Chapter 3. Continuous Distributions

A random variable $X$ is called continuous if there is a nonnegative function $f$, called the probability density function of $X$, such that for any set $B$

$$P(X \in B) = \int_B f(x)dx.$$ 

The distribution function of $X$ is defined by

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(y)dy.$$ 

We have the following formulae:

1. $\int_{-\infty}^{\infty} f(x)dx = 1.$
2. $P(a < X \leq b) = \int_a^b f(x)dx = F(b) - F(a).$
3. $F'(x) = f(x).$

Expectation and Variance of Continuous Random Variables

The expected value of a continuous random variable $X$ is defined by

$$\mu = E[X] = \int_{-\infty}^{\infty} xf(x)dx.$$ 

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The variance of $X$ is defined by
\[ \sigma^2 = \text{Var}(X) = E[(X - E[X])^2] = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx \]
and the standard deviation of $X$ is
\[ \sigma = \sqrt{\text{Var}(X)}. \]

The moment generating function $M(t)$ of $X$ is defined by
\[ M(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx, \quad -h < t < h. \]

We have the following formulae
(1) $E[aX + b] = aE[X] + b$, $a, b$ are constants.
(2) $\text{Var}(X) = E[X^2] - (E[X])^2$, $\text{Var}(aX + b) = a^2 \text{Var}(X)$.
(3) For any function $u$,
\[ E[u(X)] = \int_{-\infty}^{\infty} u(x) f(x) dx \]
(4)
\[ \mu = M'(0), \quad \sigma^2 = M''(0) - \mu^2. \]

**Special Continuous Random Variables**

(i) A random variable $X$ is said to be *uniform* over the interval $(a, b)$ if its probability density function is given by
\[ f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases} \]
Its expected value and variance are

\[ E[X] = \frac{a + b}{2}, \quad Var(X) = \frac{(b - a)^2}{12}. \]

(ii) A random variable \( X \) is said to be exponential with parameter \( \theta \) if its density function is given by

\[ f(x) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}}, & x \geq 0 \\ 0, & \text{otherwise}. \end{cases} \]

The distribution function \( F(x) \) of \( X \) is given by

\[ F(x) = \begin{cases} 1 - e^{-\frac{x}{\theta}}, & x \geq 0 \\ 0, & \text{otherwise}. \end{cases} \]

The expected value and variance of \( X \) are

\[ E[X] = \theta, \quad Var(X) = \theta^2. \]

It satisfies the memoryless property, for positive \( s \) and \( t \),

\[ P\{X > s + t|X > t\} = P\{X > s\}. \]

(iii) Gamma distribution: its p.d.f. is defined by

\[ f(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}, \quad 0 \leq x < \infty, \]

and its mean \( \mu = \alpha \theta \) and variance \( \sigma^2 = \alpha \theta^2. \)

(iv) Chi-square distribution \( \chi^2(r) \) (\( r \) is the degrees of freedom): its p.d.f. is given by

\[ f(x) = \frac{1}{\Gamma(r/2)2^{r/2}} x^{r/2-1} e^{-x/2}, \quad 0 \leq x < \infty, \]
and its mean \( \mu = r \) and variance \( \sigma^2 = 2r \).

(v) Normal distributions:
A random variable \( X \) is said to be \textit{normal} with parameters \( \mu \) and \( \sigma^2 \) (denoted by \( N(\mu, \sigma^2) \)) if its probability density function is given by

\[
f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty
\]

We have \( E[X] = \mu \) and \( Var(X) = \sigma^2 \). If \( X \) has a normal distribution \( N(\mu, \sigma^2) \), then \( Z = (X - \mu) / \sigma \) is a standard normal random variable \( N(0, 1) \) with mean 0 and variance 1. The distribution function of \( X \) can be expressed by

\[
F_X(a) = P\{X \leq a\} = P\left(\frac{X - \mu}{\sigma} \leq \frac{a - \mu}{\sigma}\right) = P \left( Z \leq \frac{a - \mu}{\sigma} \right) = \Phi \left( \frac{a - \mu}{\sigma} \right).
\]

where \( \Phi \) is the distribution function of \( Z \). We also have

\[
P(a \leq X \leq b) = F(b) - F(a) = \Phi \left( \frac{b - \mu}{\sigma} \right) - \Phi \left( \frac{a - \mu}{\sigma} \right).
\]

\textbf{Chapter 4. Bivariate Distributions:}
If \( X \) and \( Y \) are discrete random variables, the \textit{joint probability mass function} of \( X \) and \( Y \) is defined by

\[
f(x, y) = P(X = x, Y = y)
\]
The marginal probability mass functions of $X$ and $Y$ are
\[ f_1(x) = P(X = x) = \sum_y f(x, y), \quad f_2(y) = P(Y = y) = \sum_x f(x, y) \]

