha} \, dt \\
= \lim_{T \to \infty} \frac{1}{\int_0^1 \left(1 - \frac{\xi(sT)}{\xi(T)}\right)^{\alpha} \, ds}. \tag{3.17}
\]

Also, by L’Hospital’s rule, it follows that for each \( s \in [0, 1] \)
\[
\lim_{T \to \infty} \frac{\xi(sT)}{\xi(T)} = \lim_{T \to \infty} \frac{\int_0^{sT} rw_s^2 \, dr}{\int_0^{sT} r^2 \, dr} \\
= \lim_{T \to \infty} sT \frac{w_{sT}^2 - s}{w_T^2} = s^2 \lim_{T \to \infty} \frac{w_{sT}^2}{w_T^2} = s^2 g(s). \tag{3.18}
\]
Therefore, we have
\[
\lim_{T \to \infty} \frac{\xi(T)^{\alpha T}}{\tau(T)} = \frac{1}{\int_0^1 (1 - s^2 g(s))^{\alpha} \, ds}, \tag{3.19}
\]
which implies that the condition (C1) is satisfied.

We provide two classes of functions which satisfy the given condition.
(i) Let \( w_t = t^p \). Then, it is seen that \( \lim_{T \to \infty} \frac{w_{sT}^2}{w_T^2} = s^{2p} \). Thus,
\[
\lim_{T \to \infty} \frac{\xi(T)^{\alpha T}}{\tau(T)} = \frac{1}{\int_0^1 (1 - (s^{2+2p}))^{\alpha} \, ds}.
\]

Here, we need to assume that \( 2 + 2p > 0 \) or \( p > -1 \). By some basic calculation, we find
\[
\int_0^1 (1 - s^{2+2p})^{\alpha} \, ds = \frac{1}{2 + 2p} \int_0^1 (1 - z)^{\alpha} z^{\frac{1}{2+2p}-1} \, dz = \frac{1}{2 + 2p} B\left(\frac{1}{2+2p}, 1 + \alpha\right).
\]
So,
\[
\lim_{T \to \infty} \frac{\xi(T)^{T^{1/\alpha}}}{\tau(T)^{1/\alpha}} = \left( \lim_{T \to \infty} \frac{\xi^{\alpha}(T)T^{1/\alpha}}{\tau(T)^{1/\alpha}} \right)^{1/\alpha} = \left( \frac{2 + 2p}{B\left(1/(2+2p), 1 + \alpha\right)} \right)^{1/\alpha}.
\]
(ii) Let \( w_t = e^{qt} \), \( q > 0 \). Then, it is easy to see that
\[
\lim_{T \to \infty} \frac{w_{sT}^2}{w_T^2} = g(s) = \begin{cases} 
0, & \text{if } s \in [0, 1) \\
1, & \text{if } s = 1.
\end{cases}
\]
So,
\[
\lim_{T \to \infty} \frac{\xi(T)^{T^{1/\alpha}}}{\tau(T)^{1/\alpha}} = \frac{1}{\int_0^1 (1 - s^2 g(s))^{\alpha} \, ds} = 1.
\]
Remark 3.6. Suppose that the condition (C1) is satisfied. As in Remark 3.5, we still assume that there exists a nonnegative function $g(s)$ such that

$$\lim_{T \to \infty} \frac{w^2_s T}{w^2_T} = g(s), \forall s \in [0, 1].$$  \hspace{1cm} (3.20)

and $s^2 g(s) \leq 1$. Under this condition, as shown in Remark 3.5, we have

$$\lim_{T \to \infty} \frac{\xi(s T)}{\xi(T)} = s^2 g(s).$$

We claim that

$$\frac{\gamma(T)}{\tau^{1/\alpha}(T)} = O(T^{1-\frac{\alpha}{2}}).$$  \hspace{1cm} (3.21)

