An approximation scheme for impulse control with random reaction periods

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ABSTRACT

We propose an approximation scheme for impulse control models with random reaction periods (ICRRP) and show that the optimal solutions can be found by solving a sequence of optimal stopping problems. Our work enhances viability of the existing ICRRP framework for applications as well as the general literature on stochastic control theory. The efficacy of our approximation scheme is validated by applying it to compute a market-reaction-adjusted optimal central bank intervention policy for a country.

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1. Introduction

Impulse Control with Random Reaction Periods (ICRRP) is used to derive an optimal foreign exchange (forex) market intervention policy for a country when the forex market reacts to the interventions by the central bank of the country. The first model for ICRRP is proposed in [4] for diffusion processes which is later extended in [40] for jump diffusion processes. In [4], the authors incorporate the forex market reactions to the classical foreign exchange rate models studied in [13] and [32], and suggest that the central bank would intervene less frequently (more frequently) and the optimal policy is more (less) expensive than its corresponding value without the market reactions if the market reactions increase (decrease) the exchange rate volatility. The ICRRP framework is recently extended for multi-dimensional jump diffusion processes in [40] using a novel minimum cost operator that simplifies the computations of the optimal solutions. Moreover, the authors show that market reactions and the jumps in the forex market are complements when the reactions increase the forex rate volatility; otherwise, they are substitutes.

Although the ICRRP framework is introduced and established in a more general setting than the forex rate models in both [4] and [40], the solutions of the impulse control problem (i.e., the solutions of the associated quasi-variational inequalities) are provided only for the exchange rate models. However, both of the above articles explain the importance of their general framework by providing other applications of ICRRP framework such as mutual fund cash inventory management problems and machine replacement problems. Moreover, it is very well accepted that the impulse control problems are not easy to solve in general, and it is extremely difficult to solve these problems when both the market reactions and jump diffusions are incorporated (which is the framework in [40]). Hence, an approximation scheme is desirable for impulse control models with random reaction periods and that is the contribution of our research.

We show that we can approximate an impulse control problem with random reaction periods with a sequence of optimal stopping problems when the number of interventions in the original problem is finite; hence, we are able to transform the difficult (or sometimes impossible) task of solving an impulse control problem with random reaction periods into a relatively easy task where we only have to solve a sequence of optimal stopping problems.

Our approximation scheme has a similar flavor to that of [36]. However, our scheme considers impulse control models where the controller’s action affects the state as well as the dynamics of the state process for a random amount of time; in other words, we allow for random reaction periods. Thus, our approximation scheme is defined using the reaction adjusted minimum cost operator proposed in [40]. Additionally, we allow the controller to take different types of interventions at the time of an intervention as oppose to [36] where the authors only allow one type of intervention; these different types of interventions generate different types of reactions in our model. Finally, it should be noted that our model subsumes the model studied in [36] when there is no market reactions and hence our research also enhances the
existing seminal work in [36] within the area of stochastic control theory.

The rest of the article is organized as follows. A brief literature review with current applications of impulse control models is presented in Section 2. Section 3 summarizes the ICRP Model studied in [40]. In Section 4, we introduce our approximation scheme for ICRP and establish the main results. We solve a central bank intervention problem with market reactions in Section 5 to demonstrate the efficacy of our approximation scheme. Some concluding remarks are provided in Section 6.

2. Related literature and applications

Stochastic impulse control models have constantly received significant interest in the area of operations research since its introduction in [2]. In particular, impulse control models have been widely used to study problems related to inventory management, risk management, portfolio management, real options and optimal central bank intervention policies in the foreign exchange market. One of the main and arguably the most unique characteristic in the impulse control framework is that it accommodates the fixed costs associated with the actions taken by the decision maker or the controller of the underlying model. This distinctive feature (compared to the traditional stochastic control models) makes an impulse control model a better candidate for applications where the controller’s actions is associated with fixed costs. Interestingly, the fixed costs do exists virtually in every practical scenario nowadays and thus impulse control models have become a fundamental instrument in the area of operations research.

