PARAMETER ESTIMATION FOR A CLASS OF STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY SMALL STABLE NOISES FROM DISCRETE OBSERVATIONS∗

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Abstract We study the least squares estimation of drift parameters for a class of stochastic differential equations driven by small $\alpha$-stable noises, observed at $n$ regularly spaced time points $t_i = i/n$, $i = 1, \ldots, n$ on $[0, 1]$. Under some regularity conditions, we obtain the consistency and the rate of convergence of the least squares estimator (LSE) when a small dispersion parameter $\varepsilon \to 0$ and $n \to \infty$ simultaneously. The asymptotic distribution of the LSE in our setting is shown to be stable, which is completely different from the classical cases where asymptotic distributions are normal.

Key words Asymptotic distribution of LSE; consistency of LSE; discrete observations; least squares method; parameter estimation; small $\alpha$-stable noises; stable distribution; stochastic differential equations

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1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a basic probability space equipped with a right continuous and increasing family of $\sigma$-algebras $(\mathcal{F}_t, t \geq 0)$. Let $(Z_t, t \geq 0)$ be a standard $\alpha$-stable Lévy motion with $Z_1 \sim S_\alpha(1, \beta, 0)$, where $\beta \in [-1, 1]$ is a skewness parameter. For technical reasons, we assume that $1 < \alpha < 2$. The stochastic process $X = (X_t, t \geq 0)$, starting from $x_0 \in \mathbb{R}$ is defined as the unique strong solution to the following stochastic differential equation (SDE)

$$dX_t = \theta b(X_t)dt + \varepsilon \sigma(X_t-)dZ_t, \ t \in [0, 1]; \ X_0 = x_0,$$

where $b : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ are known functions. The regularity conditions on $b$ and $\sigma$ will be provided in Section 3. Assume that this process is observed at regularly spaced time points $\{t_i = i/n, \ i = 1, 2, \ldots, n\}$. The only unknown quantity in SDE (1.1) is the parameter $\theta$. We

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denote the true value of the parameter by \( \theta_0 \). The purpose of this article is to study the least squares estimator for the true value \( \theta_0 \) based on the sampling data \( (X_t)_{i=1}^n \). The SDE (1.1) was used to describe geophysical models for climate change in [2, 3].

The asymptotic theory of parametric estimation for diffusion processes with small white noise based on continuous-time observations is well developed (see, e.g., [14], [15], [32], [35], and [37]). There were many applications of small noise asymptotics to mathematical finance, see, for example, [13], [28], [29], [33], and [36]. From a practical point of view in parametric inference, it is more realistic and interesting to consider asymptotic estimation for diffusion processes with small noise based on discrete observations. Substantial progress was made along this direction. The efficient estimation of drift parameters of small diffusions from discrete observations was studied in [5] and [16]. Martingale estimating function was proposed in [25] to establish consistency and asymptotic normality of the estimators of drift and diffusion coefficient parameters when \( \varepsilon \to 0 \) and \( n \) is fixed. In [6] and [26], a contrast function was used to study the efficient estimation for unknown parameters in both drift and diffusion coefficient functions. In [30,31], martingale estimating function approach was used to study estimation of drift parameters for small diffusions under weaker conditions. Thus, in the case of small diffusions, the asymptotic distributions of the estimators are normal under suitable conditions on \( \varepsilon \) and \( n \).

The parametric estimation problems for diffusion processes with jumps based on discrete observations were studied in [23] and [24] via maximum likelihood method. Consistency and asymptotic normality for the proposed estimators were established. The driving jump processes considered in [23] and [24] include a large class of Lévy processes such as compound Poisson processes, gamma, inverse Gaussian, variance gamma, normal inverse Gaussian, or some generalized tempered stable processes. In [18], it dealt with the consistency and asymptotic normality of the TFE (trajectory-fitting estimator) and LSE when the driving process is zero-mean adapted process (including Lévy process) with finite moments. The parametric estimation for Lévy-driven Ornstein-Uhlenbeck processes was also studied in [1], [27], and [34]. However, the aforementioned papers were unable to cover an important class of driving Lévy processes, namely, \( \alpha \)-stable Lévy motions with \( \alpha \in (0,2) \). Recently, parameter estimation problem for Ornstein-Uhlenbeck processes driven by \( \alpha \)-stable Lévy motions was discussed in [7, 8].

In [17], we studied the parameter estimation problem for discretely observed Ornstein-Uhlenbeck processes with small Lévy noises. In that article, \( Z_t \) is replaced by a Lévy process \( L_t = aB_t + bZ_t \), where \( a \) and \( b \) are known constants, \( \{B_t, t \geq 0\} \) is a standard Brownian motion independent of \( \{Z_t, t \geq 0\} \). The consistency and rate of convergence of the least squares estimator were established. The asymptotic distribution of the LSE was shown to be the convolution of a normal distribution and a stable distribution. In this article, we consider a more general class of stochastic processes with small stable noises satisfying (1.1). For simplicity, we consider only pure jump \( \alpha \)-stable noise \( \{Z_t, t \geq 0\} \). Of course, the ideas presented in this article are still valid for more general Lévy noise \( \{L_t, t \geq 0\} \). The main difficulty in such cases arises from the infinite variance property of \( \alpha \)-stable processes. The important tools we shall employ are random time changes and moment inequalities for stable stochastic integrals (see [12], [19], and [20]). We are interested in the study of parameter estimation for stochastic processes satisfying SDE (1.1) based on discrete observations. We shall use the least squares
method to obtain an asymptotically consistent estimator.

To obtain the LSE, we introduce the following contrast function
\[ \rho_{n,\varepsilon}(\theta) = \sum_{i=1}^{n} \left| \frac{X_{t_i} - X_{t_{i-1}} + \theta b(X_{t_{i-1}}) \cdot \Delta t_{i-1}}{\varepsilon \sigma(X_{t_{i-1}})} \right|^2, \] (1.2)
where \( \Delta t_{i-1} = t_i - t_{i-1} = 1/n \). Then, the LSE \( \hat{\theta}_{n,\varepsilon} \) is defined as
\[ \hat{\theta}_{n,\varepsilon} = \arg \min_{\theta} \rho_{n,\varepsilon}(\theta), \] which can be explicitly represented as
\[ \hat{\theta}_{n,\varepsilon} = \sum_{i=1}^{n} \frac{\sigma^{-2}(X_{t_{i-1}})(X_{t_i} - X_{t_{i-1}})b(X_{t_{i-1}})}{\sigma^{-2}(X_{t_{i-1}})b^2(X_{t_{i-1}})}. \] (1.3)

In this article, we consider the asymptotics of the LSE \( \hat{\theta}_{n,\varepsilon} \) with high frequency \( (n \to \infty) \) and small dispersion \( (\varepsilon \to 0) \). Our goal is to prove that \( \hat{\theta}_{n,\varepsilon} \to \theta_0 \) in probability and to establish the rate of convergence and the asymptotic distributions. We obtain some new asymptotic distributions for the LSE in our setting, which are stable distributions.

The article is organized as follows. In Section 2, we present some preliminaries on \( \alpha \)-stable distributions, \( \alpha \)-stable Lévy motions, stable stochastic integrals, and related moment inequalities. In Section 3, we establish the consistency of the LSE \( \hat{\theta}_{n,\varepsilon} \). In Section 4, we study the rate of convergence for the LSE and obtain the asymptotic distribution, which is different from that in classical cases. Some interesting examples are also provided in Section 4.

2 Preliminaries

In this section, we present some preliminaries on \( \alpha \)-stable distributions, \( \alpha \)-stable Lévy processes, stable stochastic integrals, and related moment inequalities. In this article, we shall use notation \( \to_d \) to denote “convergence in distribution” and notation \( \to_p \) to denote “convergence in probability.” We write \( \equiv \) for equality in distribution. Also, for convenience, we shall use \( C \) to denote a generic constant whose value may vary from place to place.

**Definition 2.1** A random variable \( \eta \) is said to follow a stable distribution, denoted by \( \eta \sim S_\alpha(\sigma, \beta, \mu) \), if it has the characteristic function of the following form:
\[ \varphi_\eta(u) = E\exp\{iu\eta\} = \begin{cases} \exp\left\{ -\sigma|u|^{\alpha} \left( 1 - i\beta \text{sgn}(u) \tan \frac{\alpha \pi}{2} \right) + i\mu u \right\}, & \text{if } \alpha \neq 1, \\
\exp\left\{ -\sigma|u| \left( 1 + i\beta \frac{2}{\pi} \text{sgn}(u) \log |u| \right) + i\mu u \right\}, & \text{if } \alpha = 1,
\end{cases} \]
where \( \alpha \in (0, 2] \), \( \sigma \in (0, \infty) \), \( \beta \in [-1, 1] \), and \( \mu \in (-\infty, \infty) \) are the index of stability, the scale, skewness, and location parameters, respectively.

When \( \mu = 0 \), we say \( \eta \) is strictly \( \alpha \)-stable. If, in addition, \( \beta = 0 \), we call \( \eta \) symmetric \( \alpha \)-stable. Note that \( \eta \) is strictly 1-stable \( (\alpha = 1) \) if and only if \( \beta = 0 \) (symmetric case). We refer to [9], [21], and [22] for more details on stable distributions.

**Definition 2.2** An \( \mathcal{F}_t \)-adapted stochastic process \( \{Z_t\}_{t \geq 0} \) is called a standard \( \alpha \)-stable Lévy motion if
(i) \( Z_0 = 0 \), a.s.;
(ii) $Z_t - Z_s \sim S_{\alpha}((t - s)^{1/\alpha}, \beta, 0)$, $t > s \geq 0$;
(iii) For any finite time points $0 \leq s_0 < s_1 < \cdots < s_m < \infty$, the random variables $Z_{s_0}, Z_{s_1} - Z_{s_0}, \cdots, Z_{s_m} - Z_{s_m-1}$ are independent.