This is equivalent to

$$\frac{\gamma(T)}{\tau^{1/\alpha}(T)T^{1-\frac{\alpha}{2}}} = O(1)$$

or

$$\frac{\gamma^\alpha(T)}{\tau(T)T^{\alpha-1}} = O(1).$$

By some basic calculation, we find

$$\lim_{T \to \infty} \frac{\gamma^\alpha(T)}{\tau(T)T^{\alpha-1}} = \lim_{T \to \infty} \frac{\left(\int_0^T t \xi(t)\right)^\alpha}{\int_0^T (\xi(T) - \xi(t))^{\alpha} dt \cdot T^{\alpha-1}} = \lim_{T \to \infty} \frac{\left(\int_0^T (\xi(T) - \xi(t)T) ds\right)^\alpha}{\int_0^T (1 - s^2 g(s)) ds \cdot T^{\alpha-1}}. \hspace{1cm} (3.22)

As in Remark 3.5, we consider the following two special classes of weight functions:

(i) Let $w_t = t^p$. Then, $g(s) = s^{2p}$ with $p > -1$. It is easy to find

$$\lim_{T \to \infty} \frac{\gamma(T)}{\tau^{1/\alpha}(T)T^{1-1/\alpha}} = \left(\frac{2 + 2p}{3 + 2p}\right) \left(\frac{1}{2 + 2p} B(1/(2 + 2p), 1 + \alpha)\right)^{-1/\alpha}. \hspace{1cm} (3.23)

It is not hard to verify that the limit function of $p$ in the right hand side of (3.23) is increasing and is bounded by constant 1.

(ii) When $w_t = e^{qt}, q > 0$. It is known from Remark 3.5 that

$$g(s) = \begin{cases} 0, & \text{if } s \in [0, 1) \\ 1, & \text{if } s = 1 \end{cases}.$$

It follows that

$$\lim_{T \to \infty} \frac{\gamma(T)}{\tau^{1/\alpha}(T)T^{1-1/\alpha}} = 1. \hspace{1cm} (3.24)

Remark 3.7. The convergence result in Theorem 3.4 means that there is a stable random variable $\Xi$ such that

$$\hat{\theta}_T - \theta \approx \frac{\tau(T)^{1/\alpha}}{\gamma(T)} \Xi.$$
Thus the leading coefficient is proportional to $\frac{\tau(T)^{1/\alpha}}{\gamma(T)}$. In fact, when $w_t = t^p$ with $p > -1$, it is easy to see that for all $T > 0$

$$\frac{\tau^{1/\alpha}(T)T^{1-1/\alpha}}{\gamma(T)} = \left(\frac{3 + 2p}{2 + 2p}\right)\left[\frac{1}{2 + 2p}B(1/(2 + 2p), 1 + \alpha)\right]^{1/\alpha} =: f(p).$$

The above is the plot of the leading coefficient function $f(p)$ for $\alpha = 1.8$. We see there is slight improvement from $p = 0$ (without weight) to bigger $p$. Although the choice of $w_t$ will not change the rate of the convergence, it will improve the leading error term.

### 4. Non-Ergodic Case

In this section, we consider the non-ergodic case, i.e., $\theta < 0$. The solution of the SDE (1.1) is given by

$$X_t = e^{-\theta t}X_0 + me^{-\theta t} \int_0^t e^{\theta s} ds + \sigma e^{-\theta t} \int_0^t e^{\theta s} dZ_s. \quad (4.1)$$

Let $\eta_t = X_0 + m \int_0^t e^{\theta s} ds + \sigma \int_0^t e^{\theta s} dZ_s$. Then,

$$e^{\theta t} X_t = \eta_t. \quad (4.2)$$

Let $\xi_t = \int_0^t e^{\theta s} dZ_s$. Then, $\{\xi_t\}_{t \geq 0}$ is a $L^p$-bounded cadlag $\mathcal{F}_t$-martingale ($1 < p < \alpha$). Moreover, $\xi_t$ is a $\alpha$-stable random variable with distribution $S_\alpha(\tau_t^{1/\alpha}, \beta, 0)$, where

$$\tau_t = \int_0^t |e^{\theta s}|^{\alpha} ds = \frac{1}{\alpha \theta} \left(e^{\alpha \theta t} - 1\right).$$