In inventory management, it is well known that just-in-time (JIT) policies are optimal in the absence of fixed costs of replenishments (see [43] and [49]); this result is independent of the demand process given that there is ample and instantaneous supply. However, the most of the practically interesting ordering/procurement cost structures have both fixed and proportional costs, and hence JIT policies cease to be optimal under these cost functions. The simplest model with such a cost structure is the classical Economic Order Quantity (EOQ) model proposed in [20]. Surprisingly, the first publication dedicated for a rigorous proof of the optimality of EOQ formula, i.e., [6], uses the impulse control framework. When the demand processes are stochastic and more general types of ordering cost functions (with fixed costs) are considered, the optimality of the EOQ formula, i.e., [6], usesthe impulse control framework.

3. Impulse control model with random reaction periods

We consider the jump diffusion model studied in [40] as the underlying impulse control model. This model also represents an extension (to the jump diffusion case) of the impulse control model with random reaction periods proposed in [4]. Since both articles [4] and [40] explain the model dynamics in detail, we briefly summarize the model here and refer the reader to these references for a full description of the model.

Consider a complete probability space \((\Omega, \mathcal{F}, P)\) equipped with a filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) satisfying the usual assumptions. If there are no interventions, the state of the system is a \(k\)-dimensional jump diffusion process \(X(t)\) of the form

\[
dX(t) = \mu(X(t))dt + \sigma(X(t))dB(t) + \int_{\mathbb{R}} \gamma(X(t-), z)\tilde{N}(dt, dz),
\]

where \(X(0) = x \in \mathbb{R}^k\), \(\mu : \mathbb{R}^k \to \mathbb{R}^k\), \(\sigma : \mathbb{R}^k \to \mathbb{R}^{k \times m}\) and \(\gamma : \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^{k \times l}\) are measurable functions such that a unique solution—\(X(t)\)—exists; \(B(t)\) is an \(m\)-dimensional Brownian motion and \(\tilde{N}(dt, dz) := (\tilde{N}_i(dt, dz_1), \ldots, \tilde{N}_i(dt, dz_d))\), where \(\tilde{N}_i(ds, dz_j) := \nu(ds, dz_j)\) for \(j = 1, \ldots, l\), are independent Poisson random measures with corresponding Lévy measures \(\nu_j\) coming from \(l\) independent \((1\text{-dimensional})\) Lévy processes.

The controller is allowed to intervene the state process with an impulse \((i, \xi) \in \{1, 2, \ldots, p\} \times \mathbb{Z}^d \subset \mathbb{N} \times \mathbb{R}\) of type \(i\) and size \(\xi\), where \(\mathbb{Z}\) is the set of admissible impulse values associated with an intervention of type \(i\). If we apply an impulse \((i, \xi)\) when the system is in state \(x\) at time \(t\) with the dynamics \([1]\), then the state jumps immediately from \(X(t-) = x\) to \(X(t) = F^i(x, \xi)\), where \(F^i : \mathbb{R}^k \times \mathbb{Z}^d \to \mathbb{R}^k\) is a given function. Moreover, the dynamics switches to

\[
dX_i(t) = \mu^i(X(t))dt + \sigma^i(X(t))dB(t) + \int_{\mathbb{R}} \gamma^i(X(t-), z)\tilde{N}(dt, dz),
\]

where \(\mu^i : \mathbb{R}^k \to \mathbb{R}^k\), \(\sigma^i : \mathbb{R}^k \to \mathbb{R}^{k \times m}\) and \(\gamma^i : \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^{k \times l}\) are given functions such that unique solution—\(X^i(t)\)—exists, and it persists for a random period of time \(T^i\) and then reverts to the pre-intervention dynamics in \([1]\). (Here, it is implicitly assumed that the controller observes the change in dynamics of the end of the reaction period; however, the detection of the exact time of the change in dynamics still remains an open problem.) For \(j = 1, 2, \ldots, l\), let \(\tau_j\) and \(\xi_j\) respectively denote the time and the type of the \(j\)th intervention. Thus, the \(j\)th intervention will affect the state dynamics during the reaction period \((\tau_j, \tau_j + T^i_j]\), where \(T^i_j\) is a non-negative random variable for \(j \in \{1, 2, \ldots, p\}\). Moreover, we assume that \(T^i_j \sim T^i\) for all \(j\), where \(T^i\) has the distribution function \(F_i\) for \(i = 1, 2, \ldots, p\), and \(T^i_j\) is independent of the jump diffusion process.