The Itô-type stochastic integrals with respect to a $\alpha$-stable Lévy motion were extensively studied in [11], [12], and [20]. We denote $L_{a,s}^\alpha$ the family of all real-valued $(\mathcal{F}_t)$-predictable processes $\phi$ on $\Omega \times [0, \infty)$ such that for every $T > 0$, $\int_0^T |\phi(t, \omega)|^\alpha dt < \infty$ a.s. Then, by Theorem 4.1 of [20] and Theorem 3.1 of [12], a predictable process $\phi$ is integrable with respect to a strictly $\alpha$-stable Lévy process $Z$, that is, $\int_0^T \phi(t) dZ_t$ exists for every $T > 0$ if and only if $\phi \in L_{a,s}^\alpha$.

We shall use $\phi_+$ and $\phi_-$ to denote the positive and negative part of $\phi$, respectively. The method of random time change (or inner clock property) for stable stochastic integrals is a very powerful tool. The following lemma is a direct consequence of Theorem 4.1 of [12] and Theorem 3.1 of [20].

**Lemma 2.3** Let $Z$ be a strictly $\alpha$-stable Lévy process and $\phi \in L_{a,s}^\alpha$. Then,

(i) There exist some independent processes $Z', Z'' \overset{d}{=} Z$, such that

$$\int_0^t \phi(s) dZ_s = Z' \circ \int_0^t \phi_+(s) ds - Z'' \circ \int_0^t \phi_-(s) ds,$$  

a.s.  \hspace{1cm} (2.1)

(ii) If $Z$ is symmetric, that is, $\beta = 0$, then, there exists some $\alpha$-stable Lévy process $Z' \overset{d}{=} Z$, such that

$$\int_0^t \phi(s) dZ_s = Z' \circ \int_0^t |\phi(s)|^\alpha ds,$$  

a.s.  \hspace{1cm} (2.2)

We say that a continuous function $F : [0, \infty) \rightarrow [0, \infty)$ grows more slowly than $u^\alpha$ $(\alpha > 0)$ if there exist positive constants $c, \lambda_0$, and $\alpha_0 < \alpha$, such that $F(\lambda u) \leq c \lambda^\alpha F(u)$ for all $u > 0$ and all $\lambda \geq \lambda_0$. Now, we state the moment inequalities for stable stochastic integrals in the following lemma, which can be regarded as a generalization of Theorem 3.2 of [19] where the symmetric case was dealt with. This lemma will be a crucial tool in the proofs of our main results.

**Lemma 2.4** Let $\phi(t)$ be a predictable process satisfying $\int_0^T |\phi(t)|^\alpha dt < \infty$ almost surely for $T < \infty$. We assume that either $\phi$ is nonnegative or $Z$ is symmetric. If $F(u)$ grows more slowly than $u^\alpha$, then, there exist positive constants $c_1$ and $c_2$ depending only on $\alpha, \alpha_0, \beta, c$, and $\lambda_0$ such that, for each $T > 0$,

$$c_1 \mathbb{E} \left[ F(\int_0^T |\phi(t)|^\alpha dt)^{1/\alpha} \right] \leq \mathbb{E} \left[ F(\sup_{t \leq T} |\int_0^t \phi(s) dZ_s|) \right] \leq c_2 \mathbb{E} \left[ F(\int_0^T |\phi(t)|^\alpha dt)^{1/\alpha} \right].$$  \hspace{1cm} (2.3)

**Proof** When $Z$ is symmetric $\alpha$-stable, moment inequalities (2.3) for stable stochastic integrals were established in Theorem 3.1 and Theorem 3.2 of [19]. We claim that the moment inequalities in Theorem 3.1 and Theorem 3.2 of [19] remain true for non-symmetric (strictly) stable Lévy processes and stable stochastic integrals. In this case, we assume that the integrand process $\phi(\cdot)$ is non-negative and predictable so that the random time change property (or inner clock property) of stable stochastic integrals is applicable (see [12]). In the proof of Theorem 3.1 of [19], there is only one place in the probability estimate of part I where the authors used the symmetric property via Lévy inequality. However, we can replace this estimate by the following probability estimate with some constant $C > 0$

$$\sup_{\lambda > 0} \lambda^\alpha P\left( \sup_{0 \leq s \leq 1} |Z_s| \geq \lambda \right) \leq C,$$
provided in Proposition 10.2 of [4] (see also [10]). All the remaining arguments in the proof of Theorem 3.1 of [19] work throughout. Consequently, the moment inequalities in Theorem 3.2 of [19] are also true for non-symmetric case as stated in (2.3) when the integrand process is non-negative, predictable, and $L^\alpha$-integrable.

**Remark 2.5** When $Z$ is non-symmetric with $1 < \alpha < 2$, we consider the stable stochastic integral $\int_0^t \phi(s) dZ_s$, where $\phi \in L_{\alpha,s}^\alpha$. Note that

$$\sup_{t \leq T} \left| \int_0^t \phi(s) dZ_s \right| \leq \sup_{t \leq T} \left| \int_0^t \phi_+(s) dZ_s \right| + \sup_{t \leq T} \left| \int_0^t \phi_-(s) dZ_s \right|.$$ 

Thus, by letting $F(u) = u$ in Lemma 2.4, we obtain

$$E \left[ \sup_{t \leq T} \left| \int_0^t \phi(s) dZ_s \right| \right] \leq E \left[ \sup_{t \leq T} \left| \int_0^t \phi_+(s) dZ_s \right| \right] + E \left[ \sup_{t \leq T} \left| \int_0^t \phi_-(s) dZ_s \right| \right]$$

$$\leq c_2 E \left[ \left( \int_0^T \phi^\alpha_+(t) dt \right)^{1/\alpha} \right] + c_2 E \left[ \left( \int_0^T \phi^\alpha_-(t) dt \right)^{1/\alpha} \right]$$

$$\leq 2c_2 E \left[ \left( \int_0^T |\phi(t)|^\alpha dt \right)^{1/\alpha} \right]. \quad (2.4)$$

This result will be used very often in Sections 3 and 4.

## 3 Consistency of the Least Squares Estimator

We first introduce the following set of assumptions.

(A1) Lipschitz conditions on $b$ and $\sigma$, that is, there exists a constant $L > 0$ such that

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq L|x - y|, \quad x, y \in \mathbb{R}.$$ 

(A2) $\sigma(\cdot)$ is uniformly bounded from above, that is, there exists a positive constant $\sigma_1 > 0$ such that $0 < |\sigma(x)| \leq \sigma_1 < \infty$.

(A3) There exist constant $K > 0$ and $r \geq 0$, such that $\sigma^{-2}(x) \leq K(1 + |x|^r)$, $x \in \mathbb{R}$.

It is well-known that SDE (1.1) has a unique strong solution under (A1). (A2) is a technical assumption under which we can apply moment inequalities to stable stochastic integrals in our stable setting.

Let $X_t^0$ be the solution of the underlying ordinary differential equation under the true value of the drift parameter:

$$dX_t^0 = \theta_0 b(X_t^0) dt, \quad X_0^0 = x_0. \quad (3.1)$$

Note that

$$X_{t_i} - X_{t_{i-1}} = \int_{t_{i-1}}^{t_i} \theta_0 b(X_s) ds + \varepsilon \int_{t_{i-1}}^{t_i} \sigma(X_s-) dZ_s. \quad (3.2)$$

Then, we can give a more explicit decomposition for $\hat{\theta}_{n,\varepsilon}$ based on (1.3) and (3.2):

$$\hat{\theta}_{n,\varepsilon} = \frac{\theta_0 \sum_{i=1}^{n} \sigma^{-2}(X_{t_{i-1}}) b(X_{t_{i-1}}) \int_{t_{i-1}}^{t_i} b(X_s) ds}{n^{-1} \sum_{i=1}^{n} \sigma^{-2}(X_{t_{i-1}}) b^2(X_{t_{i-1}})}$$

$$+ \frac{\varepsilon \sum_{i=1}^{n} \sigma^{-2}(X_{t_{i-1}}) b(X_{t_{i-1}}) \int_{t_{i-1}}^{t_i} \sigma(X_s-) dZ_s}{n^{-1} \sum_{i=1}^{n} \sigma^{-2}(X_{t_{i-1}}) b^2(X_{t_{i-1}})}.$$
By the Markov inequality, Lemma 2.4, Remark 2.5, and condition (A2), we have, for any given 

\[
\theta_0 + \frac{n^{-1} \sum_{i=1}^{n} \sigma^{-2}(X_{t_{i-1}}) \int_{t_{i-1}}^{t_{i}} (b(X_s) - b(X_{t_{i-1}})) ds}{\varepsilon \sum_{i=1}^{n} \sigma^{-2}(X_{t_{i-1}}) \int_{t_{i-1}}^{t_{i}} (X_s - X_{t_{i-1}}) dZ_s} = \theta_0 + \frac{\Phi_2(n, \varepsilon)}{\Phi_1(n, \varepsilon)} \cdot \frac{\Phi_3(n, \varepsilon)}{\Phi_1(n, \varepsilon)}
\]

(3.3)

**Theorem 3.1** Under the conditions (A1)–(A3), we have \( \hat{\theta}_{n, \varepsilon} \rightarrow_P \theta_0 \) as \( \varepsilon \rightarrow 0 \), \( n \rightarrow \infty \) and \( n \varepsilon \rightarrow \infty \) (3.10).

Before proving Theorem 3.1, we need to establish some preliminary results. We shall study the asymptotic behavior of \( \Phi_1(n, \varepsilon) \), \( \Phi_2(n, \varepsilon) \), and \( \Phi_3(n, \varepsilon) \), respectively.

