Online Appendix:
An approximation scheme for impulse control with random reaction periods

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Abstract
This short online appendix presents the proof of Theorem 1 in Perera and Long (2017).

Appendix A. Proof of Theorem 1

Note that we do not have any smoothness conditions on \( \varphi \). Therefore, we need the lemma below in order to prove the theorem. A version of this lemma can be found in Øksendal and Sulem (2007) (see Corollary 7.4); the proof of the lemma is omitted here as it resembles the corresponding proof in Øksendal and Sulem (2007).

Lemma 1. Let \( F, G \) and \( H \) be real valued functions on \( \mathbb{R}^d \) which have at most polynomial growth. Then, (a) for a given stopping time \( \tau \geq 0 \), the process
\[
\{U(t) := \int_0^{\tau \wedge T} F(X(s)) ds + G(X(t \wedge \tau)) : t \geq 0\}
\]
is a submartingale. In particular, if \( \tau_1 \leq \tau_2 \) are stopping times, then \( E^\tau [G(X(\tau_1))] \leq E^\tau [\int_{\tau_1}^{\tau_2} F(X(s)) ds + G(X(\tau_2))] \).
(b) For \( \varepsilon > 0 \), let \( \tau^{(\varepsilon)} := \inf\{t > 0 : X(t) \notin D^{(\varepsilon)} \} \) with \( D^{(\varepsilon)} := \{y : G(y) < H(y) - \varepsilon \} \) and let \( \beta_1 \leq \beta_2 \leq \tau^{(\varepsilon)} \) be stopping times. Then \( E^\tau[G(X(\beta_1))] = E^\tau[\int_{\beta_1}^{\beta_2} F(X(s)) ds + G(X(\beta_2))] \).

Now, we are ready to provide a step-by-step proof for Theorem 1.

Proof of Theorem 1: We choose \( u_n = (\tau_1, \tau_2, \ldots, \tau_n; (i_1, \xi_1), (i_2, \xi_2), \ldots, (i_n, \xi_n)) \) with \( \tau_{n+1} = \infty \), and for \( j \in \{1, 2, \ldots, n\} \) and \( t \geq 0 \), define \( \varphi^d_j(X^{(i_n)}(t)) = e^{-\tau_i} \varphi_j(X^{(i_n)}(t)) \) and \( f^d_j(X^{(i_n)}(t)) = e^{-\tau_i} f(X^{(i_n)}(t)) \). Then, by Lemma 1(a), for \( j \in \{1, 2, \ldots, n-1\} \) and \( i_j \in \{1, 2, \ldots, p\} \), we have
\[
E^\tau\left[\varphi^d_{n-j}(X^{(i_n)}(\tau_j + T_j^i)) T_j^i = s \right] \leq E^\tau\left[\int_{\tau_j + T_j^i}^{\tau_j + T_j^i + s} f^d_j(X^{(i_n)}(t)) dt + \varphi^d_{i-j}(X^{(i_n)}(\tau_{j+1} -)) T_j^i = s \right]. \quad (A.1)
\]

Integrating (A.1) with respect to the probability measure induced by \( T^i \), we get
\[
\int_0^\infty E^\tau\left[\varphi^d_{n-j}(X^{(i_n)}(\tau_j + s)) \right] dF_{i_j}(s) \leq \int_0^\infty E^\tau\left[\int_{\tau_j + s}^{\tau_j + s + T_j^i} f^d_j(X^{(i_n)}(t)) dt \right] dF_{i_j}(s) + E^\tau\left[\varphi^d_{n-j}(X^{(i_n)}(\tau_{j+1} -)) \right]. \quad (A.2)
\]