An impulse control is defined as below.

**Definition 1.** An impulse control is a double sequence \(u = (\mathbf{r}_1, \mathbf{r}_2, \ldots ; (i_1, \xi_1), (i_2, \xi_2), \ldots , (i_p, \xi_p), \ldots\), where \(\tau_1 < \tau_2 < \ldots \) are \(\mathcal{F}_t\)-stopping times (the intervention times), and \(i_1, \xi_2, \ldots \) and \(i_p, \xi_p, \ldots\) are the corresponding intervention types and impulses, respectively, at these times such that each \(i_j\) and \(\xi_j\) are \(\mathcal{F}_\tau\)-measurable.
Now let $\Delta X(t)$ denote the jump of $X$ resulting from the jump of the random measure $N(t, \cdot)$ only. Then, if we apply an impulse control $u$ to the state process $X(t)$ with $X(0) = x$, then the resulting intervened process $X_{\cdot}(t)$ can be characterized as follows:

$$X_{\cdot}^{(u)}(t) = X_0(t); \quad 0 \leq t \leq \tau_1,$$

$$X_{\cdot}^{(u)}(\tau_j) = f_j(X_{\cdot}^{(u)}(\tau_{j-1}) + \Delta X(\tau_j), \xi_j); \quad j = 1, 2, \ldots, \tag{3}$$

$$X_{\cdot}^{(u)}(t) = X_{\cdot}^{(u)}(\tau_{j+1}) \quad \text{for} \quad \tau_j \leq t \leq \tau_j + \tau_j^3; \quad j = 1, 2, \ldots, \tag{4}$$

$$X_{\cdot}^{(u)}(t) = X_{\cdot}^{(u)}(\tau_{j+1}) \quad \text{for} \quad \tau_j + \tau_j^3 < t < \tau_{j+1}; \quad j = 1, 2, \ldots, \tag{5}$$

The performance measure in this case is the total expected discounted long-run cost which is defined by

$$f^{(u)}(x) = E^x \left[ \int_0^{\infty} e^{-rt} f(X_{\cdot}(t)) dt \right] + \sum_{j=1}^{\infty} e^{-r\tau_j} K_j(X_{\cdot}(\tau_j), \xi_j), \tag{7}$$

where $r > 0$ is the discount rate, $f : \mathbb{R}^k \rightarrow \mathbb{R}$ is the running cost function, $K_j(x, \xi) : \mathbb{R}^k \times \mathbb{Z}^l \rightarrow \mathbb{R}$ is the cost of making an intervention of type $i$ with impulse $\xi_j \in \mathbb{Z}^l$, and $E^x$ is the expectation operator with respect to the probability law of $X_{\cdot}(0)$ starting at $x$, i.e., $X_{\cdot}(0) = x$.

The set of admissible controls for the impulse control problem is defined similarly to [40].

**Definition 2.** The set $\mathcal{U}$ of impulse controls is called admissible, if for all $x \in \mathbb{R}^k$, $u \in \mathcal{U}$,

(i) a unique solution $X_{\cdot}^{(u)}(t)$, $t \geq 0$, of the system (3)–(6) exists,

(ii) $\lim_{t \rightarrow \infty} \tau_j = \infty$ a.s.,

(iii) $E \left[ \int_0^{\infty} e^{-rt} f(X_{\cdot}^{(u)}(t)) dt \right] < \infty$,

(iv) $E \left[ \sum_{j=1}^{\infty} e^{-r\tau_j} K_j(X_{\cdot}^{(u)}(\tau_j), \xi_j) \right] < \infty$, and

(v) $\tau_{j+1} - \tau_j > \tau_j^3$, $j = 1, 2, \ldots$.