**Lemma 3.2** Under conditions (A1)–(A2), we have

\[
|X_t - X_t^0| \leq \varepsilon e^{L|\theta_0|} \left| \int_0^t \sigma(X_s) dZ_s \right|.
\]

(3.4)

and

\[
\sup_{0 \leq t \leq 1} |X_t - X_t^0| \rightarrow_P 0 \text{ as } \varepsilon \rightarrow 0.
\]

(3.5)

**Proof** The solution of (3.1) is given by

\[
X_t^0 = x_0 + \theta_0 \int_0^t b(X_s^0) ds.
\]

(3.6)

Then, it is clear that

\[
X_t - X_t^0 = \theta_0 \int_0^t (b(X_s) - b(X_s^0)) ds + \varepsilon \int_0^t \sigma(X_s) dZ_s.
\]

(3.7)

By Lipschitz condition on \( b(\cdot) \) in (A1), we find

\[
|X_t - X_t^0| \leq L|\theta_0| \int_0^t |X_s - X_s^0| ds + \varepsilon \int_0^t \sigma(X_s) dZ_s.
\]

(3.8)

By Gronwall’s inequality, it yields that (3.4) holds. Then, we have

\[
\sup_{0 \leq t \leq 1} |X_t - X_t^0| \leq \varepsilon e^{L|\theta_0|} \sup_{0 \leq t \leq 1} \left| \int_0^t \sigma(X_s) dZ_s \right|.
\]

(3.9)

By the Markov inequality, Lemma 2.4, Remark 2.5, and condition (A2), we have, for any given \( \delta > 0 \),

\[
P \left( \varepsilon e^{L|\theta_0|} \sup_{0 \leq t \leq 1} \left| \int_0^t \sigma(X_s) dZ_s \right| > \delta \right) \leq \delta^{-1} e^{L|\theta_0|} \varepsilon E \left[ \sup_{0 \leq t \leq 1} \left| \int_0^t \sigma(X_s) dZ_s \right| \right]
\]

\[
\leq 2c_2 \delta^{-1} e^{L|\theta_0|} \varepsilon E \left[ \left( \int_0^1 |\sigma(X_s)|^\alpha ds \right)^{1/\alpha} \right]
\]

\[
\leq 2c_2 \delta^{-1} e^{L|\theta_0|} \varepsilon_1 \varepsilon,
\]

(3.10)
which implies that (3.5) holds.

**Proposition 3.3** Under conditions (A1)–(A3), we have \( \Phi_1(n, \varepsilon) \to \mu \int_0^1 \sigma^{-2}(X^0_t)b^2(X^0_t)dt \) as \( \varepsilon \to 0 \) and \( n \to \infty \).

**Proof** We have the following decomposition of \( \Phi_1(n, \varepsilon) \):

\[
\Phi_1(n, \varepsilon) = n^{-1} \sum_{i=1}^{n} \sigma^{-2}(X_{t_i-})b^2(X_{t_i-})
\]

\[
= n^{-1} \sum_{i=1}^{n} \sigma^{-2}(X^0_{t_i-})b^2(X^0_{t_i-}) + n^{-1} \sum_{i=1}^{n} \sigma^{-2}(X_{t_i-})(b^2(X_{t_i-}) - b^2(X^0_{t_i-}))
\]

\[
+ n^{-1} \sum_{i=1}^{n} (\sigma^{-2}(X_{t_i-}) - \sigma^{-2}(X^0_{t_i-}))b^2(X^0_{t_i-})
\]

\[
:= \Phi_{1,1}(n, \varepsilon) + \Phi_{1,2}(n, \varepsilon) + \Phi_{1,3}(n, \varepsilon).
\]

(3.11)

It is clear that \( \Phi_{1,1}(n, \varepsilon) \to \int_0^1 \sigma^{-2}(X^0_t)b^2(X^0_t)dt \) as \( n \to \infty \). For \( \Phi_{1,2}(n, \varepsilon) \), we have

\[
|\Phi_{1,2}(n, \varepsilon)|
\]

\[
\leq n^{-1} \sum_{i=1}^{n} \sigma^{-2}(X_{t_i-})|b^2(X_{t_i-}) - b^2(X^0_{t_i-})|
\]

\[
\leq n^{-1} \sum_{i=1}^{n} K(1 + |X_{t_i-}|^r)|b(X_{t_i-}) - b(X^0_{t_i-})||b(X^0_{t_i-})| + |b(X^0_{t_i-})|
\]

\[
\leq n^{-1} \sum_{i=1}^{n} K(1 + |X_{t_i-}|^r) \left( |b(X_{t_i-}) - b(X^0_{t_i-})|^2 + 2|b(X^0_{t_i-})||b(X_{t_i-}) - b(X^0_{t_i-})| \right)
\]

\[
\leq Kn^{-1} \sum_{i=1}^{n} |b(X_{t_i-}) - b(X^0_{t_i-})|^2 + 2Kn^{-1} \sum_{i=1}^{n} |b(X^0_{t_i-})||b(X_{t_i-}) - b(X^0_{t_i-})|
\]

\[
+Kn^{-1} \sum_{i=1}^{n} |X_{t_i-}|^r|b(X_{t_i-}) - b(X^0_{t_i-})|^2
\]

\[
+2Kn^{-1} \sum_{i=1}^{n} |b(X^0_{t_i-})||X_{t_i-}|^r|b(X_{t_i-}) - b(X^0_{t_i-})|
\]

\[
:= \sum_{i=1}^{4} \Phi_{1,2}^{i}(n, \varepsilon).
\]

(3.12)

Using Lipschitz condition on \( b(\cdot) \) and the fact that, for \( C_r = 2r^{-1} \lor 1 \),

\[
|X_{t_i-}|^r \leq C_r \left( |X_{t_i-} - X^0_{t_i-}|^r + |X^0_{t_i-}|^r \right),
\]

(3.13)

we obtain

\[
\Phi_{1,2}^{(1)}(n, \varepsilon) \leq KL^2n^{-1} \sum_{i=1}^{n} |X_{t_i-} - X^0_{t_i-}|^2,
\]

(3.14)

\[
\Phi_{1,2}^{(2)}(n, \varepsilon) \leq 2KLn^{-1} \sum_{i=1}^{n} |b(X^0_{t_i-})||X_{t_i-} - X^0_{t_i-}|,
\]

(3.15)

\[
\Phi_{1,2}^{(3)}(n, \varepsilon) \leq KL^2C_rn^{-1} \sum_{i=1}^{n} |X_{t_i-} - X^0_{t_i-}|^r+2
\]
\[+K L^2 C_r n^{-1} \sum_{i=1}^{n} |X_{t_{i-1}}^0|^r |X_{t_{i-1}} - X_{t_{i-1}}^0|^2, \quad (3.16)\]

\[\Phi_{1,2}^{(1)}(n, \varepsilon) \leq 2 K L C_r n^{-1} \sum_{i=1}^{n} |X_{t_{i-1}} - X_{t_{i-1}}^0||r+1|b(X_{t_{i-1}}^0)|\]

\[+2 K L C_r n^{-1} \sum_{i=1}^{n} |X_{t_{i-1}}^0|^r |b(X_{t_{i-1}}^0)||X_{t_{i-1}} - X_{t_{i-1}}^0|. \quad (3.17)\]

Now, it is seen that

\[|\Phi_{1,2}^{(1)}(n, \varepsilon)| \leq K L^2 \left( \sup_{0 \leq t \leq 1} |X_t - X_t^0| \right)^2, \quad (3.18)\]

which converges to zero in probability as \(\varepsilon \to 0\) by (3.5). For \(\Phi_{1,2}^{(2)}(n, \varepsilon)\), we have

\[|\Phi_{1,2}^{(2)}(n, \varepsilon)| \leq 2 K L \sup_{0 \leq t \leq 1} |X_t - X_t^0| n^{-1} \sum_{i=1}^{n} |b(X_{t_{i-1}}^0)|, \quad (3.19)\]

which converges to zero in probability as \(\varepsilon \to 0\) and \(n \to \infty\) by (3.5) and the fact that \(n^{-1} \sum_{i=1}^{n} |b(X_{t_{i-1}}^0)| \to 1 \int_0^1 |b(X_t^0)| \, dt\). Similarly, we have \(\Phi_{1,2}^{(3)}(n, \varepsilon) \to P 0\) and \(\Phi_{1,2}^{(4)}(n, \varepsilon) \to P 0\) as \(\varepsilon \to 0\) and \(n \to \infty\) by (3.5). For \(\Phi_{1,3}(n, \varepsilon)\), by conditions (A1)-(A3) and (3.13), we have

\[|\Phi_{1,3}(n, \varepsilon)| \leq n^{-1} \sum_{i=1}^{n} |\sigma^{-2}(X_{t_{i-1}}) - \sigma^{-2}(X_{t_{i-1}}^0)|b^2(X_{t_{i-1}}^0)\]

\[\leq n^{-1} \sum_{i=1}^{n} \sigma^{-2}(X_{t_{i-1}})|\sigma^{-2}(X_{t_{i-1}}^0)|\sigma^2(X_{t_{i-1}}^0) - \sigma^2(X_{t_{i-1}}^0)|b^2(X_{t_{i-1}}^0)\]

\[\leq 2 K L \sigma_1 n^{-1} \sum_{i=1}^{n} (1 + |X_{t_{i-1}}|^r)|X_{t_{i-1}} - X_{t_{i-1}}^0| |\sigma^{-2}(X_{t_{i-1}}^0)|b^2(X_{t_{i-1}}^0)\]

\[\leq 2 K L \sigma_1 n^{-1} \sum_{i=1}^{n} |X_{t_{i-1}} - X_{t_{i-1}}^0| |\sigma^{-2}(X_{t_{i-1}}^0)|b^2(X_{t_{i-1}}^0)\]

\[+2 K L C_r \sigma_1 n^{-1} \sum_{i=1}^{n} |X_{t_{i-1}} - X_{t_{i-1}}^0| |r+1| \sigma^{-2}(X_{t_{i-1}}^0)|b^2(X_{t_{i-1}}^0)\]

\[+2 K L C_r \sigma_1 n^{-1} \sum_{i=1}^{n} |X_{t_{i-1}} - X_{t_{i-1}}^0| |X_{t_{i-1}}^0|^r \sigma^{-2}(X_{t_{i-1}}^0)|b^2(X_{t_{i-1}}^0)\]

\[\leq 2 K L \sigma_1 \left( \sup_{0 \leq t \leq 1} |X_t - X_t^0| + C_r \sup_{0 \leq t \leq 1} |X_t - X_t^0|^r \right) n^{-1} \sum_{i=1}^{n} |\sigma^{-2}(X_{t_{i-1}}^0)|b^2(X_{t_{i-1}}^0)\]

\[+2 K L C_r \sigma_1 \sup_{0 \leq t \leq 1} |X_t - X_t^0| n^{-1} \sum_{i=1}^{n} |X_{t_{i-1}}^0|^r |\sigma^{-2}(X_{t_{i-1}}^0)|b^2(X_{t_{i-1}}^0). \quad (3.20)\]

Again (3.5) implies that \(\Phi_{1,3}(n, \varepsilon) \to P 0\) as \(\varepsilon \to 0\) and \(n \to \infty\).

