Necessary and sufficient conditions for the symmetrizability of differential operators over infinite dimensional state spaces

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Abstract. In this paper, we establish some necessary and sufficient conditions for the symmetrizability of second-order differential operators over infinite dimensional state spaces including locally convex spaces and separable metrizable Banach manifolds. In particular, we provide an abstract and simple criterion which can be applied to verify many important known examples.

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1 Introduction

It is well known that symmetric diffusion operators play an important role in the theory of Malliavin calculus and Dirichlet forms on finite or infinite dimensional state spaces (cf. Bell [9], Fukushima, Oshima and Takeda [34], Ma and Röckner [44] and references therein). Symmetric diffusion operators are essentially related to reversible diffusion processes and invariant measures. This topic was first investigated by Kolmogorov [41] and then also by Nelson [45] independently. Let \( \{X_t\}_{t \geq 0} \) be a diffusion process on a compact Riemannian manifold \( M \) with transition semigroup \( \{P_t\}_{t \geq 0} \) which is generated by \( L = \Delta_M + b \), where \( \Delta_M \) is the Laplace-Beltrami operator and \( b \) is a smooth vector field on \( M \). Kolmogorov and Nelson proved that \( (P_t, B_b(M)) \) (here \( B_b(M) \) denotes the set of bounded Borel measurable functions on \( M \)) is symmetric on \( L^2(M, \text{dvol}) \) (\( \text{dvol} \) is the Riemannian volume measure on \( M \)) if and only if \( b \) is a gradient vector field. Obviously, in this case, the symmetrizability of \( (P_t, B_b(M)) \) on \( L^2(M, \text{dvol}) \) is equivalent to the symmetrizability of \( (L, C^\infty_0(M)) \) on \( L^2(M, \text{dvol}) \). One important property for symmetric diffusion processes is that they are reversible (cf. Fukushima [32] [33] and some references therein). In the non-compact case we need some additional conditions to ensure the existence of an invariant measure (cf. Bogachev and Röckner [15] and references therein). Symmetric diffusions with singular drifts in finite-dimensional spaces were studied by Albeverio, Hoegh-Krohn and Streit [4] and Fukushima [33] (see also some references therein).
So far there have many examples and extensions concerning the symmetrizability of transition semigroups and differential operators in infinite dimensional settings. In [19] and [20], Daletskii proved that a class of differential operator with non-constant diffusion part with respect to Gaussian measure on a scale of Hilbert spaces is symmetric and essentially self-adjoint by using the finite dimensional approximation approach. In [27], Elworthy defined the divergence of admissible vector field with respect to Gaussian measure on an abstract Wiener manifold (AWM) in terms of the integral flow of vector field and the transformation of integral formulae on abstract Wiener space (AWS) due to Kuo [42] and proved the divergence theorem on AWM and that the corresponding second order differential operator defined on $C^2$-smooth function space via Wiener data is formally self-adjoint and negative definite. In [49], Shigekawa considered the diffusion generated by the perturbed Ornstein-Uhlenbeck operator $A = \frac{1}{2}L + b$, where $L$ is the O-U operator and $b$ is a bounded $H$-valued vector field on an abstract Wiener space $(E, H, \mu)$. He proved the existence of invariant measure of diffusion and obtained the criterion for the symmetrizability of Markov semigroup by using the Malliavin calculus and Gross’s Logarithmic Sobolev Inequality. In [52], by using finite dimensional approximation approach, Zabczyk discussed the symmetrizability of diffusion processes generated by the semilinear stochastic evolution equations and provided that the non-linear perturbation term to the linear equations is of gradient form. In [6], Albeverio and Röckner proved that vector logarithmic derivatives are exactly the drifts to infinite dimensional symmetric diffusion processes via Dirichlet form theory. In [38], Hu and Kallianpur considered the analogous questions as in [6] with much more singular drift term in the dual of a nuclear space via Cameron-Martin-Girsanov theory and provided an application to stochastic quantization. The most recent and new result is due to Bogachev and Röckner [15]. They provided an extension to locally convex spaces (e.g. separable Banach spaces) and diffusion operators with singular drift part in terms of the notion of vector logarithmic derivative in the sense of Fomin (cf. [8]). Consider the following operator

$$L f = A_H f + E, \langle f', b \rangle_E, \quad f \in \mathcal{F} C^\infty_b(E)$$