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We next proceed to derive an upper bound on the second term of the right-hand side of (A.2). For this, note that $T_{i_{j+1}}$ is independent of $X^{(i_{j+1})}_{i_{j+1}}$ for $i_{j+1} \in \{1, 2, \ldots, p\}$. Therefore, by the definition of $\mathcal{M}_\delta$, we have

$$
\mathcal{M}_\delta \varphi_{n-j-1}(\tilde{X}^{(i_{j+1})}_{i_{j+1}}(\tau_{j+1}-)) \leq K_{i_{j+1}}(\tilde{X}^{(i_{j+1})}_{i_{j+1}}(\tau_{j+1}-), \zeta_{i_{j+1}}),
\mathcal{M}_\delta \varphi_{n-j-1}(\tilde{X}^{(i_{j+1})}_{i_{j+1}}(\tau_{j+1}-)) \leq K_{i_{j+1}}(\tilde{X}^{(i_{j+1})}_{i_{j+1}}(\tau_{j+1}-), \zeta_{i_{j+1}}) + \int_0^{\tau_{j+1}} E^{T_{i_{j+1}}(\tilde{X}^{(i_{j+1})}_{i_{j+1}}(\tau_{j+1}-), \zeta_{i_{j+1}})} \left[ e^{-\tau_{j+1} f(X(t))} dt + e^{-\tau_{j+1} \varphi_{n-j-1}(X(t))} \right] dF_{i_{j+1}}(s). \quad (A.3)
$$

Since $X^{(i_{j+1})}_{i_{j+1}}(\tau_{j+1}) = \Gamma_{i_{j+1}}(\tilde{X}^{(i_{j+1})}_{i_{j+1}}(\tau_{j+1}-), \zeta_{i_{j+1}})$, it follows that

$$
\mathcal{M}_\delta \varphi_{n-j-1}(\tilde{X}^{(i_{j+1})}_{i_{j+1}}(\tau_{j+1}-)) \leq K_{i_{j+1}}(\tilde{X}^{(i_{j+1})}_{i_{j+1}}(\tau_{j+1}-), \zeta_{i_{j+1}}) + \int_0^{\tau_{j+1}} E^{\tilde{X}^{(i_{j+1})}_{i_{j+1}}(\tau_{j+1})} \left[ e^{-\tau_{j+1} f(X(t))} dt \right] dF_{i_{j+1}}(s)
+ \int_0^{\tau_{j+1}} E^{\tilde{X}^{(i_{j+1})}_{i_{j+1}}(\tau_{j+1})} \left[ e^{-\tau_{j+1} \varphi_{n-j-1}(X(t))} \right] dF_{i_{j+1}}(s). \quad (A.4)
$$

Now recall that $X^{(i_{j+1})}_{i_{j+1}}(t) = \tilde{X}^{(i_{j+1})}_{i_{j+1}}(\tau_{j+1}-) + \int_0^t e^{-\tau_{j+1} f(X(t))} dt + e^{-\tau_{j+1} \varphi_{n-j-1}(X(t))} dt \quad (A.5)$

and

$$
\int_0^{\tau_{j+1}} E^{\tilde{X}^{(i_{j+1})}_{i_{j+1}}(\tau_{j+1})} \left[ e^{-\tau_{j+1} \varphi_{n-j-1}(X(t))} \right] dF_{i_{j+1}}(s) = \int_0^{\tau_{j+1}} E^{\tilde{X}^{(i_{j+1})}_{i_{j+1}}(\tau_{j+1})} \left[ e^{-\tau_{j+1} \varphi_{n-j-1}(X(t))} \right] dF_{i_{j+1}}(s)
= \int_0^{\tau_{j+1}} E^{\tilde{X}^{(i_{j+1})}_{i_{j+1}}(\tau_{j+1})} \left[ e^{-\tau_{j+1} \varphi_{n-j-1}(X(t))} \right] dF_{i_{j+1}}(s).
$$

Then, from (A.4), we can obtain that

$$
\mathcal{M}_\delta \varphi_{n-j-1}(\tilde{X}^{(i_{j+1})}_{i_{j+1}}(\tau_{j+1}-)) \leq K_{i_{j+1}}(\tilde{X}^{(i_{j+1})}_{i_{j+1}}(\tau_{j+1}-), \zeta_{i_{j+1}}) + \int_0^{\tau_{j+1}} E^{\tilde{X}^{(i_{j+1})}_{i_{j+1}}(\tau_{j+1})} \left[ e^{-\tau_{j+1} f(X(t))} dt \right] dF_{i_{j+1}}(s)
+ \int_0^{\tau_{j+1}} E^{\tilde{X}^{(i_{j+1})}_{i_{j+1}}(\tau_{j+1})} \left[ e^{-\tau_{j+1} \varphi_{n-j-1}(X(t))} \right] dF_{i_{j+1}}(s). \quad (A.6)
$$