**Proposition 3.4** Under conditions (A1)-(A3), we have \(\Phi_2(n, \varepsilon) \to P 0\) as \(\varepsilon \to 0\) and \(n \to \infty\).

**Proof** For \(t_{i-1} \leq t \leq t_i, i = 1, 2, \ldots, n\), we have

\[X_t = X_{t_{i-1}} + \int_{t_{i-1}}^{t} \theta_0 b(X_s) \, ds + \varepsilon \int_{t_{i-1}}^{t} \sigma(X_s) \, dZ_s. \quad (3.21)\]
It follows that

\[
|X_t - X_{t_i - 1}| \leq \int_{t_i-1}^{t} |\theta_0|(b(X_s) - b(X_{t_i-1})) + |b(X_{t_i-1})|)ds + \varepsilon \int_{t_i-1}^{t} \sigma(X_{s-})dZ_s \leq |\theta_0| L \int_{t_i-1}^{t} |X_s - X_{t_i-1}|ds + n^{-1}|\theta_0||b(X_{t_i-1})| + \varepsilon \sup_{t_i-1 \leq t \leq t_i} \left|\int_{t_i-1}^{t} \sigma(X_{s-})dZ_s\right|.
\]

By Gronwall inequality, we obtain

\[
|X_t - X_{t_i - 1}| \leq e^{\theta_0 L (t - t_i)} \left[ n^{-1}|\theta_0||b(X_{t_i-1})| + \varepsilon \sup_{t_i-1 \leq t \leq t_i} \left|\int_{t_i-1}^{t} \sigma(X_{s-})dZ_s\right| \right],
\]

which yields that

\[
\sup_{t_i-1 \leq t \leq t_i} |X_t - X_{t_i - 1}| \leq e^{n^{-1}|\theta_0| L} \left[ n^{-1}|\theta_0||b(X_{t_i-1})| + \varepsilon \sup_{t_i-1 \leq t \leq t_i} \left|\int_{t_i-1}^{t} \sigma(X_{s-})dZ_s\right| \right].
\]

By (A1), (A3), and (3.24), we have

\[
|\Phi_2(n, \varepsilon)| \leq |\theta_0| \sum_{i=1}^{n} K(1 + |X_{t_i-1}|)\left|\int_{t_i-1}^{t} (b(X_s) - b(X_{t_i-1}))ds \right| \leq K|\theta_0| \sum_{i=1}^{n} (1 + |X_{t_i-1}|)\left|\int_{t_i-1}^{t} L|X_s - X_{t_i-1}|ds \right| \leq KL|\theta_0| \sum_{i=1}^{n} (1 + |X_{t_i-1}|)\left|b(X_{t_i-1})\right| n^{-1} \sup_{t_i-1 \leq t \leq t_i} |X_t - X_{t_i - 1}| \leq KL|\theta_0|^2 e^{\theta_0 L n} n^{-2} \sum_{i=1}^{n} (1 + |X_{t_i-1}|)\left|b(X_{t_i-1})\right|^2 + K L|\theta_0|^2 e^{\theta_0 L n} n^{-2} \varepsilon \sum_{i=1}^{n} (1 + |X_{t_i-1}|)\left|b(X_{t_i-1})\right| \sup_{t_i-1 \leq t \leq t_i} \left|\int_{t_i-1}^{t} \sigma(X_{s-})dZ_s\right| \leq \Phi_{2,1}(n, \varepsilon) + \Phi_{2,2}(n, \varepsilon).
\]

We first consider \(\Phi_{2,1}(n, \varepsilon)\). Using the basic inequality (3.13) and Lipschitz condition on \(b\) in (A1), we find

\[
\Phi_{2,1}(n, \varepsilon) \leq KL|\theta_0|^2 e^{\theta_0 L n} n^{-2} \sum_{i=1}^{n} (1 + |X_{t_i-1}|)\left|b(X_{t_i-1})\right|^2 \leq KL|\theta_0|^2 e^{\theta_0 L n} n^{-2} \sum_{i=1}^{n} (1 + C_r|X_{t_i-1}|) \left|X_{t_i-1} - X_{t_i-1}^0\right|^2 + KL|\theta_0|^2 e^{\theta_0 L n} \varepsilon \sum_{i=1}^{n} (1 + C_r|X_{t_i-1}|) \left|X_{t_i-1} - X_{t_i-1}^0\right|^2 \left|\int_{t_i-1}^{t} \sigma(X_{s-})dZ_s\right| \leq C n^{-2} \sum_{i=1}^{n} |b(X_{t_i-1}^0)|^2 (1 + C_r|X_{t_i-1}|) + C n^{-2} \sum_{i=1}^{n} |b(X_{t_i-1})|^2 |X_{t_i-1} - X_{t_i-1}^0| + C n^{-2} \sum_{i=1}^{n} |X_{t_i-1} - X_{t_i-1}^0|^{2+r}.
\]
\[ \leq C n^{-2} \sum_{i=1}^{n} |b(X_{t_{i-1}}^0)|^2 (1 + C_r |X_{t_{i-1}}^0|^r) + C \sup_{0 \leq t \leq 1} |X_t - X_t^0|^r n^{-2} \sum_{i=1}^{n} |b(X_{t_{i-1}}^0)|^2 \\
+ C \sup_{0 \leq t \leq 1} |X_t - X_t^0|^2 n^{-2} \sum_{i=1}^{n} (1 + C_r |X_{t_{i-1}}^0|^r) + C \sup_{0 \leq t \leq 1} |X_t - X_t^0|^{2+r} n^{-1}, \quad (3.26) \]

which converges to zero in probability as \( n \to \infty \) and \( \varepsilon \to 0 \) using (3.5). Finally, for \( \Phi_{2,2}(n, \varepsilon) \), we have

\[ \Phi_{2,2}(n, \varepsilon) \leq KL|\theta_0|e^{\frac{|\theta_0|L}{n}} \varepsilon n^{-1} \sum_{i=1}^{n} (1 + |X_{t_{i-1}}^0|) b(X_{t_{i-1}}) \sup_{t_{i-1} \leq t \leq t_i} \left| \int_{t_{i-1}}^{t} \sigma(X_s) dZ_s \right| \]

\[ \leq KL|\theta_0|e^{\frac{|\theta_0|L}{n}} \varepsilon n^{-1} \sum_{i=1}^{n} (1 + C_r |X_{t_{i-1}}^0|^r + C_r |X_{t_{i-1}} - X_{t_{i-1}}^0|^r) \]

\[ \times (|b(X_{t_{i-1}}^0)| + L|X_{t_{i-1}} - X_{t_{i-1}}^0|) \sup_{t_{i-1} \leq t \leq t_i} \left| \int_{t_{i-1}}^{t} \sigma(X_s) dZ_s \right| \]

\[ \leq C \varepsilon n^{-1} \sum_{i=1}^{n} \left[ b(X_{t_{i-1}}^0) (1 + C_r |X_{t_{i-1}}^0|^r) \sup_{t_{i-1} \leq t \leq t_i} \left| \int_{t_{i-1}}^{t} \sigma(X_s) dZ_s \right| \right] \\
+ C \varepsilon n^{-1} \sum_{i=1}^{n} \left[ b(X_{t_{i-1}}^0) |X_{t_{i-1}} - X_{t_{i-1}}^0|^r \sup_{t_{i-1} \leq t \leq t_i} \left| \int_{t_{i-1}}^{t} \sigma(X_s) dZ_s \right| \right] \\
+ C \varepsilon n^{-1} \sum_{i=1}^{n} \left[ |X_{t_{i-1}} - X_{t_{i-1}}^0| (1 + C_r |X_{t_{i-1}}^0|^r) \sup_{t_{i-1} \leq t \leq t_i} \left| \int_{t_{i-1}}^{t} \sigma(X_s) dZ_s \right| \right] \\
+ C \varepsilon n^{-1} \sum_{i=1}^{n} \left[ |X_{t_{i-1}} - X_{t_{i-1}}^0|^{1+r} \sup_{t_{i-1} \leq t \leq t_i} \left| \int_{t_{i-1}}^{t} \sigma(X_s) dZ_s \right| \right] \]

\[ \leq C \varepsilon n^{-1} \sum_{i=1}^{n} \left[ b(X_{t_{i-1}}^0) (1 + C_r |X_{t_{i-1}}^0|^r) \sup_{t_{i-1} \leq t \leq t_i} \left| \int_{t_{i-1}}^{t} \sigma(X_s) dZ_s \right| \right] \\
+ C \sup_{0 \leq t \leq 1} |X_t - X_t^0|^r \varepsilon n^{-1} \sum_{i=1}^{n} \left[ b(X_{t_{i-1}}^0) \sup_{t_{i-1} \leq t \leq t_i} \left| \int_{t_{i-1}}^{t} \sigma(X_s) dZ_s \right| \right] \\
+ C \sup_{0 \leq t \leq 1} |X_t - X_t^0|^{1+r} \varepsilon n^{-1} \sum_{i=1}^{n} \sup_{t_{i-1} \leq t \leq t_i} \left| \int_{t_{i-1}}^{t} \sigma(X_s) dZ_s \right| \]

\[ := \sum_{j=1}^{4} \Phi_{2,2}^{(j)}(n, \varepsilon). \quad (3.27) \]