Letting $t \to \infty$, we find that $\xi_t$ converges to a $\alpha$-stable random variable with distribution $S_\alpha((-\alpha \theta)^{-1/\alpha}, \beta, 0)$. Therefore, by martingale convergence theorem, it follows that

$$\lim_{t \to \infty} e^{\theta t} X_t = X_0 - \frac{m}{\theta} + \sigma \int_0^\infty e^{\theta s} dZ_s := \eta_\infty, \quad \mathbb{P} - a.s. \quad (4.3)$$
Similar to Section 3, we can define the weighted TFE of $\theta$ as follows
\[ \hat{\theta}_T = \frac{\int_0^T w_t^2 (X_t - X_0 - mt) A_t dt}{\int_0^T w_t^2 A_t^2 dt}, \]
where $A_t = \int_0^t X_s ds$. We also have
\[ \hat{\theta}_T - \theta = -\sigma \frac{\int_0^T w_t^2 A_t Z_t dt}{\int_0^T w_t^2 A_t^2 dt}. \]

Denote
\[ h_1(T) = \int_0^T w_t^2 e^{-2\theta t} dt \quad \text{and} \quad h_2(T) = \int_0^T w_t^2 e^{-\theta t} dt. \]

In this section, we always assume that the weight function $w_t$ is given so that $h_i(T) \to \infty$ and $h_i(K)/h_i(T) \to 0$ as $T \to \infty$ for each $K > 0$ and $i = 1, 2$.

We first prove the consistency of the weighted TFE.

**Theorem 4.1.** If $\theta < 0$, then
\[ \lim_{T \to \infty} (\hat{\theta}_T - \theta) = 0 \quad \mathbb{P} - \text{a.s.} \tag{4.4} \]

**Proof.** By the Toeplitz’s Lemma 3.2, it follows that
\[
\lim_{t \to \infty} \frac{A_t}{e^{-\theta t}} = \lim_{t \to \infty} \frac{\int_0^t (e^{\theta s} X_s - \eta) e^{-\theta s} ds + \eta \int_0^t e^{-\theta s} ds}{e^{-\theta t}} = \frac{\lambda(t)}{e^{-\theta t}} \int_0^t (e^{\theta s} X_s - \eta) e^{-\theta s} ds + \eta \int_0^t e^{-\theta s} ds \quad \mathbb{P} - \text{a.s.} \tag{4.5}
\]
where $\lambda(t) = \int_0^t e^{-\theta s} ds$. Then, by the Toeplitz’s Lemma 3.2 again, we have
\[
\lim_{T \to \infty} (\hat{\theta}_T - \theta) = \lim_{T \to \infty} -\sigma \frac{\int_0^T w_t^2 A_t Z_t dt}{\int_0^T w_t^2 A_t^2 dt} = \lim_{T \to \infty} -\sigma \frac{\int_0^T \left( \frac{A_t}{e^{-\theta t}} \right) \cdot \left( \frac{Z_t}{e^{-\theta t}} \right) \int_0^T \frac{w_t^2 e^{-2\theta t}}{h_1(T)} dt}{\int_0^T \left( \frac{A_t}{e^{-\theta t}} \right)^2 \frac{w_t^2 e^{-2\theta t}}{h_1(T)} dt} = 0, \quad \mathbb{P} - \text{a.s.} \tag{4.6}
\]
since $\lim_{t \to \infty} Z_t/e^{-\theta t} = 0, \quad \mathbb{P} - \text{a.s.}$ (from (3.5)). This completes the proof. \(\square\)

Next, we are going to discuss the asymptotic distribution of $\hat{\theta}_T$. We assume that the weight function $w_t$ satisfies the following condition:

**C2** There exist constants $C_0 > 0, b < 0$ such that when $T$ is large enough,
\[ \frac{w_t^2 e^{-\theta t}}{h_2(T)} \leq C_0 e^{b (T-t)}, \quad \forall t \in [0, T]. \tag{4.7} \]
We have the following result:

**Theorem 4.2.** If $\theta < 0$ and (C2) holds, then

$$\frac{h_1(T)}{h_2(T)T^{1/\alpha}}(\hat{\theta}_T - \theta) \Rightarrow -\sigma|\theta| \frac{\zeta}{\eta_\infty},$$

where $\zeta$ is a random variable with $\alpha$-stable distribution $S_{\alpha}(1, \beta, 0)$ independent of $\eta_\infty$.

**Proof.** We have

$$\frac{h_1(T)}{h_2(T)T^{1/\alpha}}(\hat{\theta}_T - \theta) = -\frac{\sigma h_2^{-1}(T)T^{-\frac{1}{\alpha}} \int_0^T w_t^2 A_t dt}{h_1^{-1}(T) \int_0^T w_t^2 A_t dt}$$

where $\zeta$ is a random variable with $\alpha$-stable distribution $S_{\alpha}(1, \beta, 0)$ independent of $\eta_\infty$.

From the Toeplitz's Lemma 3.2, it follows that

$$\lim_{T \to \infty} h_1^{-1}(T) \int_0^T w_t A_t dt = \lim_{T \to \infty} \int_0^T \left( \frac{A_t}{e^{-\theta t}} \right)^2 \frac{w_t^2 e^{-2\theta t}}{h_1(T)} dt = \frac{\eta_\infty^2}{|\theta|^2} \mathbb{P} - \text{a.s.} \quad (4.10)$$

Thus, we have

$$\lim_{T \to \infty} F_T = |\theta|^2 \mathbb{P} - \text{a.s.} \quad (4.11)$$

Now let us consider $G_T$. Note that

$$G_T = \frac{(-\sigma) h_2^{-1}(T) \int_0^T w_t^2 A_t dt}{\eta_T} \cdot \frac{T^{\frac{1}{\alpha}} Z_T}{\eta_T}$$

By the Toeplitz's Lemma 3.2 again, we get

$$\lim_{T \to \infty} h_2^{-1}(T) \int_0^T w_t^2 A_t dt = \lim_{T \to \infty} \int_0^T \left( \frac{A_t}{e^{-\theta t}} \right)^2 \frac{w_t^2 e^{-\theta t}}{h_2(T)} dt = \frac{\eta_\infty}{|\theta|} \mathbb{P} - \text{a.s.} \quad (4.13)$$

Consequently

$$\lim_{T \to \infty} \frac{(-\sigma) h_2^{-1}(T) \int_0^T w_t^2 A_t dt}{\eta_T} = -\frac{\sigma}{|\theta|} \mathbb{P} - \text{a.s.} \quad (4.14)$$

For the second factor in $G_T$, we have

$$\frac{T^{\frac{1}{\alpha}} Z_T}{\eta_T} = \frac{T^{\frac{1}{\alpha}} (Z_T - Z_{T, \frac{1}{\alpha}}) + T^{\frac{1}{\alpha}} Z_{T, \frac{1}{\alpha}}}{\eta_T + (\eta_T - \eta_T^{1/\alpha})}$$

We have the following claims.
The random variable \( T^{-\frac{1}{\alpha}}(Z_T - Z_T^\frac{1}{\alpha}) \) has an \( \alpha \)-stable distribution \( S_{\alpha}(\sigma (1 - T^{\frac{1}{\alpha}} - 1)^{1/\alpha}, \beta, 0) \), which converges weakly to a random variable \( \zeta \) with stable distribution \( S_{\alpha}(1, \beta, 0) \) as \( T \to \infty \).