The impulse control problem is to minimize $f^{(u)}$ over $\mathcal{U}$. Let $\Phi$ denote the value function of the impulse control problem. Then, for all $x \in \mathbb{R}^k$,

$$\Phi(x) = \inf_{u \in \mathcal{U}} \{ f^{(u)}(x) \} = f^{(\ast u)}(x).$$

Thus, the impulse control problem (with random reaction periods) is to find the minimum cost $\Phi(x)$ and the corresponding optimal control $u^* \in \mathcal{U}$. This problem is completely studied in [40]. In [40], the authors present the Quasi Integro-Variational Inequalities (QIVI) for this impulse control problem and provide an inductive method for obtaining the QIVI-control associated with their QIVI; moreover, they provide a verification theorem for impulse control model with random reaction periods. However, the solution of the QIVI is presented only for the central bank intervention application. In general, the solution of the QIVI is not easy to compute and hence an approximation scheme is desirable. We present our approximation scheme next.

**4. An approximation scheme for impulse control with random reaction periods**

Our approximation scheme is defined parallel to that of [36]. We also assume that only (up to) a finite number of interventions are allowed; let $n$ be the number of interventions allowed. In [36], the minimum cost operator does not incorporate market reactions whereas the reaction adjusted minimum cost operator (proposed in [40]) is used in our approximation scheme; this is the main difference between our approximation scheme and that of [36]. Although this replacement of the operator appears to be very natural and straightforward, the proof of the main result involves more complicated and somewhat different arguments due to the general nature of our approximation scheme. However, the classical result in [36] is still subsumed in our work. Thus, our research not only enhances the viability of the existing ICRRP framework but also generalizes the existing work in [36] within the area of optimal control theory.

We now introduce several definitions and an important result that is needed in order to prove our main result. Let $\mathcal{L}_n$ be the set of all admissible controls with at most $n$ interventions. That is, for $n = 1, 2, \ldots$,

$$\mathcal{L}_n := \{ u \in \mathcal{U} : u = (\tau_1, \ldots, \tau_n, \tau_{n+1}; (i_1, \xi_1), \ldots, (i_n, \xi_n)) \}$$

with $\tau_{n+1} = \infty$ a.s.

Observe that if there is no admissible $u \in \mathcal{U}$ with at most $n$ interventions, then $\mathcal{L}_n$ will be empty, and thus, our approximation scheme will not be valid. Therefore, we assume that $\mathcal{L}_n$ is non-empty. However, this assumption is only needed for simplicity of our analysis. For example, one could relax this assumption by defining the underlying impulse control problem as in Chapter 7 of [36]; then, somewhat intricate yet parallel arguments to ours will yield the results of this paper.

Moreover, for $n = 1, 2, \ldots$, let

$$\Phi_n(x) := \inf_{u \in \mathcal{L}_n} f^{(u)}(x); \quad u \in \mathcal{L}_n.$$

Then, clearly $\mathcal{L}_n \subseteq \mathcal{L}_{n+1} \subseteq \mathcal{U}$ for all $n$, and thus we have $\Phi_n(x) \geq \Phi_{n+1}(x) \geq \Phi(x)$. Furthermore, with obvious modifications to the proof of Lemma 7.1 in [36], we can deduce the following result.

**Proposition 1.** For all $x \in \mathbb{R}^k$, $\lim_{n \rightarrow \infty} \Phi_n(x) = \Phi(x)$.

Next, we recall the definition of the reaction adjusted minimum cost operator, $M_r$, proposed in [40].

**Definition 3.** Let $M_r \phi(x) := \inf_{\mathcal{L}} \{ K(x, \xi) + G(x, \xi) ; \xi \in \mathbb{Z}^l \}$, where $G(x, \xi) := \int_0^{\tau} E^{(x, \xi)} \left[ \int_0^{T_i} e^{-rT} f(X(T)) dt \right] dF_i(t)$.

Then, $M_r \phi(x) := \min \{ M_r \phi(x) ; i \in \{ 1, 2, \ldots, p \} \}$.