By (A2), the Markov inequality, Lemma 2.4, and Remark 2.5, we find that, for any given \( \delta > 0 \),

\[ \mathbb{P}(\Phi_{2,2}^{(1)}(n, \varepsilon) > \delta) \leq C \delta^{-1} \varepsilon n^{-1} \sum_{i=1}^{n} \left[ b(X_{t_{i-1}}^0) (1 + C_r |X_{t_{i-1}}^0|^r) \mathbb{E} \left[ \sup_{t_{i-1} \leq t \leq t_i} \left| \int_{t_{i-1}}^{t} \sigma(X_s) dZ_s \right| \right] \right] \\
\leq C \delta^{-1} \varepsilon n^{-1} \sum_{i=1}^{n} \left[ b(X_{t_{i-1}}^0) (1 + C_r |X_{t_{i-1}}^0|^r) 2c_2 \mathbb{E} \left( \int_{t_{i-1}}^{t_i} |\sigma(X_s)|^a ds \right)^{\frac{1}{a}} \right] \\
\leq 2c_2 C \sigma \delta^{-1} \varepsilon n^{-\frac{a}{a-1}} \sum_{i=1}^{n} \left[ b(X_{t_{i-1}}^0) (1 + C_r |X_{t_{i-1}}^0|^r) \right], \quad (3.28) \]
which implies that $\Phi_{\alpha,2,2}(n, \varepsilon) \to 0$ as $\varepsilon \to 0$ and $n \to \infty$. By (3.5) and similar techniques used in (3.28), it immediately follows that $\Phi_{\alpha,2,j}(n, \varepsilon) \to 0$ as $\varepsilon \to 0$ and $n \to \infty$, for $j = 2, 3, 4$.

**Proposition 3.5** Under the conditions (A1)–(A3), we have $\Phi_3(n, \varepsilon) \to 0$ as $\varepsilon \to 0$, $n \to \infty$ and $n \varepsilon \to 0$.

**Proof** Recall that $\Phi_3(n, \varepsilon) = \varepsilon \sum_{i=1}^{n} \sigma^{-2}(X_{t_{i-1}}) b(X_{t_{i-1}}) \int_{t_{i-1}}^{t_i} \sigma(X_{s-}) dZ_s$. By (A1), (A3), and (3.13), we have

$$
|\Phi_3(n, \varepsilon)| \leq \varepsilon \sum_{i=1}^{n} K(1 + |X_{t_{i-1}}|) |b(X_{t_{i-1}})| \int_{t_{i-1}}^{t_i} \sigma(X_{s-}) dZ_s
$$

$$
\leq \varepsilon \sum_{i=1}^{n} K(1 + C_r |X_{t_{i-1}}| | + C_r |X_{t_{i-1}} - X_{t_{i-1}}|)
$$

$$
\times |b(X_{t_{i-1}})| + L|X_{t_{i-1}} - X_{t_{i-1}}| \int_{t_{i-1}}^{t_i} \sigma(X_{s-}) dZ_s
$$

$$
\leq K \varepsilon \sum_{i=1}^{n} (1 + C_r |X_{t_{i-1}}| |b(X_{t_{i-1}}))| \int_{t_{i-1}}^{t_i} \sigma(X_{s-}) dZ_s
$$

$$
\leq K \varepsilon \sum_{i=1}^{n} (1 + C_r |X_{t_{i-1}}| |b(X_{t_{i-1}})|) \int_{t_{i-1}}^{t_i} \sigma(X_{s-}) dZ_s
$$

$$
K L \varepsilon \sum_{i=1}^{n} (1 + C_r |X_{t_{i-1}}| |X_{t_{i-1}} - X_{t_{i-1}}|) \int_{t_{i-1}}^{t_i} \sigma(X_{s-}) dZ_s
$$

$$
\leq K \varepsilon \sum_{i=1}^{n} (1 + C_r |X_{t_{i-1}}| |b(X_{t_{i-1}})|) \int_{t_{i-1}}^{t_i} \sigma(X_{s-}) dZ_s
$$

$$
\leq K \varepsilon \sum_{i=1}^{n} (1 + C_r |X_{t_{i-1}}| |b(X_{t_{i-1}})|) \int_{t_{i-1}}^{t_i} \sigma(X_{s-}) dZ_s
$$

$$
+ C_r K \varepsilon \sum_{i=1}^{n} |X_{t_{i-1}} - X_{t_{i-1}}| ^{1+\varepsilon} \int_{t_{i-1}}^{t_i} \sigma(X_{s-}) dZ_s
$$

$$
= 4 \Phi_{3,j}(n, \varepsilon).
$$

(3.29)

By (A2), the Markov inequality, Lemma 2.4, and Remark 2.5, we find that for any given $\delta > 0$

$$
P(|\Phi_{3,1}(n, \varepsilon)| > \delta) \leq \delta^{-1} K \varepsilon \sum_{i=1}^{n} (1 + C_r |X_{t_{i-1}}| |b(X_{t_{i-1}})|) \int_{t_{i-1}}^{t_i} \sigma(X_{s-}) dZ_s
$$

$$
\leq \delta^{-1} K \varepsilon \sum_{i=1}^{n} (1 + C_r |X_{t_{i-1}}| |b(X_{t_{i-1}})|) \cdot 2c_2 \mathbb{E} \left[ \left( \int_{t_{i-1}}^{t_i} |\sigma(X_{s-})| dZ_s \right)^{\frac{1}{\alpha}} \right]
$$

$$
\leq 2c_2 \delta^{-1} K \sigma \varepsilon n^{1-\frac{1}{\alpha}} \cdot n^{-1} \sum_{i=1}^{n} (1 + C_r |X_{t_{i-1}}| |b(X_{t_{i-1}})|),
$$

(3.30)
which implies that \( \Phi_{3,1}(n, \varepsilon) \rightarrow P 0 \) as \( n \rightarrow \infty \) and \( n\varepsilon \frac{\alpha}{2} \rightarrow 0 \). Similarly, using (3.5) and ideas in the proof of (3.30), it follows that \( \Phi_{3,j}(n, \varepsilon) \rightarrow P 0 \) as \( n \rightarrow \infty \) and \( n\varepsilon \frac{\alpha}{2} \rightarrow 0 \), for \( j = 2, 3, 4 \).

Finally, we are in a position to prove Theorem 3.1.

**Proof of Theorem 3.1** Using Propositions 3.3, 3.4, and 3.5, we have \( \frac{\Phi_2(n, \varepsilon)}{\Phi_1(n, \varepsilon)} \rightarrow P 0 \) as \( \varepsilon \rightarrow 0 \) and \( n \rightarrow \infty \), and \( \frac{\Phi_3(n, \varepsilon)}{\Phi_1(n, \varepsilon)} \rightarrow P 0 \) as \( \varepsilon \rightarrow 0 \), \( n \rightarrow \infty \) and \( n\varepsilon \frac{\alpha}{2} \rightarrow 0 \). It immediately follows that

\[
\hat{\theta}_{n, \varepsilon} = \theta_0 + \frac{\Phi_2(n, \varepsilon)}{\Phi_1(n, \varepsilon)} + \frac{\Phi_3(n, \varepsilon)}{\Phi_1(n, \varepsilon)} \rightarrow P \theta_0
\]

as \( \varepsilon \rightarrow 0 \), \( n \rightarrow \infty \) and \( n\varepsilon \frac{\alpha}{2} \rightarrow 0 \).

\[\text{Figure 1} \quad \text{Figure 2} \quad \text{Figure 3}\]

4 **Asymptotics of the Least Squares Estimator**

Our main result of this section is as follows.

**Theorem 4.1** Under conditions (A1)–(A3), we have

\[
\varepsilon^{-1}(\hat{\theta}_{n, \varepsilon} - \theta_0) = \frac{f_0^1 |\sigma(X_s^0)|^{-2\alpha}(b(X_s^0)\sigma(X_s^0))^0 ds}{f_0^1 \sigma^{-2}(X_s^0)b^2(X_s^0)ds} U_1 - \frac{f_0^1 |\sigma(X_s^0)|^{-2\alpha}(b(X_s^0)\sigma(X_s^0))^0 ds}{f_0^1 \sigma^{-2}(X_s^0)b^2(X_s^0)ds} U_2
\]

(4.1)

as \( n \rightarrow \infty , \varepsilon \rightarrow 0 , n\varepsilon \rightarrow \infty \) and \( n\varepsilon \frac{\alpha}{2} \rightarrow 0 \), where \( U_1 \) and \( U_2 \) are independent random variables with \( \alpha \)-stable distribution \( S_a(1, \beta, 0) \).

**Remark 4.2** (i) If \( \sigma(\cdot) \) is a positive function, then, the asymptotic distribution in (4.1) becomes

\[
\left( f_0^1 \sigma^{-\alpha}(X_s^0)b^2(X_s^0)ds \right)^{\frac{1}{\alpha}} U_1 - \left( f_0^1 \sigma^{-\alpha}(X_s^0)b^2(X_s^0)ds \right)^{\frac{1}{\alpha}} U_2
\]

as \( \varepsilon \rightarrow 0 \) and \( n\varepsilon \rightarrow \infty \) with \( n\varepsilon \frac{\alpha}{2} \rightarrow 0 \).