Moreover, by the definition of $\varphi_j(x)$, for $j = 0, 1, \ldots, n-1$, we have

$$
\varphi_{n-j}(x) = \inf_{\tau \geq 0} E^{\tau} \left[ \int_0^\tau e^{-\tau f(X(t))} dt + \mathcal{M}_\delta \varphi_{n-j-1}(X(\tau)) \right] \leq E^{\tau} \left[ \int_0^\tau e^{-\tau f(X(t))} dt + \mathcal{M}_\delta \varphi_{n-j-1}(X(0)) \right] = \mathcal{M}_\delta \varphi_{n-j-1}(x).
$$

Combining (A.5) and (A.6) yields

$$
\varphi_{n-j}(\tilde{X}^{(i_{j+1})}_{i_{j+1}}(\tau_{j+1}-)) \leq K_{i_{j+1}}(\tilde{X}^{(i_{j+1})}_{i_{j+1}}(\tau_{j+1}-), \zeta_{i_{j+1}}) + \int_0^{\tau_{j+1}} E^{\tilde{X}^{(i_{j+1})}_{i_{j+1}}(\tau_{j+1})} \left[ e^{-\tau_{j+1} f(X(t))} dt \right] dF_{i_{j+1}}(s)
+ \int_0^{\tau_{j+1}} E^{\tilde{X}^{(i_{j+1})}_{i_{j+1}}(\tau_{j+1})} \left[ e^{-\tau_{j+1} \varphi_{n-j-1}(X(t))} \right] dF_{i_{j+1}}(s).
$$
Multiplying the above inequality by $e^{-\tau_j t}$ and then taking the expectation, we obtain the inequality below.

$$E^x \left[ \varphi_{n-j}^d \left( \hat{X}^{(u_n)}(\tau_{j+1}^-) \right) \right] \leq E^x \left[ e^{-\tau_j t_j} K_{ij} \left( \hat{X}^{(u_n)}(\tau_{j+1}^-), \zeta_{j+1} \right) \right] + \int_0^{\tau_j} E^x \left[ \int_{\tau_{j+1}}^{\tau_{j}+\tau} f^d \left( X^{(u_n)}(t) \right) dt \right] dF_{ij}(s)
+ \int_0^{\tau_j} E^x \left[ \varphi_{n-j-1}^d \left( X^{(u_n)}(\tau_{j+1}^- + s) \right) \right] dF_{ij}(s).
\quad (A.7)$$

Thus, we have an upper bound on $E^x \left[ \varphi_{n-j}^d \left( \hat{X}^{(u_n)}(\tau_{j+1}^-) \right) \right]$. Now, combining (A.2) and (A.7), for $j = 1, ..., n - 1$, we deduce that

$$\int_0^{\tau_j} E^x \left[ \varphi_{n-j}^d \left( X^{(u_n)}(\tau_j + s) \right) \right] dF_{ij}(s) \leq \int_0^{\tau_j} E^x \left[ \int_{\tau_{j+1}}^{\tau_j} f^d \left( X^{(u_n)}(t) \right) dt \right] dF_{ij}(s) + E^x \left[ e^{-\tau_j t_j} K_{ij} \left( \hat{X}^{(u_n)}(\tau_{j+1}^-), \zeta_{j+1} \right) \right]
+ \int_0^{\tau_j} E^x \left[ \varphi_{n-j-1}^d \left( X^{(u_n)}(\tau_{j+1}^- + s) \right) \right] dF_{ij}(s).
\quad (A.8)$$

Equivalently, we have

$$\int_0^{\tau_j} E^x \left[ \varphi_{n-j}^d \left( X^{(u_n)}(\tau_j + s) \right) \right] dF_{ij}(s) - \int_0^{\tau_j} E^x \left[ \varphi_{n-j-1}^d \left( X^{(u_n)}(\tau_{j+1}^- + s) \right) \right] dF_{ij}(s)
\leq \int_0^{\tau_j} E^x \left[ \int_{\tau_{j+1}}^{\tau_j} f^d \left( X^{(u_n)}(t) \right) dt \right] dF_{ij}(s) + E^x \left[ e^{-\tau_j t_j} K_{ij} \left( \hat{X}^{(u_n)}(\tau_{j+1}^-), \zeta_{j+1} \right) \right]
+ \int_0^{\tau_j} E^x \left[ \varphi_{n-j-1}^d \left( X^{(u_n)}(\tau_{j+1}^- + s) \right) \right] dF_{ij}(s).$$