By strong law of large numbers, we have
\[
\lim_{T \to \infty} T^{-\frac{1}{\alpha}}Z_T^\frac{1}{\alpha} = 0 \quad P\text{-a.s.}
\]

It is clear that
\[
\lim_{T \to \infty} \eta_T^\frac{1}{\alpha} = \eta_\infty \quad P\text{-a.s.}
\]

\( T^{-\frac{1}{\alpha}}(Z_T - Z_T^\frac{1}{\alpha}) \) and \( \eta_T^\frac{1}{\alpha} \) are independent.

We have that \( \eta_T - \eta_T^\frac{1}{\alpha} \) converges to zero in probability as \( T \to \infty \).

**Proof of (5).** By the definition of \( \eta_t \), we find
\[
\eta_T - \eta_T^\frac{1}{\alpha} = m \int_{T^\frac{1}{\alpha}}^T e^{\theta s} ds + \int_{T^\frac{1}{\alpha}}^T e^{\theta s} dZ_s.
\]

It follows that
\[
|\eta_T - \eta_T^\frac{1}{\alpha}| \leq |m| \int_{T^\frac{1}{\alpha}}^T e^{\theta s} ds + \int_{T^\frac{1}{\alpha}}^T e^{\theta s} dZ_s
\]
\[
\leq |m| \left( e^{\theta T} - e^{\theta T^\frac{1}{\alpha}} \right) + \int_{T^\frac{1}{\alpha}}^T e^{\theta s} dZ_s. \tag{4.16}
\]

The first term on the right hand side of (4.16) converges to zero as \( T \to \infty \). The second term converges to zero in probability as \( T \to \infty \), since
\[
P \left\{ \int_{T^\frac{1}{\alpha}}^T e^{\theta s} dZ_s > \varepsilon \right\} \leq \frac{E \left| \int_{T^\frac{1}{\alpha}}^T e^{\theta s} dZ_s \right|}{\varepsilon}
\]
\[
\leq C \left( \int_{T^\frac{1}{\alpha}}^T e^{\alpha \theta s} ds \right)^\frac{1}{\alpha}
\]
\[
\leq C \frac{e^{\alpha \theta T} - e^{\alpha \theta T^\frac{1}{\alpha}}}{\alpha \theta}, \tag{4.17}
\]

which tends to zero as \( T \to \infty \) for any given \( \varepsilon > 0 \) and some constant \( C > 0 \).

From all the claims (1)-(5), we conclude that
\[
\frac{T^{-\frac{1}{\alpha}}Z_T}{\eta_T} \Rightarrow \frac{\zeta}{\eta_\infty} \tag{4.18}
\]

where \( \zeta \) and \( \eta_\infty \) are independent. Combining (4.14) and (4.18), we find
\[
G_T \Rightarrow -\frac{\sigma}{\theta} \frac{\zeta}{\eta_\infty} \tag{4.19}
\]
as \( T \to \infty \). Finally, we shall prove that \( H_T \to 0 \) in probability as \( T \to \infty \). We have

\[
\left| \sigma T^{-\frac{\alpha}{2}} h_2^{-1}(T) \int_0^T w_t^2 A_t(Z_T - Z_t) \, dt \right|
\]

\[
\leq \sigma T^{-\frac{\alpha}{2}} h_2^{-1}(T) \int_0^T w_t^2 \left| \int_0^t X_s \, ds \right| |Z_T - Z_t| \, dt
\]

\[
\leq \sigma T^{-\frac{\alpha}{2}} h_2^{-1}(T) \int_0^T \left( \int_0^t |e^{\theta_s} X_s| e^{-\theta_s} \, ds \right) |Z_T - Z_t| w_t^2 \, dt
\]

\[
\leq \frac{\sigma}{|\theta|} \sup_{t \geq 0} |e^{\theta t} X_t| \cdot T^{-\frac{\alpha}{2}} h_2^{-1}(T) \int_0^T |Z_T - Z_t| w_t^2 e^{-\theta t} \, dt. \quad (4.20)
\]