(ii) If \( Z \) is symmetric (that is \( \beta = 0 \)), then, using some basic properties of stable distributions in [9], we find that the asymptotic distribution in (4.1) is equivalent to

\[
\left( f_0^1 |\sigma(X_s^0)|^{-\alpha}|b(X_s^0)|^0 ds \right)^{\frac{1}{\alpha}} S_a(1, 0, 0).
\]

Theorem 4.1 will be proved by establishing some preliminary results. By (3.3), we obtain

\[
\varepsilon^{-1}(\hat{\theta}_{n, \varepsilon} - \theta_0) = \varepsilon^{-1}\Phi_2(n, \varepsilon)\Phi_1(n, \varepsilon) + \varepsilon^{-1}\Phi_3(n, \varepsilon)\Phi_1(n, \varepsilon) := \Psi_2(n, \varepsilon) + \Psi_3(n, \varepsilon).
\]

(4.2)

We shall study the asymptotic behaviors of \( \Psi_i(n, \varepsilon), i = 2, 3 \), respectively.

**Proposition 4.3** Under conditions (A1)–(A3), as \( n \rightarrow \infty , \varepsilon \rightarrow 0 \) and \( n\varepsilon \rightarrow \infty \), we have

\[
\Psi_2(n, \varepsilon) \rightarrow P 0.
\]

(4.3)

**Proof** We may use all the estimates for \( \Phi_2(n, \varepsilon) \) derived in Proposition 3.4. By (3.25), we obtain

\[
|\Psi_2(n, \varepsilon)| = \varepsilon^{-1}|\Phi_2(n, \varepsilon)| \leq \varepsilon^{-1}\Phi_2,1(n, \varepsilon) + \varepsilon^{-1}\Phi_2,2(n, \varepsilon)
\]

:= \Psi_2,1(n, \varepsilon) + \Psi_2,2(n, \varepsilon).

(4.4)
From the estimate provided in (3.26), we can see that $\Psi_{2,1}(n, \varepsilon) \to p \ 0$ as $n \to \infty$, $\varepsilon \to 0$ and $n \varepsilon \to \infty$. Similarly, on the basis of (3.27) and (3.28) as well as (3.5), it is clear that $\Psi_{2,2}(n, \varepsilon) \to p \ 0$ as $n \to \infty$ and $\varepsilon \to 0$.

**Proposition 4.4** Under conditions (A1)–(A3), we have

$$
\Psi_3(n, \varepsilon) \Rightarrow \left( \int_0^1 |\sigma(X_s^0)|^{-2\alpha}(b(X_s^0)\sigma(X_s^0))_+^\alpha \, ds \right)^{\frac{1}{\alpha}} U_1
\quad - \left( \int_0^1 |\sigma(X_s^0)|^{-2\alpha}(b(X_s^0)\sigma(X_s^0))_-^\alpha \, ds \right)^{\frac{1}{\alpha}} U_2.
$$

as $n \to \infty$, $\varepsilon \to 0$ and $n \varepsilon \to p \ 0$.

**Proof** Note that

$$
\Psi_3(n, \varepsilon) = \sum_{i=1}^{n} \sigma^{-2}(X_{t_{i-1}}) b(X_{t_{i-1}}) \int_{t_{i-1}}^{t_i} \sigma(X_s) \, dZ_s
\quad = \sum_{i=1}^{n} \sigma^{-2}(X_{t_{i-1}}) b(X_{t_{i-1}}) \int_{t_{i-1}}^{t_i} \sigma(X_s) \, dZ_s
\quad + \sum_{i=1}^{n} \sigma^{-2}(X_{t_{i-1}}) b(X_{t_{i-1}}) \int_{t_{i-1}}^{t_i} (\sigma(X_s) - \sigma(X_{t_{i-1}})) \, dZ_s
\quad + \sum_{i=1}^{n} \sigma^{-2}(X_{t_{i-1}}) (b(X_{t_{i-1}}) - b(X_{t_{i-1}})) \int_{t_{i-1}}^{t_i} \sigma(X_s) \, dZ_s
\quad + \sum_{i=1}^{n} [\sigma^{-2}(X_{t_{i-1}}) - \sigma^{-2}(X_{t_{i-1}})] (b(X_{t_{i-1}}) - b(X_{t_{i-1}})) \int_{t_{i-1}}^{t_i} \sigma(X_s) \, dZ_s
\quad := \sum_{j=1}^{5} \Psi_{3,j}(n, \varepsilon).
$$

Define a deterministic process $V(s)$ by $V(s) = \sum_{i=1}^{n} \sigma^{-2}(X_{t_{i-1}}) b(X_{t_{i-1}}) \sigma(X_{t_{i-1}}^+ 1_{(t_{i-1}, t_i)}(s))$. Let $V_+(s)$ and $V_-(s)$ denote the positive and negative part of $V(s)$, respectively. By Lemma 2.3, there exist two independent processes $Z', Z'' \overset{d}{=} Z$, such that

$$
\Psi_{3,1}(n, \varepsilon) = \int_0^1 V(s) \, dZ_s = Z' \circ \int_0^1 V_+^\alpha(s) \, ds - Z'' \circ \int_0^1 V_-^\alpha(s) \, ds, \ \text{a.s.}
$$

Note that

$$
V_+^\alpha = \sum_{i=1}^{n} [\sigma(X_{t_{i-1}}^0)]^{-2\alpha}(b(X_{t_{i-1}}^0)\sigma(X_{t_{i-1}}^0))_+^\alpha 1_{(t_{i-1}, t_i)}(s)
$$

and

$$
V_-^\alpha = \sum_{i=1}^{n} [\sigma(X_{t_{i-1}}^0)]^{-2\alpha}(b(X_{t_{i-1}}^0)\sigma(X_{t_{i-1}}^0))_-^\alpha 1_{(t_{i-1}, t_i)}(s).
$$

Thus, it is clear that

$$
\int_0^1 V_+^\alpha(s) \, ds \to \int_0^1 [\sigma(X_s^0)]^{-2\alpha}(b(X_s^0)\sigma(X_s^0))_+^\alpha \, ds
$$

and

$$
\int_0^1 V_-^\alpha(s) \, ds \to \int_0^1 [\sigma(X_s^0)]^{-2\alpha}(b(X_s^0)\sigma(X_s^0))_-^\alpha \, ds
$$
and
\[
\int_{0}^{1} V^\alpha(s) \, ds \to \int_{0}^{1} |\sigma(X^0_s)|^{-2\alpha} (b(X^0_s) \sigma(X^0_s))^\alpha \, ds
\]
as \( n \to \infty \). Hence, we have
\[
Z' \circ \int_{0}^{1} V^\alpha(s) \, ds \to Z' \circ \int_{0}^{1} |\sigma(X^0_s)|^{-2\alpha} (b(X^0_s) \sigma(X^0_s))^\alpha \, ds, \quad \text{a.s.} \tag{4.10}
\]
and
\[
Z' \circ \int_{0}^{1} V^\alpha(s) \, ds \to Z' \circ \int_{0}^{1} |\sigma(X^0_s)|^{-2\alpha} (b(X^0_s) \sigma(X^0_s))^\alpha \, ds, \quad \text{a.s.} \tag{4.11}
\]
It immediately follows that
\[
\Psi_{3,1}(n, \varepsilon) \Rightarrow \left( \int_{0}^{1} |\sigma(X^0_s)|^{-2\alpha} (b(X^0_s) \sigma(X^0_s))^\alpha \, ds \right)^{\frac{1}{\alpha}} U_1
\]
\[-\left( \int_{0}^{1} |\sigma(X^0_s)|^{-2\alpha} (b(X^0_s) \sigma(X^0_s))^\alpha \, ds \right)^{\frac{1}{\alpha}} U_2 \tag{4.12}
\]
as \( n \to \infty \), where \( U_1 \) and \( U_2 \) are two independent random variables with \( \alpha \)-stable distribution \( S_\alpha(1, \beta, 0) \). For \( \Psi_{3,2}(n, \varepsilon) \), using the Markov inequality, Lemma 2.4, Remark 2.5, Lipschitz condition on \( \sigma(\cdot) \), and (3.4) in Lemma 3.2, we have, for any given \( \delta > 0 \),
\[
P(\{|\Psi_{3,2}(n, \varepsilon)| > \delta\})
\]
\[\leq \delta^{-1} \mathbb{E} \left[ \sum_{i=1}^{n} \sigma^{-2}(X^0_{t_{i-1}})|b(X^0_{t_{i-1}})| \int_{t_{i-1}}^{t_i} (\sigma(X_s) - \sigma(X^0_s)) \, dZ_s \right]
\[\leq \delta^{-1} \sum_{i=1}^{n} \sigma^{-2}(X^0_{t_{i-1}})|b(X^0_{t_{i-1}})| \mathbb{E} \left[ \int_{t_{i-1}}^{t_i} (\sigma(X_s) - \sigma(X^0_s)) \, dZ_s \right]
\[\leq \delta^{-1} \sum_{i=1}^{n} \sigma^{-2}(X^0_{t_{i-1}})|b(X^0_{t_{i-1}})| \cdot 2c_2 \mathbb{E} \left[ \left( \int_{t_{i-1}}^{t_i} |\sigma(X_s) - \sigma(X^0_s)|^\alpha \, ds \right)^{1/\alpha} \right]
\[\leq 2c_2 \delta^{-1} \sum_{i=1}^{n} \sigma^{-2}(X^0_{t_{i-1}})|b(X^0_{t_{i-1}})| \mathbb{E} \left[ \sup_{t_{i-1} \leq t \leq t_i} |X_t - X^0_t|^{\frac{\alpha}{2}} \right]
\[\leq 2c_2 \delta^{-1} \sum_{i=1}^{n} \sigma^{-2}(X^0_{t_{i-1}})|b(X^0_{t_{i-1}})| \mathbb{E} \left[ \sup_{t_{i-1} \leq t \leq t_i} |X_t - X^0_t| n^{-\frac{\alpha}{2}} \right]
\[\leq 4c_2 \delta^{-1} \sum_{i=1}^{n} \sigma^{-2}(X^0_{t_{i-1}})|b(X^0_{t_{i-1}})| \mathbb{E} \left[ \left( \int_{0}^{t} |\sigma(X_s)|^{-\alpha} \, dZ_s \right)^{1/\alpha} \right]
\[\leq O(\varepsilon n^{\frac{\alpha-1}{2}}), \tag{4.13}
\]
which tends to zero as \( n \to \infty, \varepsilon \to 0 \) and \( n\varepsilon^{-\alpha-1} \to 0 \). For \( \Psi_{3,3}(n, \varepsilon) \), we make the following decomposition:
\[
\Psi_{3,3}(n, \varepsilon) = \sum_{i=1}^{n} \sigma^{-2}(X^0_{t_{i-1}})(b(X_{t_{i-1}}) - b(X^0_{t_{i-1}})) \int_{t_{i-1}}^{t_i} \sigma(X_s) \, dZ_s
\]
\[\begin{align*}
&= \sum_{i=1}^{n} \sigma^{-2}(X^0_{t_{i-1}})(b(X_{t_{i-1}}) - b(X^0_{t_{i-1}})) \int_{t_{i-1}}^{t_i} \sigma(X^0_s) dZ_s \\
&\quad + \sum_{i=1}^{n} \sigma^{-2}(X^0_{t_{i-1}})(b(X_{t_{i-1}}) - b(X^0_{t_{i-1}})) \int_{t_{i-1}}^{t_i} (\sigma(X_s) - \sigma(X^0_s)) dZ_s \\
&:= \Psi^{(1)}_{3,3}(n, \varepsilon) + \Psi^{(2)}_{3,3}(n, \varepsilon). \\
\end{align*}\]