Summing from $j = 1$ to $j = n - 1$, we get

$$\int_0^{\tau_1} E^x \left[ \varphi_{n-1}^d \left( X^{(u_n)}(\tau_1 + s) \right) \right] dF_{i1}(s) - \int_0^{\tau_1} E^x \left[ \varphi_0^d \left( X^{(u_n)}(\tau_n + s) \right) \right] dF_{i1}(s)
\leq \sum_{j=1}^{n-1} \int_0^{\tau_j} E^x \left[ \int_{\tau_{j+1}}^{\tau_j} f^d \left( X^{(u_n)}(t) \right) dt \right] dF_{ij}(s) + \sum_{j=2}^{n} E^x \left[ e^{-\tau_j t_j} K_{ij} \left( \hat{X}^{(u_n)}(\tau_j^-), \zeta_j \right) \right]
+ \sum_{j=2}^{n} \int_0^{\tau_j} E^x \left[ \varphi_{n-j-1}^d \left( X^{(u_n)}(\tau_{j+1}^- + s) \right) \right] dF_{ij}(s).
\quad (A.8)$$

Next, again by Lemma 1(a), we have

$$E^x \left[ \varphi_{n}^d \left( X^{(u_n)}(0) \right) \right] \leq E^x \left[ \int_0^{\tau_1} f^d \left( X^{(u_n)}(t) \right) dt + \varphi_{n}^d \left( \hat{X}^{(u_n)}(\tau_1^-) \right) \right].
\quad (A.9)$$

Using $E^x \left[ \varphi_{n}^d \left( X^{(u_n)}(0) \right) \right] = \varphi_n(x)$ and (A.7) with $j = 0$, we get

$$\varphi_n(x) - \int_0^{\tau_1} E^x \left[ \varphi_{n-1}^d \left( X^{(u_n)}(\tau_1 + s) \right) \right] dF_{i1}(s)
\leq E^x \left[ \int_0^{\tau_1} f^d \left( X^{(u_n)}(t) \right) dt \right] + E^x \left[ e^{-\tau_1 t_1} K_{i1} \left( \hat{X}^{(u_n)}(\tau_1^-), \zeta_1 \right) \right]
+ \int_0^{\tau_1} E^x \left[ \varphi_{n-1}^d \left( X^{(u_n)}(\tau_1^- + s) \right) \right] dF_{i1}(s).
\quad (A.10)$$
Adding (A.8) and (A.10) yields
\[
\varphi_n(x) = \int_0^\infty E^x \left[ \varphi_0^\varepsilon \left( X^{(i_n)}(\tau_n + s) \right) \right] dF_{i_n}(s)
\]
\[
\leq \sum_{j=1}^{n-1} \int_0^\infty E^x \left[ \int_{\tau_{j+1}}^{\tau_j} f^d \left( X^{(i_n)}(t) \right) dt \right] dF_{i_j}(s) + \sum_{j=1}^n E^x \left[ e^{-\tau_j} K_i \left( \hat{X}^{(i_n)}(\tau_j), \zeta_j \right) \right]
\]
\[
+ \sum_{j=1}^n \int_0^\infty E^x \left[ \int_{\tau_j}^{\tau_{j+1}} f^d \left( X^{(i_n)}(t) \right) dt \right] dF_{i_j}(s) + \int_0^\infty E^x \left[ \int_{\tau_1}^{\tau_2} f^d \left( X^{(i_n)}(t) \right) dt \right] .
\]  
(A.11)

Moreover, we observe that
\[
\int_0^\infty E^x \left[ \varphi_0^\varepsilon \left( X^{(i_n)}(\tau_n + s) \right) \right] dF_{i_n}(s) = \int_0^\infty E^x \left[ e^{-\tau_n} E^{X^{(i_n)}(\tau_n + s)} \left[ \int_0^\infty e^{-\tau} f \left( X^{(i_n)}(t) \right) dt \right] \right] dF_{i_n}(s)
\]
\[
= \int_0^\infty E^x \left[ \int_{\tau_n}^{\tau_{n+1}} f^d \left( X^{(i_n)}(t) \right) dt \right] dF_{i_n}(s),
\]
where the second equality is due to the strong Markov property.