Now, for $j = 1, 2, \ldots, n$, consider

$$\psi_j(x) = \inf_{\tau \geq 0} E^x \left[ \int_0^{\tau} e^{-rT} f(X(T)) dt + M_r \psi_{j-1}(X(T)) \right]$$

with $\psi_0(x) := E^x \left[ \int_0^{\infty} e^{-rT} f(X(T)) dt \right]$. Under some mild conditions on $r$ and $M_r \psi_{j-1}$, we will prove that $\psi_n = \Phi_n$. Thus, from Proposition 1, we have that $\lim_{n \rightarrow \infty} \psi_n(x) = \Phi(x)$. Since each $\psi_j$ is associated with an optimal stopping problem, we can approximate our original impulse control problem by a sequence of optimal stopping problems. We next state an important definition and the main approximation result.

**Definition 4.** Let $h$ be a real-valued function. We say that $h$ has at most polynomial growth if there exists constants $C$ and $m = m(h)$ such that $|h(x)| \leq C (1 + |x|^m)$ for all $x \in \mathbb{R}^k$.

**Theorem 1.** Let $f$ and $M_r \psi_{j-1}$, for $j = 1, 2, \ldots, n$, have at most polynomial growth. Then $\psi_n = \Phi_n$.

The proof of Theorem 1 is presented in an online appendix [42], and the discussion below uses the equation numbers from the
appendix. First note that the online proof subsumes a search
mechanism for the optimal impulse control. Specifically, if we can find an impulse control \( \hat{u}_n \) such that the equality holds in (A.15) and (A.16), with \( \epsilon = 0 \) in (A.14)-(A.16), then \( \hat{u}_n \) is indeed optimal. The optimal control problem with this \( \hat{u}_n = (\hat{t}_1, \ldots, \hat{t}_n; \hat{\xi}_1, \hat{\xi}_2) \) is implemented as follows. The first intervention, \( \hat{t}_1 \), is applied when \( X_n \) exits the set \( D_0^{(0)} \) for the first time; the type and the size of this intervention (i.e., \( \hat{\xi}_1, \hat{\xi}_2 \)) will be determined such that the equality holds in (A.15) with \( \epsilon = 0 \). Then, wait until the duration period is over (i.e., \( t > t_1^* + T_1^* \)) and the process \( X_1^{(1)} \) exits the set \( D_0^{(0)} \) for the first time to implement the second intervention, \( \hat{t}_2^* \); moreover, \( (\hat{t}_2^*, \hat{\xi}_2) \) will be determined such that the equality holds in (A.16) with \( \epsilon = 0 \) and \( j = 1 \). We continue this procedure until the \( n \)th intervention, \( \hat{t}_n \), which is applied when the process \( X_{n-1}^{(n-1)} \) exits the set \( D_0^{(0)} \) for the first time after \( \hat{t}_n^{*} + T_{n-1}^* \); the values of \( \hat{t}_n \) and \( \hat{\xi}_n \) will be determined such that the equality holds in (A.16) with \( \epsilon = 0 \) and \( j = n - 1 \). Finally, observe that each step of the construction of the optimal policy involves an optimal stopping (or exit) problem which is much easier to solve compared to an impulse control problem. Thus, our approximation scheme can be adapted to solve practical applications (such as central bank intervention problem) more efficiently.

5. Computational example: a central bank intervention problem

In this section, we illustrate how we can apply our approximation scheme to solve the central bank intervention problem studied in [40]. We first briefly introduce their model.

5.1. Central bank intervention (CBI) problem in a Lévy market

In [40], the authors assume that the exchange rate \( X_n(t) \) at time \( t \), when there is no interventions, satisfies the following stochastic differential equation with Lévy jumps

\[
\frac{dX_n(t)}{X(t-)} = \mu dt + \sigma dB(t) + \int_{\mathbb{R}} \eta(z)N(dt, dz); \quad X_n(0) = x,
\]

where \( x > 0, \mu \in \mathbb{R}, \sigma > 0 \) are constants, and \( \eta(z) \) is a function satisfying \( \eta(z) > -1 \). Some typical choices of \( \eta(z) \) are \( \eta(z) = e^z - 1 \) with \( z \in \mathbb{R} \), and \( \eta(z) = \delta z \) with \( z > -1/\delta \) for \( \delta > 0 \).