It is easy to see that \( \sup_{t \geq 0} |e^{\theta t} X_t| \) is almost surely finite. We claim that the last factor \( T^{-\frac{\alpha}{2}} h_2^{-1}(T) \int_0^T |Z_T - Z_t| w_t^2 e^{-\theta t} \, dt \) in the above inequality converges to zero in probability. Indeed, we have for \( T \) large enough

\[
E \left[ T^{-\frac{\alpha}{2}} h_2^{-1}(T) \int_0^T |Z_T - Z_t| w_t^2 e^{-\theta t} \, dt \right]
\]

\[
\leq T^{-\frac{\alpha}{2}} \int_0^T E \left[ |Z_T - Z_t| w_t^2 e^{-\theta t} \right] h_2(T) \, dt
\]

\[
\leq C_0 T^{-\frac{\alpha}{2}} \int_0^T C(1, \alpha) (T-t)^{\frac{\alpha}{2}} e^{h(T-t)} \, dt
\]

\[
= C_0 T^{-\frac{\alpha}{2}} \int_0^T C(1, \alpha) u^{\frac{\alpha}{2}} e^{h u} \, du
\]

\[
\leq C(1, \alpha) C_0 T^{-\frac{\alpha}{2}} \int_0^{\infty} u^{\frac{\alpha}{2}} e^{h u} \, du
\]

\[
\leq C(1, \alpha) C_0 T^{-\frac{\alpha}{2}} \Gamma(1+\frac{1}{\alpha}) |b|^{-1+\frac{1}{\alpha}}, \quad (4.21)
\]

which tends to zero as \( T \to \infty \), where \( C(1, \alpha) = \frac{4^\alpha(-1/\alpha)}{\alpha \sqrt{\pi} \Gamma(-1/2)} \) (see Zolotarev [18]). This implies that \( \sigma T^{-\frac{\alpha}{2}} h_2^{-1}(T) \int_0^T w_t^2 A_t(Z_T - Z_t) \, dt \) converges to zero in probability as \( T \to \infty \). So, \( H_T \to 0 \) in probability as \( T \to \infty \). From (4.9), (4.11), (4.19), (4.20) and (4.21), we conclude that

\[
\frac{h_1(T)}{h_2(T) T^{\frac{\alpha}{2}}} (\hat{\theta}_T - \theta) \Rightarrow -\sigma |\theta| \frac{\zeta}{\eta\infty}, \quad (4.22)
\]

where \( \zeta \) is a \( S_\alpha(1, \beta, 0) \) random variable independent of \( \eta\infty \). This completes the proof. \( \square \)

**Remark 4.3.** We consider two special classes of weight functions.

(i) Let \( w_t = t^p, p \geq 0 \). Some basic calculation yields

\[
h_2(T) = \int_0^T w_t^2 e^{-\theta t} \, dt = \int_0^T t^{2p} e^{-\theta t} \, dt \geq C_1 T^{2p} e^{-\theta T}, \quad (4.23)
\]
for each $T \geq T_0$ with some $T_0 > 0$ and some $C_1 > 0$. It follows that for $T$ large enough
\[
\frac{w_t^2 e^{-\theta t}}{h_2(T)} \leq \frac{t^2 e^{-\theta t}}{C_1 T^2 e^{-\theta T}} \leq \frac{1}{C_1} e^{\theta (T-t)}, \quad t \in [0, T]. \tag{4.24}
\]
This implies that (C2) is satisfied with $b = \theta$.

(ii) Let $w_t^2 = e^{rt}$. Then,
\[
h_2(T) = \int_0^T e^{rt} e^{-\theta t} dt = \frac{e^{(r-\theta)T} - 1}{r - \theta}. \tag{4.25}
\]
We have
\[
\frac{w_t^2 e^{-\theta t}}{h_2(T)} \leq C_2 e^{(r-\theta)(T-t)}, \tag{4.26}
\]
for $T$ large enough and $b = \theta - r < 0$. Thus, if $r > \theta$, then the condition (C2) is satisfied.