Now, substituting the above equality in (A.11) and collecting the similar terms together, we obtain
\[
\varphi_n(x) \leq E^x \left[ \int_0^\tau f^d \left( X^{(i_n)}(t) \right) dt \right] + \sum_{j=1}^{n-1} \int_0^\tau E^x \left[ \int_{\tau_{j+1}}^{\tau_j} f^d \left( X^{(i_n)}(t) \right) dt \right] dF_{i_j}(s)
\]
\[
+ \int_0^\tau E^x \left[ \int_{\tau_n}^{\infty} f^d \left( X^{(i_n)}(t) \right) dt \right] dF_{i_n}(s) + \sum_{j=1}^n E^x \left[ e^{-\tau_j} K_i \left( \hat{X}^{(i_n)}(\tau_j), \zeta_j \right) \right]
\]
\[
= E^x \left[ \int_0^\tau f^d \left( X^{(i_n)}(t) \right) dt \right] + \sum_{j=1}^n E^x \left[ e^{-\tau_j} K_i \left( \hat{X}^{(i_n)}(\tau_j), \zeta_j \right) \right]
\]
\[
= E^x \left[ \int_0^\tau e^{-\tau} f \left( X^{(i_n)}(t) \right) dt \right] + \sum_{j=1}^n e^{-\tau_j} K_i \left( \hat{X}^{(i_n)}(\tau_j), \zeta_j \right)
\]
\[
= e^{\tau_n} f^{(i_n)}(x),
\]  
(A.12)

where we have used the facts that \( \mathcal{T}_j^\varepsilon \) is independent of the jump diffusion process and \( \int_0^\infty dF_{i_j}(s) = 1, \ j = 1, ..., n. \)

Finally, since \( u_0 \in \mathcal{U}_0 \) is an arbitrary element, we have
\[
\varphi_n(x) \leq \inf \{ f^{(i_n)}(x); u_0 \in \mathcal{U}_0 \} = \Phi_n(x).
\]  
(A.13)

Next, we prove the opposite of the above inequality, i.e., \( \varphi_n(x) \geq \Phi_n(x) \). For this, we construct an increasing sequence of stopping times, \( 0 = \tau_0 < \tau_1 < \cdots < \tau_n \), inductively. First, let \( \varepsilon > 0 \), and for \( j = 1, 2, ..., n \), consider
\[
D_\varepsilon^j := \left\{ y : \varphi_j(y) < M_j \varphi_j - 1(y) - \varepsilon \right\}.
\]  
(A.14)

Define \( \tau_1 = \inf \left\{ t > 0 : X_j(t) \in D_\varepsilon^{(i_1)} \right\} \), and choose \( (\hat{i}_1, \xi_1) \) such that
\[
M_{\varphi_n-1} \left( \hat{X}^{(i_n)}(\tau_1) \right) \geq K_i \left( \hat{X}^{(i_n)}(\tau_1), \xi_1 \right)
\]
\[
+ \int_0^{\tau_1} E^{\hat{i}_1} \left[ \int_0^{\tau_1} e^{-\tau} f \left( \hat{X}^{(i_1)}(t) \right) dt \right] + e^{-\tau_1} \varphi_n-1 \left( \hat{X}^{(i_1)}(\tau_1) \right) dF_{i_1}(s) - \varepsilon.
\]  
(A.15)

The idea behind this construction is similar to \( \varepsilon \)-optimality idea introduced in Øksendal and Sulem (2007). However, the construction in our case is more general than theirs as we now have random reaction periods as well as the multiple types of interventions.
Now, we extend this idea for an arbitrary term \((j + 1)\). For this, suppose that \(0 = \hat{\tau}_0, \ldots, \hat{\tau}_j\) and \((\hat{\tau}_1, \hat{\xi}_1), \ldots, (\hat{\tau}_j, \hat{\xi}_j)\) have been defined, where \(j \leq n - 1\). Let \(X^{(i)}_j(t)\) be the process obtained by applying the impulse control \(\hat{u}_j = (\hat{\tau}_1, \ldots, \hat{\tau}_j, (\hat{\tau}_i, \hat{\xi}_i), \ldots, (\hat{\tau}_j, \hat{\xi}_j))\) to \(X_j(t)\). Then, we let \(\hat{\tau}_{j+1} := \inf\{t > \hat{\tau}_j + T_{j+1} : X^{(i)}_j(t) \not\in D^{(e)}_{n-j}\}\), and for \(j = 0, 1, \ldots, n - 1\), choose \((\hat{\tau}_{j+1}, \hat{\xi}_{j+1})\) such that