By Lipschitz condition on \(b(\cdot)\) and (3.4) in Lemma 3.2, we have

\[|\Psi^{(1)}_{3,3}(n, \varepsilon)| \leq \left| \sum_{i=1}^{n} \sigma^{-2}(X^0_{t_{i-1}})(b(X_{t_{i-1}}) - b(X^0_{t_{i-1}})) \int_{t_{i-1}}^{t_i} \sigma(X^0_s) dZ_s \right| \]

\[\leq L \sum_{i=1}^{n} \sigma^{-2}(X^0_{t_{i-1}})|X_{t_{i-1}} - X^0_{t_{i-1}}| \left| \int_{t_{i-1}}^{t_i} \sigma(X^0_s) dZ_s \right| \]

\[\leq L \sum_{i=1}^{n} \sigma^{-2}(X^0_{t_{i-1}})\varepsilon \left| \int_{0}^{t_{i-1}} \sigma(X^0_s) dZ_s \right| \left| \int_{t_{i-1}}^{t_i} \sigma(X^0_s) dZ_s \right|. \tag{4.15}\]

By the Markov inequality, Lemma 2.4, Remark 2.5, and the independence of \(\int_{0}^{t_{i-1}} \sigma(X^0_s) dZ_s\) and \(\int_{t_{i-1}}^{t_i} \sigma(X^0_s) dZ_s\), we find that, for any given \(\delta > 0\),

\[P(|\Psi^{(1)}_{3,3}(n, \varepsilon)| > \delta) \leq P \left( \sum_{i=1}^{n} \sigma^{-2}(X^0_{t_{i-1}})\varepsilon \left| \int_{0}^{t_{i-1}} \sigma(X^0_s) dZ_s \right| \left| \int_{t_{i-1}}^{t_i} \sigma(X^0_s) dZ_s \right| > \delta \right) \]

\[\leq \delta^{-1} L \varepsilon \sum_{i=1}^{n} \sigma^{-2}(X^0_{t_{i-1}})\varepsilon \left| \int_{0}^{t_{i-1}} \sigma(X^0_s) dZ_s \right| \left| \int_{t_{i-1}}^{t_i} \sigma(X^0_s) dZ_s \right| \]

\[\leq 4c_2 \delta^{-1} L \varepsilon \sum_{i=1}^{n} \sigma^{-2}(X^0_{t_{i-1}})\varepsilon \left[ \left( \int_{0}^{t_{i-1}} |\sigma(X^0_s)|^\alpha dZ_s \right)^{1/\alpha} \right] \]

\[\leq C \varepsilon n^{-\frac{\alpha}{\alpha}} \sum_{i=1}^{n} \sigma^{-2}(X^0_{t_{i-1}})^{1/\alpha} \leq O(n^{-\frac{\alpha-1}{\alpha}} \varepsilon), \tag{4.16}\]

which tends to zero as \(n \to \infty\) and \(n \varepsilon^{\frac{\alpha}{\alpha-1}} \to 0\). For \(\Psi^{(2)}_{3,3}(n, \varepsilon)\), using Lipschitz condition on \(b(\cdot)\), (3.5) and the same arguments as used in (4.13) for the convergence of \(\Psi^{(2)}_{3,3}(n, \varepsilon)\), we find

\[|\Psi^{(2)}_{3,3}(n, \varepsilon)| = \left| \sum_{i=1}^{n} \sigma^{-2}(X^0_{t_{i-1}})(b(X_{t_{i-1}}) - b(X^0_{t_{i-1}})) \int_{t_{i-1}}^{t_i} (\sigma(X_s) - \sigma(X^0_s)) dZ_s \right| \]

\[\leq L \sum_{i=1}^{n} \sigma^{-2}(X^0_{t_{i-1}})|X_{t_{i-1}} - X^0_{t_{i-1}}| \left| \int_{t_{i-1}}^{t_i} (\sigma(X_s) - \sigma(X^0_s)) dZ_s \right| \]

\[\leq L \sup_{0 \leq t \leq 1} |X_t - X^0_t| \cdot \sum_{i=1}^{n} \sigma^{-2}(X^0_{t_{i-1}}) \left| \int_{t_{i-1}}^{t_i} (\sigma(X_s) - \sigma(X^0_s)) dZ_s \right|. \tag{4.17}\]

which converges to zero in probability as \(\varepsilon \to 0\), \(n \to \infty\), and \(n \varepsilon^{\frac{\alpha}{\alpha-1}} \to 0\). Thus, (4.16) and (4.17) immediately follow that \(\Psi_{3,3}(n, \varepsilon) \to_p 0\) as \(\varepsilon \to 0\), \(n \to \infty\) and \(n \varepsilon^{\frac{\alpha}{\alpha-1}} \to 0\).
\[\Psi_{3,4}(n, \varepsilon), \text{ by (A1)--(A3) and (3.13), we have}\]
\[
\begin{align*}
|\Psi_{3,4}(n, \varepsilon)| \\
&\leq \sum_{i=1}^{n} \left| \sigma^{-2}(X_{t_{i-1}}) - \sigma^{-2}(X_{t_{i-1}}^0) \right| |b(X_{t_{i-1}}^0)| \left| \int_{t_{i-1}}^{t_i} \sigma(X_{s-}) dZ_s \right| \\
&\leq \sum_{i=1}^{n} \sigma^{-2}(X_{t_{i-1}}) |\sigma^{-2}(X_{t_{i-1}}) - \sigma^{-2}(X_{t_{i-1}}^0)| |b(X_{t_{i-1}}^0)| \left| \int_{t_{i-1}}^{t_i} \sigma(X_{s-}) dZ_s \right| \\
&\leq \sum_{i=1}^{n} K(1 + |X_{t_{i-1}}|) \sigma^{-2}(X_{t_{i-1}}^0) |b(X_{t_{i-1}}^0)| |X_{t_{i-1}} - X_{t_{i-1}}^0| \left| \int_{t_{i-1}}^{t_i} \sigma(X_{s-}) dZ_s \right| \\
&\leq 2\sigma_1 K \sigma_2 (X_{t_{i-1}}) |b(X_{t_{i-1}}^0)| |X_{t_{i-1}} - X_{t_{i-1}}^0| \left| \int_{t_{i-1}}^{t_i} \sigma(X_{s-}) dZ_s \right| \\
&\quad + 2\sigma_1 K \sigma_2 (X_{t_{i-1}}) |b(X_{t_{i-1}}^0)| |X_{t_{i-1}} - X_{t_{i-1}}^0| |X_{t_{i-1}} - X_{t_{i-1}}^0| \left| \int_{t_{i-1}}^{t_i} \sigma(X_{s-}) dZ_s \right| \\
&\quad + 2\sigma_1 K \sigma_2 (X_{t_{i-1}}) |b(X_{t_{i-1}}^0)| |X_{t_{i-1}} - X_{t_{i-1}}^0| |X_{t_{i-1}} - X_{t_{i-1}}^0| \left| \int_{t_{i-1}}^{t_i} \sigma(X_{s-}) dZ_s \right| \\
&:= \sum_{j=1}^{3} \Psi_{3,4}^{(j)}(n, \varepsilon). \quad (4.18)
\end{align*}
\]