where $H$ is a fixed Hilbert space which is continuously and densely embedded into a locally convex space $E$, $A_H$ is Gross’s Laplacian (cf. Gross [36]) and $b$ is a Borel measurable vector field on $E$, $\mathcal{F} C^\infty_b(E)$ denotes the collection of smooth cylindrical functions on $E$ which is dense in $L^2(E, \mu)$ and $\mu$ is a finite Borel measure on $(E, \mathcal{B}(E))$. Bogachev and Röckner proved that $(L, \mathcal{F} C^\infty_b(E))$ is symmetrizable on $L^2(E, \mu)$ if and only if the vector logarithmic derivative $\beta^\mu$ of the measure $\mu$ exists and $b = \beta^\mu$. Note that as described above, there are two kinds of symmetrizability. One is for the differential operators and another is for the diffusion processes or transition semigroups. In this paper, we only consider the symmetrizability for differential operators. In here, we should point out that many known results are only adapted to differential operators with constant diffusion part. In this paper we aim to study the symmetrizability for general differential operators with non-constant diffu-
sion part and Radon measures on locally convex spaces or separable Banach manifolds which are not necessarily finite.

The contents of this paper are organized as follows. In section 2 we present some basic notions, notation and lemma concerning differentiable measures and differential operators over infinite dimensional state spaces, which will be used throughout the paper. In section 3 we are going to provide and prove some necessary and sufficient conditions for the symmetrizability of second order differential operators with non-constant diffusion part. In particular, we give a simple and abstract criterion which guarantees the symmetrizability for a class of second-order differential operators over locally convex spaces or separable metrizable Banach manifolds including path and loop spaces. Finally, in section 4 we shall provide some important and interesting examples which can be verified via our general criteria in section 3.

We conclude the introduction by mentioning the fact that after carrying out the preliminary results of this paper, we received a preprint with the title "On transformations of smooth measure related to parabolic and hyperbolic differential equations in infinite dimensions" from Prof. Yu. L. Daleksii, in which some similar questions have been discussed in the setting of a rigged Hilbert space with Hilbert-Schmidt embedding.

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2 Preliminaries of differentiable measures and differential operators

In this section, we would like to present some basic notions, notation and lemma about differentiable measures and differential operators on certain locally convex spaces and separable metrizable Banach manifolds. For the definitions of Banach manifold, vector bundles, connection on vector bundles, covariant derivative of section over vector bundles, and Finsler structure we refer to Lang [43], Eells [24], Eiasson [26] and Palais [47] [48].

Let us first introduce some notation. Let $E$ be a locally convex space with dual space $E^*$, we denote by $\mathcal{F} C^k_c (E, \mathbb{R})$ the class of functions $f = \varphi(l_1, \ldots, l_n)$, $n \in \mathbb{N}$, $l_i \in E^*$, $\varphi \in C^k_c (\mathbb{R}^n)$. For a separable metrizable Banach manifold $M$ with Finsler structure, we denote by $C^k_b (M, \mathbb{R})$ the space of function $f : M \to \mathbb{R}$ with uniformly bounded derivative, i.e. \( \|df\| := \sup_{x \in M} \|df(x)\|_{T_x M} < +\infty \). For a pair of Banach spaces $E$ and $F$ and a subset $E_0 \subseteq E$, we denote by $C^k_b (E, E_0, F)$ the space of functions on $E$ taking values in $F$ and possessing continuous and bounded derivative along $E_0$ up to order $k$ inclusively. If $E_0 = E$, we simply use $C^k_b (E, F)$ instead of $C^k_b (E, E, F)$.

Let $M$ be a locally convex space or separable metrizable Banach manifold with Finsler structure and $A$ a collection of real-valued functions on $M$. Let $\mu$ be a positive Radon measure on $(M, \mathcal{B}(M))$. In this paper, we will consider a family of operators $\partial_2$ on $A$ indexed by a set $V$ of measurable vector fields on $M$. As examples:
(i) \( A \subseteq C^1(M, \mathbb{R}) \) and \( \partial_\zeta f = \langle df, \zeta \rangle \).

(ii) Each \( \zeta \in V \) has global flow, i.e. there exists a transformation \( T^\zeta_t : M \to M \) for all \( t \geq 0 \) such that

\[
\frac{d(T^\zeta_t(x))}{dt} = \xi(T^\zeta_t(x)), \quad T^\zeta_0(x) = x, \mu \text{ - a.s.}
\]

then we can define \( \partial_\zeta f \) by

\[
\partial_\zeta f = L^1(\mu) - \lim_{t \to 0} \frac{f(T^\zeta_t(x)) - f(x)}{t}
\]

if the limit exists.