\[
M_i \varphi_{n-j-1} (\hat{X}^{(i)}_n(\hat{\tau}_{j+1}^-), \hat{\xi}_{j+1}) + \int_0^{\hat{\tau}_{j+1}^-} E^\gamma_i (\hat{X}^{(i)}_n(\hat{\tau}_{j+1}^-), \hat{\xi}_{j+1}) \left[ \int_0^t e^{-r\tau} f (X^{(i)}_j(t)) \, dt + e^{-r\hat{\tau}_{j+1}^-} \varphi_{n-j-1} (X^{(i)}_j(s)) \right] dF_{j+1} (s) = \varepsilon. \tag{A.16}
\]

Finally, let \(\hat{\tau}_{n+1} := \infty\) and define \(\hat{u}_n = (\hat{\tau}_1, \ldots, \hat{\tau}_n, (\hat{\tau}_1, \hat{\xi}_1), \ldots, (\hat{\tau}_n, \hat{\xi}_n)) \in \mathcal{U}_n\). Then, applying an argument similar to that leading up to (A.13) with the impulse control \(\hat{u}_n\), we can prove the desired result. Thus, we will be brief and only highlight the differences in the proof.

First note that, by applying Lemma 1(b), we now have the equality in (A.1). Hence, for \(j = 1, \ldots, n - 1\), (A.2) becomes

\[
\int_0^{\tau_{j+1}^-} E^i \left[ \varphi_{n-j} (X^{(i)}_j(\hat{\tau}_j + s)) \right] dF_{j+1} (s) = \int_0^{\tau_{j+1}^-} E^i \left[ \int_{\hat{\tau}_{j+1}^-}^{\tau_{j+1}^-} f^d (X^{(i)}_j(t)) \, dt \right] dF_{j+1} (s) + E^i \varphi_{n-j} (X^{(i)}_j(\tau_{j+1}^-)). \tag{A.17}
\]

The equation (A.3) will now be replaced by the optimality equation in (A.16). Then, for \(j = 0, 1, \ldots, n - 1\), the argument in (A.3)–(A.5) gives the following inequality

\[
M_i \varphi_{n-j-1} (\hat{X}^{(i)}_n(\hat{\tau}_{j+1}^-), \hat{\xi}_{j+1}) + \int_0^{\tau_{j+1}^-} E^i \left[ \int_{\hat{\tau}_{j+1}^-}^{\tau_{j+1}^-} e^{-r\tau} f (X^{(i)}_j(t)) \, dt \right] dF_{j+1} (s) + \int_0^{\tau_{j+1}^-} E^i \varphi_{n-j-1} (X^{(i)}_j(\hat{\tau}_{j+1}^-)) dF_{j+1} (s) = \varepsilon. \tag{A.18}
\]

Next, note that \(\hat{X}^{(i)}_n(\hat{\tau}_{j+1}^-) \not\in D^{(e)}_{n-j}\). Then, by the definition of \(D^{(e)}_{n-j}\), for \(j = 0, 1, \ldots, n - 1\), we conclude that

\[
\varphi_{n-j} (\hat{X}^{(i)}_n(\hat{\tau}_{j+1}^-)) = M_i \varphi_{n-j-1} (\hat{X}^{(i)}_n(\hat{\tau}_{j+1}^-)) - \varepsilon \tag{A.19}
\]

Combining (A.18) and (A.19) first, and then multiplying the resulting inequality by \(e^{-r\hat{\tau}_{j+1}^-}\) and taking the expectation, for \(j = 0, 1, \ldots, n - 1\), we can derive the inequality below.