If $X$ and $Y$ have a joint probability mass function $f(x, y)$, then
\[ E[u(X, Y)] = \sum_y \sum_x u(x, y) f(x, y) \]

The random variables $X$ and $Y$ are said to be jointly continuous if there is a nonnegative function $f(x, y)$, called the joint probability density function, such that for any set $C$ in the $xy$-plane,
\[ P\{(X, Y) \in C\} = \int \int_C f(x, y) \, dx \, dy \]

When $C = \{(x, y)|x \in A, y \in B\}$, then
\[ P\{X \in A, Y \in B\} = \int_A \int_B f(x, y) \, dx \, dy \]

The marginal density functions of $X$ and $Y$ are given by
\[ f_1(x) = \int_{-\infty}^{\infty} f(x, y) \, dy, \quad f_2(y) = \int_{-\infty}^{\infty} f(x, y) \, dx \]

If $X$ and $Y$ have a joint density function $f(x, y)$, then
\[ E[u(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x, y) f(x, y) \, dx \, dy \]

**Conditional Distributions**
If $X$ and $Y$ are discrete random variables, then the conditional
probability mass function of \( X \) given that \( Y = y \) is defined by

\[
g(x|y) = P(X = x|Y = y) = \frac{f(x, y)}{f_2(y)}
\]

where \( f(x, y) \) is the joint probability mass function of \( X \) and \( Y \), and \( f_2(y) \) is the marginal probability mass function of \( Y \). Then the conditional probability mass function of \( Y \) given that \( X = x \) is defined by

\[
h(y|x) = P(Y = y|X = x) = \frac{f(x, y)}{f_1(x)}
\]

where \( f_1(x) \) is the marginal probability mass function of \( X \).

Some related formulas:
(1) \( E[u(Y)|X = x] = \sum_y u(y)h(y|x) \).
(2) \( \mu_{Y|x} = E(Y|x) = \sum_y yh(y|x) \).
(3) \( \sigma^2_{Y|x} = E\{[Y - E(Y|x)]^2|x\} = \sum_y[y - E(Y|x)]^2h(y|x) \).

If \( X \) and \( Y \) are jointly continuous with joint density function \( f \), then the conditional probability density function of \( X \) given that \( Y = y \) is given by

\[
g(x|y) = \frac{f(x, y)}{f_2(y)}
\]

and the conditional probability density function of \( Y \) given that \( X = x \) is given by

\[
h(x|y) = \frac{f(x, y)}{f_1(x)}
\]
Some related formulas:

1. \( P(a < X < b|Y = y) = \int_a^b g(x|y)dx. \)
2. \( P(a < Y < b|X = x) = \int_a^b h(y|x)dy. \)
3. \( E(Y|x) = \int_{-\infty}^{\infty} yh(y|x)dy. \)
4. \( Var(Y|x) = \int_{-\infty}^{\infty} [y - E(Y|x)]^2 h(y|x)dx. \)

**Independent Random Variables**

The random variables \( X \) and \( Y \) are *independent* if for all sets \( A \) and \( B \)

\[
P\{X \in A, Y \in B\} = P\{X \in A\}P\{Y \in B\}
\]

The independence of \( X \) and \( Y \) is equivalent to \( f(x, y) = f_1(x)f_2(y) \) for both discrete and continuous cases. If the joint density (or mass) function factors into a part depending only on \( x \) and a part depending only on \( y \), then \( X \) and \( Y \) are independent.

**Chapter 5. Distributions of Functions of Random Variables**

**Transformation of Variables:**

Let the random variable \( X \) have probability density function \( f(x) \) with support \( S_x \). For \( Y = u(X) \), either an increasing or decreasing function of \( X \) with the inverse function \( X = v(Y) \) whose support is \( S_y \), the probability density function for \( Y \) is

\[
g(y) = f(v(y)) \left| v'(y) \right|.
\]
Sums of Independent Random Variables

If $X_1, X_2, \cdots, X_n$ are independent continuous random variables with respective p.m.f. (or p.d.f.) $f_1(x_1), f_2(x_2), \cdots, f_n(x_n)$, then the joint p.m.f. (or p.d.f.) is

$$f(x_1, x_2, \cdots, x_n) = f_1(x_1)f_2(x_2)\cdots f_n(x_n).$$

and we have the following property

$$E[u_1(X_1)u_2(X_2)\cdots u_n(X_n)] = E[u_1(X_1)]E[u_2(X_2)]\cdots E[u_n(X_n)].$$

If $X_i, i = 1, 2, \cdots, n$, are independent random variables with respective means $\mu_i$ and variances $\sigma_i^2, i = 1, 2, \cdots, n$, then for $Y = \sum_{i=1}^n a_i X_i$, we have

$$\mu_Y = E[Y] = \sum_{i=1}^n a_i \mu_i$$

and

$$\sigma_Y^2 = Var(Y) = \sum_{i=1}^n a_i^2 \sigma_i^2.$$