(iii) If only the local flow of \( \zeta \) exists, i.e.

\[
\frac{dT^\zeta_t(x)}{dt} \bigg|_{t=0} = \xi(x) \quad \mu \text{ - a.s.,}
\]

we can still define \( \partial_\zeta f \) by (2.2). (cf. Bismut [13] and Fang and Malliavin [31]).

**Definition 2.1.** We say that \( \zeta \) is \( A \)-admissible if \( \partial_\zeta f \) exists for any \( f \in A \) and satisfies

1. \( f \in A, f = 0 \mu \text{-a.e.} \implies \partial_\zeta f = 0 \mu \text{-a.e.} \)
2. \( \partial_\zeta f \in L^1(M, \mu) \) for all \( f \in A \).

In addition, if \( A \) is an algebra on \((M, \mathcal{B}(M))\) which is dense in \( L^p(M, \mu) \) \( (p \geq 1) \) and \( \partial_\zeta f \) satisfies

3. \( \partial_\zeta (f + g) = \partial_\zeta f + \partial_\zeta g \) for any \( f, g \in A \);
4. \( \partial_\zeta (fg) = \partial_\zeta f \cdot g + f \partial_\zeta g \),

then \( \partial_\zeta \) is a differentiation on \( A \).

**Definition 2.2.** (1) The measure \( \mu \) on \( M \) is said to be \( A \)-differentiable along the \( A \)-admissible vector field \( \zeta \) if there exists a measure \( \nu \) such that for each function \( f \in A \subseteq L^1(M, \nu) \) one has

\[
\int_M \partial_\zeta f(x) \mu(dx) = - \int_M f(x) \nu(dx).
\]

If \( \nu \) is absolutely continuous with respect to \( \mu \), then its density is called the logarithmic derivative of \( \mu \) along the vector field \( \zeta \) and denoted by \( p_\mu(\zeta; x) \), i.e. \( p_\mu(\zeta; x) = \frac{\nu(dx)}{\mu(dx)} \).
(2) Let $M = E$ be a locally convex space. Further we assume that there exists a separable Hilbert space $H$ such that $H \subset E$ continuously and densely. Let $i : H \to E$ be the embedding mapping. After identifying $E^*$ and $H$ with their images in $E$, we get the triple

$$E^* \subset H \subset E.$$  

In (1) if the vector field $\xi$ is the constant vector field $h \in E$, then we say that the measure $\mu$ is $A$-differentiable along the vector $h \in E$. If a measure $\mu$ is $A$-differentiable along all directions from $E^*$ and there exists a Borel measurable mapping $\beta : E \to E$ such that

$$\rho_\mu(l, x) = E \cdot \langle l, \beta(x) \rangle_E \quad \forall l \in E^*,$$

then we call $\beta$ the vector logarithmic derivative of $\mu$ with respect to $H$.

**Remark 2.3.** The above definition is due to Daletskii and Maryanin [22] and Norin [46] which are motivated from Fomin [8] and Skorohod [50].

(i) For constant vector fields on locally convex spaces, if $A$ is the collection of indicator functions of Borel subsets of $E$ we obtain differentiability in the sense of Fomin. If $A$ is the collection of bounded continuous functions, we obtain differentiability in the sense of Skorohod. Note that there is a definition given in Bogachev and Röckner [15] for $A = \mathcal{F} C_0^\infty (E, \mathbb{R})$ which was applied to deal with the regularity of invariant measures on infinite dimensional spaces. For detailed discussion and comments on different kinds of differentiability of measures on locally convex spaces we refer to the survey paper [16] by Bogachev and Smolyanov and monograph [21] by Daletskii and Fomin. For some extensions to differentiability of a one-parameter family of measures on locally convex spaces one can see Smolyanov and Weizsäcker [51]. Many interesting examples can be found in [15].

(ii) For non-constant vector fields on Banach manifolds, in here we should point out that in Elworthy [27], when he dealt with the divergence theorem and Laplace-Beltrami operators on abstract Wiener manifolds (AWM), he gave a definition of divergence of vector field $\xi$ with respect to Gaussian measure $\mu$, i.e.

$$\text{div}_\mu \xi = \frac{d}{dt} \frac{dT_t^\xi(\mu)}{d\mu} |_{t=0},$$

which coincides with the above definition of logarithmic derivative of $\mu$ along the vector field $\xi$ for suitable chosen $A$ (e.g. $C_0^\infty (M, \mathbb{R})$).