\[
E^i \left[ \varphi_{n-j} (X^{(i)}_j(\hat{\tau}_j - s)) \right] \geq E^i \left[ e^{-r\tau_{j+1}^-} K_{j+1} (\hat{X}^{(i)}_n(\hat{\tau}_{j+1}^-), \hat{\xi}_{j+1}) \right] + \int_0^{\tau_{j+1}^-} E^i \left[ \int_{\hat{\tau}_{j+1}^-}^{\tau_{j+1}^-} f^d (X^{(i)}_j(t)) \, dt \right] dF_{j+1} (s) + \int_0^{\tau_{j+1}^-} E^i \varphi_{n-j} (X^{(i)}_j(\tau_{j+1}^-)) dF_{j+1} (s) = 2E^i \left[ e^{-r\tau_{j+1}^-} \right]. \tag{A.20}
\]

Now, applying the argument in (A.7)–(A.8) with (A.17) replacing (A.2), we can easily deduce that

\[
\int_0^{\tau_{j+1}^-} E^i \left[ \varphi_{n-j} (X^{(i)}_j(\hat{\tau}_j + s)) \right] dF_{j+1} (s) - \int_0^{\tau_{j+1}^-} E^i \left[ \varphi_{n} (X^{(i)}_j(\hat{\tau}_j + s)) \right] dF_{j+1} (s) \geq \sum_{j=1}^{n-1} \int_0^{\tau_{j+1}^-} E^i \left[ \int_{\hat{\tau}_{j+1}^-}^{\tau_{j+1}^-} f^d (X^{(i)}_j(t)) \, dt \right] dF_{j+1} (s) + \sum_{j=1}^{n} E^i \left[ e^{-r\tau_{j+1}^-} K_{j+1} (\hat{X}^{(i)}_n(\hat{\tau}_{j+1}^-), \hat{\xi}_{j+1}) \right] \nonumber
\]

\[
+ \sum_{j=1}^{n-1} \int_0^{\tau_{j+1}^-} E^i \left[ \int_{\hat{\tau}_{j+1}^-}^{\tau_{j+1}^-} f^d (X^{(i)}_j(t)) \, dt \right] dF_{j+1} (s) - 2E^i \left[ e^{-r\tau_{j+1}^-} \right]. \tag{A.21}
\]

Note again that, by applying Lemma 1(b), we now have the equality in (A.9), and it follows that

\[
\varphi_{n} (x) = E^i \left[ \varphi_{n-j} (X^{(i)}_j(\hat{\tau}_j - s)) \right] = E^i \left[ \int_{\hat{\tau}_j}^{\tau_{j+1}^-} f^d (X^{(i)}_j(t)) \, dt \right]. \tag{A.22}
\]
We then combine (A.20) when $j = 0$, with (A.22) and obtain that
\[
\phi_n(x) - \int_0^\infty E^x \left[ \phi_{n-1} \left( X^{(i_k)}(\hat{\tau}_1 + s) \right) \right] dF_i(s) \geq E^x \left[ \int_0^{\hat{\tau}_1} f^d \left( X^{(i_k)}(t) \right) dt \right] + E^x \left[ e^{-r\hat{\tau}_j} K_i \left( X^{(i_k)}(\hat{\tau}_1 -), \hat{\xi}_1 \right) \right] \\
+ \int_0^\infty E^x \left[ \int_{\hat{\tau}_1}^{\hat{\tau}_1 + s} f^d \left( X^{(i_k)}(t) \right) dt \right] dF_i(s) - 2\epsilon E^x \left[ e^{-r\hat{\tau}_1} \right].
\] (A.23)

Adding (A.21) and (A.23), and then following the argument in (A.11)–(A.12), we obtain that
\[
\phi_n(x) \geq j^{(i_k)}(x) - 2\epsilon \sum_{j=1}^n E^x \left[ e^{-r\hat{\tau}_j} \right] \\
\geq j^{(i_k)}(x) - 2\epsilon n,
\]
where the last inequality is true as $E^x \left[ e^{-r\hat{\tau}_j} \right] \leq 1$ for all $j \in \{1, 2, \cdots, n\}$.

Since $\epsilon > 0$ is an arbitrary number, from the above inequality, we have
\[
\phi_n(x) \geq \Phi_n(x).
\] (A.24)

Consequently, from (A.13) and (A.24), we have the desired equality, i.e., $\phi_n(x) = \Phi_n(x).$ $\square$

References