**Remark 4.4.** We claim that
\[
\frac{h_1(T)}{h_2(T)} = O(e^{-\theta T^{1/\alpha}}). \tag{4.27}
\]
This is equivalent to
\[
\frac{h_1(T)}{h_2(T)} = O(e^{-\theta T}).
\]
We assume that $w_t$ is a continuously differentiable function satisfying
\[
\lim_{T \to \infty} \frac{w'_T}{w_T} = a, \tag{4.28}
\]
where $a$ is a constant such that $-2\theta + 2a > 0$ and $2a - \theta > 0$. Then, by L'Hospital's rule, we have
\[
\lim_{T \to \infty} \frac{h_1(T) e^{\theta T}}{h_2(T)} = \lim_{T \to \infty} \frac{w_T^2 e^{-2\theta T} e^{\theta T} + h_1(T) e^{\theta T}}{w_T^2 e^{-\theta T}} = 1 + \theta \lim_{T \to \infty} \frac{h_1(T)}{w_T^2 e^{-2\theta T}} = 1 + \theta \lim_{T \to \infty} \frac{w_T^2 e^{-2\theta T}}{w_T^2 e^{-2\theta T} (-2\theta) + 2w_T w'_T e^{-2\theta T}} = 1 + \frac{\theta}{2(a - \theta)} = \frac{2a - \theta}{2(a - \theta)}. \tag{4.29}
\]
Two special cases: (i) Let $w_t = t^p, p \geq 0$. Then,
\[
\lim_{T \to \infty} \frac{w'_T}{w_T} = \lim_{T \to \infty} \frac{p T^{p-1}}{T^p} = 0.
\]
So,
\[
\lim_{T \to \infty} \frac{h_1(T) e^{\theta T}}{h_2(T)} = 1/2. \tag{4.30}
\]
(ii) Let $w_t = e^{r/2}$, $r > \theta$. Then,

$$\lim_{T \to \infty} \frac{w'_T}{w_T} = \lim_{T \to \infty} \frac{\frac{r}{2} e^{rT/2}}{e^{rT/2}} = \frac{r}{2}.$$ 

So,

$$\lim_{T \to \infty} \frac{h_1(T)e^{\theta T}}{h_2(T)} = \frac{r-\theta}{r - 2\theta}. \quad (4.31)$$

5. Simulation

In this section we perform a few numerical simulations to test our estimators.

First we simulate the $\alpha$ stable process $Z_t$ according to [1]. Then we use the so-called Euler scheme to simulate $X_t$. Let $T > 0$ and $n$ are given and denote $t_k = \frac{kT}{n}$. The equation (1.1) can be approximated by

$$X_{t_{k+1}}^\pi = X_{t_k}^\pi - \theta X_{t_k}^\pi \Delta_k + m \Delta_k + \sigma \Delta Z_{t_k}$$

where $\Delta_k = t_{k+1} - t_k$ and $\Delta Z_{t_k} = Z_{t_{k+1}} - Z_{t_k}$. The Euler scheme for stochastic differential equations driven by Lévy processes has been well studied (see [5], [6], [7], [13] and the references therein).

Then we use our weighted trajectory fitting estimator (1.2) to estimate $\theta$. We plot $\hat{\theta}_T$ as a function of $T$ to see how $\hat{\theta}_T$ approximates $\theta$ as $T$ becomes bigger for different weights.

The following two figures describe the estimator $\hat{\theta}_T$ ($T \leq 300$) for

$$dX_t = -3X_t dt + 2dt + 0.1dZ_t, \quad X_0 = 1$$

We use $\alpha = 1.8$, $w_t = 1$ (figure on the left) and $w_t = t^8$ (figure on the right).

The case $\theta < 0$ has a fast convergence which we will not describe here.

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References


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