1. A box contains 4 red and 6 blue marbles. Two marbles are withdrawn randomly. If they are the same color, then you win $2.00; if they are different colors, then you win -$2.00 (that is, you lose $2.00). Let \( X \) be the amount you win. 

(7) (a) Find the probability mass function of \( X \).

(8) (b) Find \( E[X] \) and \( \text{Var}(X) \).

Solution: (a) \( X \) takes value 2, -2, and \( \{X = 2\} \) = \{two marbles are the same color\}, \( \{X = -2\} \) = \{two marbles are different colors\}. We can find that 

\[
f(2) = P(X = 2) = \frac{\binom{4}{2} + \binom{6}{2}}{\binom{10}{2}} = \frac{7}{15}
\]

and

\[
f(-2) = P(X = -2) = \frac{4 \cdot 6}{\binom{10}{2}} = \frac{8}{15}.
\]

(b) So, \( E[X] = 2P(X = 2) + (-2)P(X = -2) = (2)\frac{7}{15} - (2)\frac{8}{15} = -\frac{2}{15} \).

Note that \( \text{Var}(X) = E[X^2] - (E[X])^2 \). We first calculate 

\[
E[X^2] = 2^2P(X = 2) + (-2)^2P(X = -2) = 4P(X = 2) + 4P(X = -2) = 4.
\]

So,

\[
\text{Var}(X) = 4 - (-\frac{2}{15})^2 = \frac{896}{225}.
\]

2. (i) Find the probability mass function of \( X \), (ii) find \( \mu = E[X] \) and \( \sigma^2 = \text{Var}(X) \) when the moment generating function \( M(t) \) of \( X \) is given by

(15) (a) \( M(t) = \frac{0.4e^t}{1 - 0.6e^t}, \quad t < -\ln(0.6) \).

(15) (b) \( M(t) = 0.4e^t + 0.3e^{2t} + 0.2e^{3t} + 0.1e^{4t} \).

Solution. (a) (i) We first find an expansion for \( M(t) \). Note that

\[
\sum_{x=0}^{\infty} z^x = \frac{1}{1 - z}, \text{ if } |z| < 1.
\]

So, we have

\[
M(t) = 0.4e^t (1 + 0.6e^t + (0.6)^2e^{2t} + \cdots)
\]

\[
= 0.4e^t + (0.4)(0.6)e^{2t} + (0.4)(0.6)^2e^{3t} + \cdots
\]

\[
= \sum_{x=1}^{\infty} (0.4)(0.6)^{x-1}e^{xt}.
\]

This implies that \( S = \{1, 2, 3, \cdots\} \) and

\[
f(x) = P(X = x) = (0.4)(0.6)^{x-1}, \quad x = 1, 2, 3, \cdots.
\]
Therefore $X$ is a geometric random variable with parameter $p = 0.4$.

(ii) $\mu = 1/p = 1/(0.4) = 10/4$ and $\sigma^2 = (1-p)/p^2 = (1 - 0.4)/(0.4)^2 = 15/4$.

(b) (i) It is easy to see that $S = \{1, 2, 3, 4\}$ and the p.m.f is given by $f(1) = P(X = 1) = 0.4, f(2) = P(X = 2) = 0.3, f(3) = P(X = 3) = 0.2$ and $f(4) = P(X = 4) = 0.1$. We can also write $f(x) = P(X = x) = (5-x)/10, x = 1, 2, 3, 4.$

(ii) $\mu = E(X) = \sum_{x=1}^{4} x f(x) = 1(0.4) + 2(0.3) + 3(0.2) + 4(0.1) = 2, \sigma^2 = \sum_{x=1}^{4} (x - \mu)^2 f(x) = (1-2)^2(0.4) + (2-2)^2(0.3) + (3-2)^2(0.2) + (4-2)^2(0.1) = 1.$

3. It is claimed that 65% of Americans under the age of 65 have private health insurance. Suppose this is true, and 8 Americans under age 65 are selected at random. Let $X$ be the number of Americans under age 65 with private health insurance in the sample.

(a) Assuming independence, how is $X$ distributed?

(b) Find (i) $P(X \geq 2)$, (ii) $P(X = 1)$, and (iii) $P(X \leq 3)$.

(c) Find the values of $E(X)$ and $Var(X)$.

Solution. (a) $X$ has a binomial distribution $b(8, 0.65)$ with parameters $n = 8$ and $p = 0.65$.

(b) (i)

\[
P(X \geq 2) = 1 - P(X \leq 1) = 1 - (P(X = 0) + P(X = 1))
\]

\[
= 1 - \sum_{x=0}^{1} \binom{8}{x} (0.65)^x (0.35)^{8-x} = 1 - 0.0035 = 0.9965.
\]

(ii)

\[
P(X = 1) = \binom{8}{1} (0.65)(0.35)^7 = 0.0033.
\]

(iii)

\[
P(X \leq 3) = \sum_{x=0}^{3} \binom{8}{x} (0.65)^x (0.35)^{8-x} = 0.1060.
\]

(c) $E[X] = np = 8(0.65) = 5.2, \text{ and } Var(X) = np(1-p) = 8(0.65)(0.35) = 1.82.$

4. A basketball player can make a free throw 50% of the time. Assume that all the free throws can be thought of as independent Bernoulli trials.

(a) Let $X$ be the number of free throws needed to make one shot. Find $P(X \geq 3)$, the mean $\mu$ and variance $\sigma^2$ of $X$.

(b) Let $X$ be the number of free throws needed to make a total of 5 shots. Find $P(X = 8)$, the mean $\mu$ and variance $\sigma^2$ of $X$.

Solution. (a) $X$ has a geometric distribution with $p = 0.5$ and the p.m.f. is given by $f(x) = (0.5) \cdot (0.5)^{x-1} = (0.5)^x, x = 1, 2, \cdots$. Then, we have

\[
P(X \geq 3) = 1 - P(X \leq 2) = 1 - [P(X = 1)+P(X = 2)] = 1 - [(0.5)+(0.5)(0.5)] = 0.25
\]

and $\mu = 1/p = 1/0.5 = 2$ and $\sigma^2 = \frac{1-p}{p^2} = \frac{1-0.5}{(0.5)^2} = 2$.

(b) $X$ has a negative binomial distribution with $r = 5$ and $p = 0.5$. So, we have

\[
P(X = 8) = \binom{8-1}{5-1} (0.5)^5 (0.5)^{8-5} = 0.1367.
\[
\mu = \frac{r}{p} = \frac{5}{0.5} = 10, \quad \sigma^2 = \frac{r(1 - p)}{p^2} = \frac{5 \cdot (0.5)}{(0.5)^2} = 10.
\]