The central bank can perform one of two possible interventions. Let \( G_1(x, \xi) = x + \xi, \ G_2(x, \xi) = x - \xi \) and \( Z^1 = Z^2 = \mathcal{K} = [0, \infty) \). After the 1st intervention, the exchange rate process \( X_n(t) \) switches to \( X'(t) \) given by (2) with an intervention of type \( i \), where \( i = 1, 2 \); the corresponding coefficient functions are \( \mu_i(x) = \mu x, \sigma_i(x) = \sigma x, \) and \( \nu_i(x, z) = \nu_i(x) \). For convenience, \( \eta_i(x) = \eta(x) = e^z - 1 \) for \( i = 1, 2 \).

It is further assumed in [40] that \( f(x) = (x - \rho^2)^2 \) is the running cost, \( K_1(\xi) = C + c \xi \) and \( K_2(\xi) := D + d \xi \) are the corresponding costs of making interventions of type 1 and type 2, respectively. As discussed in Section 4, we need to find an optimal impulse control \( u^* \) in \( \mathcal{U} \) such that for all \( x \in \mathbb{R} \)

\[
\Phi(x) = \inf \left\{ J^{(u)}(x); \ u \in \mathcal{U} \right\} = \int f^{(u)}(x),
\]

where \( f^{(u)}(x) \) is defined in (7).

5.2. Approximation scheme for the CBI problem with market reaction

Consider the CBI problem associated with the impulse control problem given by

\[
\Phi_n(x) := \inf \left\{ f^{(u)}(x); \ u \in \mathcal{U}_{n-1} \right\}.
\]

We can solve this problem by constructing a sequence of functions \( \{\varphi_i(x)\}_{i=1}^n \), where

\[
\varphi_i(x) = \inf_{r \geq 0} E^x \left[ \int_0^r e^{-\eta r f(X(t))dt} + \mathcal{M}_r \varphi_{i-1}(X(r)) \right]
\]

with \( \varphi_0(x) := E^x \left[ \int_0^\infty e^{-\eta r f(X(t))dt} \right] \). We next proceed with this construction.

First note that the solution \( X(t) \) of (8) can be explicitly expressed as

\[
X_n(t) = x \exp[(\mu - \sigma^2/2) t - \int_0^t \sigma B(s) + \int_0^t \int_{|z| < 1} z \nu(ds, dz)].
\]

Now let \( \delta := \mu - \sigma^2/2 - \int_0^1 (e^{z^2} - 1) \nu(ds, dz) \). Then, we have

\[
X_n(t) = x \exp[(\delta t + \sigma B(t) + \int_0^t \int_{|z| < 1} z \nu(ds, dz)]
\]

Using the exponential moments of Lévy process, it follows that, for \( q \in \mathbb{R} \),

\[
E \left[ (X_n(t))^q \right]
\]

Similarly, for the regime-switching process \( X'_n(t) \) defined by (2) under the intervention of type \( i = 1, 2 \), we have for \( q \in \mathbb{R} \),

\[
E \left[ (X'_n(t))^q \right]
\]

Observe that, in this case, the infinitesimal generator \( A \) of the exchange rate process \( X(t) \) with discount rate \( r \), and the minimum cost operator \( \mathcal{M}_r \), can be expressed in the following forms:

\[
A \phi(x) = \mu x \frac{\partial \phi}{\partial x} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 \phi}{\partial x^2} + \int_{\mathbb{R}} \left[ \phi(x + \xi) - \phi(x) - \xi \phi'(x) \right] \nu(ds) - r \phi(x),
\]

\[
\mathcal{M}_r \phi(x) = \min \{M^1 \phi(x), M^2 \phi(x) \}, \quad \mathcal{M}_r \phi(x) = \inf \{C + c \xi + R_1(x + \xi); \ \xi > 0 \},
\]

\[
\mathcal{M}_r \phi(x) = \inf \{D + d \xi + R_2(x - \xi); \ \xi > 0 \}, \quad \text{and, for } i \in \{1, 2\},
\]

\[
R_i(x) = \int_0^\infty E^y \int_0^t e^{-\eta r f(X(s))ds} + e^{-\eta r \phi(X'(t'))} | T' = t | \mathcal{M}_r \phi(X(t')).
\]