The same arguments used for the convergence of \(\Psi_{3,3}(n, \varepsilon)\) yield that \(\Psi_{3,4}^{(1)}(n, \varepsilon) \rightarrow P 0\) and \(\Psi_{3,4}^{(3)}(n, \varepsilon) \rightarrow P 0\) as \(n \rightarrow \infty, \varepsilon \rightarrow 0\), and \(n \varepsilon^{-\frac{\alpha}{\beta}} \rightarrow 0\). For \(\Psi_{3,4}^{(2)}(n, \varepsilon)\), we have
\[
\begin{align*}
\Psi_{3,4}^{(2)}(n, \varepsilon) \\
&\leq \sup_{0 \leq t \leq 1} |X_t - X_t^0|^2 \sigma_1 K \sigma_2 \sum_{i=1}^{n} \sigma^{-2}(X_{t_{i-1}}) |b(X_{t_{i-1}}^0)| |X_{t_{i-1}} - X_{t_{i-1}}^0| \left| \int_{t_{i-1}}^{t_i} \sigma(X_{s-}) dZ_s \right|, \quad (4.19)
\end{align*}
\]

which converges to zero in probability, because \(\sup_{0 \leq t \leq 1} |X_t - X_t^0|^r\) converges to zero in probability as \(\varepsilon \rightarrow 0\) by (3.5) when \(r > 0\) or is bounded by 1 if \(r = 0\), and
\[
2\sigma_1 K \sigma_2 \sum_{i=1}^{n} \sigma^{-2}(X_{t_{i-1}}) |b(X_{t_{i-1}}^0)| |X_{t_{i-1}} - X_{t_{i-1}}^0| \left| \int_{t_{i-1}}^{t_i} \sigma(X_{s-}) dZ_s \right| \rightarrow P 0
\]
as \(\varepsilon \rightarrow 0, n \rightarrow \infty, n \varepsilon^{-\frac{\alpha}{\beta}} \rightarrow 0\), using the same arguments for the convergence of \(\Psi_{3,3}(n, \varepsilon)\). Hence, \(\Psi_{3,4}(n, \varepsilon) \rightarrow P 0\) as \(\varepsilon \rightarrow 0, n \rightarrow \infty\) and \(n \varepsilon^{-\frac{\alpha}{\beta}} \rightarrow 0\). Finally, for \(\Psi_{3,5}(n, \varepsilon)\), we have
\[
\begin{align*}
|\Psi_{3,5}(n, \varepsilon)| \\
&\leq \sum_{i=1}^{n} |\sigma^{-2}(X_{t_{i-1}}) - \sigma^{-2}(X_{t_{i-1}}^0)||b(X_{t_{i-1}}) - b(X_{t_{i-1}}^0)| \left| \int_{t_{i-1}}^{t_i} \sigma(X_{s-}) dZ_s \right| \\
&\leq \sum_{i=1}^{n} |\sigma^{-2}(X_{t_{i-1}}) - \sigma^{-2}(X_{t_{i-1}}^0)||\sigma^{-2}(X_{t_{i-1}}) - \sigma^{-2}(X_{t_{i-1}}^0)||b(X_{t_{i-1}}) - b(X_{t_{i-1}}^0)| \\
&\quad \times |b(X_{t_{i-1}}) - b(X_{t_{i-1}}^0)| \left| \int_{t_{i-1}}^{t_i} \sigma(X_{s-}) dZ_s \right| \\
&\rightarrow P 0.
\end{align*}
\]
\[
\leq C \sum_{i=1}^{n} \sigma^{-2}(X^0_{t_{i-1}})(1 + |X_{t_{i-1}}|)|X_{t_{i-1}} - X^0_{t_{i-1}}|^2 \int_{t_{i-1}}^{t_i} \sigma(X_s)\text{d}Z_s \\
\leq \sup_{0 \leq t \leq 1} |X_t - X^0_t| \cdot C \sum_{i=1}^{n} \sigma^{-2}(X^0_{t_{i-1}})(1 + |X_{t_{i-1}}|)|X_{t_{i-1}} - X^0_{t_{i-1}}| \int_{t_{i-1}}^{t_i} \sigma(X_s)\text{d}Z_s,
\]

(4.20)

which converges to zero in probability, because \( \sup_{0 \leq t \leq 1} |X_t - X^0_t| \to_P 0 \) as \( \varepsilon \to 0 \) by (3.5), and

\[
\varepsilon^{-1}(\hat{\theta}_{n, \varepsilon} - \theta_0) = \frac{\Psi_2(n, \varepsilon)}{\Phi_1(n, \varepsilon)} + \frac{\Psi_3(n, \varepsilon)}{\Phi_1(n, \varepsilon)}
\]

\[
\to \left( \int_0^1 |\sigma(X^0_s)|^{-2\alpha}(b(X^0_s)\sigma(X^0_s))^\alpha \text{d}s \right)^{1/\alpha} U_1 - \left( \int_0^1 |\sigma(X^0_s)|^{-2\alpha}(b(X^0_s)\sigma(X^0_s))^\alpha \text{d}s \right)^{1/\alpha} U_2
\]

\[
\int_0^1 \sigma^{-2}(X^0_s)b^2(X^0_s)\text{d}s
\]

as \( n \to \infty, \varepsilon \to 0, n\varepsilon \to \infty \) and \( n\varepsilon \to 0 \) using the same arguments for the convergence of \( \Psi_{3, 4}(n, \varepsilon) \).

Now, we are in a position to prove Theorem 4.1.

**Proof of Theorem 4.1** Using Proposition 3.3, Proposition 4.3, Proposition 4.4, and Slutsky’s theorem, we conclude that

\[
\varepsilon^{-1}(\hat{\theta}_{n, \varepsilon} - \theta_0) = \frac{\Psi_2(n, \varepsilon)}{\Phi_1(n, \varepsilon)} + \frac{\Psi_3(n, \varepsilon)}{\Phi_1(n, \varepsilon)}
\]

\[
\to \left( \int_0^1 |\sigma(X^0_s)|^{-2\alpha}(b(X^0_s)\sigma(X^0_s))^\alpha \text{d}s \right)^{1/\alpha} U_1 - \left( \int_0^1 |\sigma(X^0_s)|^{-2\alpha}(b(X^0_s)\sigma(X^0_s))^\alpha \text{d}s \right)^{1/\alpha} U_2
\]

\[
\int_0^1 \sigma^{-2}(X^0_s)b^2(X^0_s)\text{d}s
\]

as \( n \to \infty, \varepsilon \to 0, n\varepsilon \to \infty \) and \( n\varepsilon \to 0 \).

**Example 4.5** We consider a generalized Ornstein-Uhlenbeck process driven by small \( \alpha \)-stable noise:

\[
dX_t = \theta X_t dt + \varepsilon \sigma dZ_t, \quad t \in [0, 1]; \quad X_0 = x_0,
\]

(4.21)

where \( x_0, \sigma, \) and \( \varepsilon \) are known constants and \( \theta \neq 0 \) is an unknown parameter. Let \( \theta_0 \) be the true value of \( \theta \). For simplicity, let \( \sigma > 0 \) and \( x_0 > 0 \). By setting \( \varepsilon = 0 \), we obtain an ODE

\[
dX^0_t = \theta_0 X^0_t dt, \quad t \in [0, 1]; \quad X^0_0 = x_0
\]

with solution \( X^0_t = x_0e^{\theta_0 t} \). In this case, the asymptotic distribution in (4.1) is

\[
\sigma x_0^{-1} \left( \frac{2\theta_0}{e^{2\theta_0} - 1} \right) \left( \frac{e^{\alpha\theta_0} - 1}{\alpha \theta_0} \right)^{1/\alpha} S_\alpha(1, \beta, 0).
\]

This result was proved in [17].

**Example 4.6** We consider the following nonlinear SDE driven by small \( \alpha \)-stable noise:

\[
dX_t = \theta X_t dt + \frac{\varepsilon}{1 + X^2_t} dZ_t, \quad t \in [0, 1]; \quad X_0 = x_0,
\]

(4.22)

where \( x_0 \) and \( \varepsilon \) are known constants, and \( \theta \neq 0 \) is an unknown parameter. Obviously, the drift function \( b(x) = x \) and the dispersion function \( \sigma(x) = 1/(1 + x^2) \) satisfy conditions (A1)–(A3). Let \( \theta_0 \) be the true value of \( \theta \). For simplicity, let \( x_0 > 0 \). By setting \( \varepsilon = 0 \), we obtain the ODE

\[
dX^0_t = \theta_0 X^0_t dt, \quad t \in [0, 1]; \quad X^0_0 = x_0,
\]
with solution $X^0_t = x_0 e^{\theta_0 t}$. In this case, the asymptotic distribution in (4.1) is
\[
\left( \int_0^1 (1 + x^2_0 e^{2\theta_0 s})^{\alpha_0} e^{\alpha_0 \theta_0 s} ds \right)^{\frac{1}{2}} \int_0^1 (1 + x^2_0 e^{2\theta_0 s})^{\beta_0} e^{\beta_0 \theta_0 s} ds \sim S_\alpha(1, \beta, 0).
\]
In particular, if let $x_0 = 1$, $\theta_0 = 1$, $\alpha = 1.5$, and $\beta = 0$, then, we find that the asymptotic distribution is
\[
\left( \int_0^1 (1 + e^{2s})^{1.5} e^{1.5s} ds \right)^{\frac{2}{3}} S_{1.5}(1,0,0) \approx \left( \frac{26.9502}{97.0651} \right)^{2/3} S_{1.5}(1,0,0) = 0.0926 \cdot S_{1.5}(1,0,0).
\]

**Example 4.7** We consider a hyperbolic process defined by the following SDE:
\[
dX_t = \theta \frac{X_t}{\sqrt{1 + X_t^2}} dt + \varepsilon dZ_t, \; t \in [0,1]; \; X_0 = x_0,
\] (4.23)
where $x_0$ and $\varepsilon$ are known constants, and $\theta \neq 0$ is an unknown parameter. The drift function $b(x) = \frac{\varepsilon}{\sqrt{1+x^2}}$ and the dispersion function $\sigma(x) = 1$ satisfy conditions (A1)–(A3). Let $\theta_0$ be the true value of $\theta$. By setting $\varepsilon = 0$, we obtain the ODE
\[
dX^0_t = \theta_0 \frac{X^0_t}{\sqrt{1 + (X^0_t)^2}} dt, \; t \in [0,1]; \; X^0_0 = x_0,
\]
which has a unique solution $X^0_t$. For simplicity, let $\beta = 0$. Then, the asymptotic distribution in (4.1) is
\[
\left( \int_0^1 [1 + (X^0_t)^2]^{-\alpha/2} |X^0_t|^{\alpha} ds \right)^{\frac{1}{\alpha}} \int_0^1 [1 + (X^0_t)^2]^{-1} (X^0_t)^2 ds \sim S_\alpha(1,0,0).
\]

**References**


[18] Masuda H. Simple estimators for non-linear Markovian trend from sampled data: I. ergodic cases[Preprint]. Fukuoka: Kyushu University, 2005