In the first step of the approximation, we have

\[
\phi_1(x) = \inf_{r \geq 0} E^x \left[ \int_0^r e^{-\eta r f(X(t))dt} + \mathcal{M}_r \phi_0(X(r)) \right].
\]
This is an optimal stopping problem that can be solved by exploiting the following QVI:

\[ A\varphi_i(x) + f(x) \geq 0, \quad \text{for all } x, \]

\[ \varphi_i(x) \leq M \tau_i \varphi_i(x), \quad \text{for all } x, \]

\[ A\varphi_i(x) + f(x) = 0 \quad \text{on } D_i = \{ x : \varphi_i(x) < M \tau_i \varphi_i(x) \}, \]

where \( D_i \) is the continuation region. We guess that \( D_i \) has the form \( D_1 = \{ L_1 < x < W_i \} \). Similar to [40], we need to find a two-band control characterized by four parameters \( L_1, l_i, w_i, \) and \( W_i \) such that \( 0 < L_1 < l_i \leq w_i < W_i < \infty \) and

\[
\varphi_i(x) = \begin{cases} 
C + c(l_1 - x) + R_{i-1}^{(0)}(l_1), & \text{if } x \leq L_1, \\
\varphi_i(x) + a_i x^{a_i} + b_i x^{b_i}, & \text{if } L_1 < x < W_i, \\
D + d(x - w_i) + R_{i-1}^{(0)}(w_i), & \text{if } x \geq W_i,
\end{cases}
\]

where \( a_i, b_i \) are constants to be determined, and \( R_{i-1}^{(0)}(x) \) is a certain function depending on \( x \).

5.3. Numerical computations

In this subsection, we conduct some numerical computations to find an approximate optimal solution to the CBI problem in the Kou jump–diffusion Lévy market (cf. [28]). For the exchange rate and its regime–switching dynamics, following [28], we assume that the distribution of jump sizes is an asymmetric double-exponential distribution with Lévy density of the form

\[ \nu(dx) = \gamma \left[ p \gamma_x e^{-\gamma_x} I_{x>0} + (1-p) \gamma_x e^{-\gamma_x} I_{x<0} \right] dx, \]

where \( \gamma_+ > 0 \) and \( \gamma_- > 0 \) are parameters governing the decay of the tails for the distribution of positive and negative jump sizes, respectively, and \( p \in [0, 1] \) is the probability of an upward jump. Here \( \gamma \) is the intensity of the Poisson process that counts the jumps of the jump–diffusion process.

For simplicity, we assume that the random reaction time period \( T_i \sim \text{Exponential}(2) \), i.e., \( F_i(t) = 1 - e^{-t/2} \) for \( t > 0 \). Observe that with the choice of \( T_i \sim \text{Exponential}(\lambda) \), \( \text{Prob}(T_i > t_0) = e^{-\lambda t_0} \approx 0.0025 \approx 0 \) for \( t_0 > 6 \lambda \). Therefore, while exponential distribution theoretically can take any non-negative value, the chance of observing a value greater than \( 6 \lambda \) is almost zero. Hence, if we choose the time units appropriately, then exponential distribution very closely approximates the short-lived behavior of the reaction period and should work very well for the CBI model.

Next, following some similar calculation (with minor modifications) as in [40], we find that, for \( j = 1, 2, \ldots, n \) and \( i = 1, 2, \ldots, n \),

\[ R_{i-1}^{(j-1)}(x) = \frac{a_j x^{a_j} + b_j x^{b_j} + \Theta_j L_j^2}{1 - 2C_i(\alpha_i)} + \frac{\Theta_j L_j^2}{1 - 2C_i(\alpha_j)} - \frac{2\rho}{1 - 2C_i(\alpha_j)} \left( \frac{2 + \rho}{2 - \rho} \right) x^2 - \frac{2(\rho - 1)}{2 - \rho} \right) \]

\[ R_{i-1}^{(j-1)}(x) = \frac{a_j x^{a_j} + b_j x^{b_j} + \Theta_j L_j^2}{1 - 2C_i(\alpha_i)} + \frac{\Theta_j L_j^2}{1 - 2C_i(\alpha_j)} - \frac{2\rho}{1 - 2C_i(\alpha_j)} \left( \frac{2 + \rho}{2 - \rho} \right) x^2 - \frac{2(\rho - 1)}{2 - \rho} L_j^2 \]

where \( a_j, b_j \) are constants to be determined, and \( R_{i-1}^{(j-1)}(x) \) is a certain function depending on \( x \).
$$- \frac{2p\rho}{1 - 2C_1(1)} \left( 2 + \frac{1}{r - \mu} \right)$$  \hspace{1cm} (15)$$

$$a_j W_{j+1}^{\alpha j} + b_j W_{j+2}^{\alpha j} + \Theta W_j^2 = \left( \frac{2\rho}{r - \mu} \right) W_j = D + d(W_j - w_j)$$

$$+ \frac{a_{j-1} W_{j+1}^{\alpha j}}{1 - 2C_1(\alpha_1)} + \frac{b_{j-1} W_{j+2}^{\alpha j}}{1 - 2C_2(\alpha_2)} + \frac{(\Theta + 2)w_j^2}{1 - 2C_2(2)}$$

$$- \frac{2\rho w_j}{1 - 2C_1(1)} \left( 2 + \frac{1}{r - \mu} \right)$$  \hspace{1cm} (16)$$

We can solve this nonlinear system of equations numerically by using Newton’s method. However, observe that it is relatively easy to solve the above system than the corresponding system of (18)–(23) in [40]. In particular, the last two equations of the above system are single variable equations whereas its counterpart in [40] has three unknowns in each equation. Moreover, since the terms $a_{j-1}$ and $b_{j-1}$ will be known constants at the $j$th iteration, the appearance of $a_{j-1}$ and $b_{j-1}$ in the above system (see (15) and (16), (19) and (20)) diminishes the non-linearity of the system. This simplification reduces the computational complexity of the solution of the system of nonlinear equations. This is a main advantage of our iterative scheme.

We computationally validated the efficacy of approximation scheme by simulating various scenarios and confirmed the convergence of the optimal bands. Table 1 summarizes the optimal bands for three different cases that we investigated during our computational study. The following parameters are used in these computations: $r = 0.06$, $\rho = 1.4$, $\mu = \mu_1 = \mu_2 = 0.4$, $C = 0.5$, $c = 0.2$, $D = 0.7$, $d = 0.4$, $p = 0.3$, $\gamma = 3$, $\gamma_+ = 30$ and $\gamma_- = 20$; volatility numbers are given in the table. Fig. 1 illustrates the convergence of the optimal bands for the case where $\sigma = 0.3$ and $\sigma_1 = \sigma_2 = 0.15$. Table 1 and Fig. 1 clearly confirm the convergence of our approximation scheme. Moreover, our approximation scheme with market reactions converges as rapid as that of [36]; thus, we have not lost any efficiency by incorporating reactions to the extant model.

### 6. Concluding remarks

In this paper, we have developed an efficient approximation scheme for impulse control model with random reaction periods. Our approximation scheme not only helps practitioners to use ICRRP framework more efficiently for applications such as central bank intervention problems, mutual fund cash management problems and machine maintenance problems, but also enhances the existing literature on stochastic control theory. We believe that this new development will encourage researchers to use the ICRRP framework for applications in other domains as well.

Finally, note that, although our approximation scheme is developed in a multi-dimensional setting, the CBI example is one dimensional. In general, impulse control and optimal stopping problems are very difficult to solve in a multi-dimensional setting; thus, the efficacy of our scheme will also be limited in a multi-dimensional setting.

### Acknowledgments

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### Appendix A. Supplementary data

Supplementary material related to this article can be found online at [http://dx.doi.org/10.1016/j.orl.2017.08.014](http://dx.doi.org/10.1016/j.orl.2017.08.014).

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**Table 1**

Convergence of iterative optimal bands.

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**Fig. 1.** Convergence of iterative bands towards the optimal bands.
References