(iii) For the uniqueness of the logarithmic derivative $p_\mu(\xi, \cdot)$, if $p_\mu(\xi, \cdot) \in L^p(M, \mu)$ $(p \geq 1)$ and $A$ is dense in $L^q(M, \mu)$ with $q$ being the conjugate number of $p$, i.e.
\[
\frac{1}{p} + \frac{1}{q} = 1, \text{ then we can easily see that } \rho_\mu(\xi, \cdot) \text{ is uniquely determined by (2.3) } \mu\text{-a.e.}. \text{ However, generally speaking, it is nontrivial to find sufficient conditions which guarantee } \rho_\mu(\xi, \cdot) \in L^p(M, \mu). \text{ For the conditions such that } \\
\rho_\mu(\xi, \cdot) \in L^2(M, \mu), \text{ we refer to Norin [46], Belopolskaya and Daletskii [11] and Albeverio, Kusuoka and Röckner [5].}
\]

(iv) When \((E, H, \mu)\) is the classical Wiener space, then the logarithmic derivative of classical Wiener measure \(\mu\) along nonanticipating (or anticipating) vector field in \(H\) is exactly Ito's (or Skorohod's) stochastic integral. Therefore sometimes logarithmic derivative is also called extended stochastic integral (cf. [22] and [46]).

If we assume that \(A\) is an algebra on \((M, \mathcal{B}(M))\) which is dense in \(L^p(M, \mu)\) for any \(p \geq 1\), then we have the following lemma which was originally proved in [22] (see also Belopolskaya and Daletskii [11]).

**Lemma 2.4.** Suppose that \(\xi\) is a \(A\)-admissible vector field and \(\partial_\xi\) is a differentiation on \(A\) in the \(L^1(M, \mu)\) sense. Let \(\mu\) be \(A\)-differentiable and have logarithmic derivative \(\rho_\mu(\xi; x)\) along an admissible vector field \(\xi\) and let \(x \in A\). Then \(\mu\) is \(A\)-differentiable along the vector field \(\partial_\xi\) and

\[
(2.4) \quad \rho_\mu(\partial_\xi; x) = \alpha(x)\rho_\mu(\xi; x) + \partial_\xi \alpha(x).
\]

**Proof.** From the Definition 2.2 of the logarithmic derivative of \(\mu\) along \(A\)-admissible vector field, we have for any \(f \in A\)

\[
\int_M \alpha(x)f(x)\nu(dx) = \int_M \partial_\xi(\alpha f)(x)\mu(dx) \\
= -\int_M [\alpha(x)\partial_\xi f(x) + f(x)\partial_\xi \alpha(x)]\mu(dx) \\
= -\int_M [\partial_\xi f(x) + f(x)\partial_\xi \alpha(x)]\mu(dx).
\]

This is equivalent to

\[
\int_M \partial_\xi f(x)\mu(dx) = -\int_M f(x)[\alpha(x)\rho_\mu(\xi; x) + \partial_\xi \alpha(x)]\mu(dx).
\]

Therefore \(\mu\) is \(A\)-differentiable along the vector field \(\partial_\xi\) and has logarithmic derivative

\[
\rho_\mu(\partial_\xi; x) = \alpha(x)\rho_\mu(\xi; x) + \partial_\xi \alpha(x).
\]

This completes the proof. \(\square\)
Definition 2.5. Let \( \mu \) be \( A \)-differentiable along a vector field \( \xi \) on \( M \) and has logarithmic derivative \( \rho_{\mu}(\xi) \). We say that the divergence theorem holds for the vector field \( \xi \) with respect to measure \( \mu \) if \( \rho_{\mu}(\xi) \) is \( \mu \)-integrable and the following equality

\[
(2.5) \quad \int_{M} \rho_{\mu}(\xi(x)) \mu(dx) = 0
\]

holds. Obviously, if \( 1 \in A \), then the divergence theorem always holds for any \( A \)-admissible vector field which satisfies (2.3).

Now we introduce the directional derivative on \( M \) with respect to a separable Hilbert space \( H \). Let \( X(\cdot) \in \mathcal{L}(H, M) \) (or \( \mathcal{L}(H, TM) \)) and set \( X_{h}(\cdot) = X(\cdot)h \in M \) (or \( TM \)). Assume that \( \partial_{X_{h}}f \) is well defined for some real-valued function \( f \) on \( M \). If for any \( h \in H \), there exists a measurable mapping \( D_{X}f : M \to H \) such that

\[
(D_{X}f, h)_{H} = \partial_{X_{h}}f,
\]

then \( D_{X}f \) is called the \( H \)-gradient of \( f \) under the action of \( X \) with respect to \( H \). Assume that \( \mathcal{D} \) is an algebra on \( M \) which is dense in \( L^{2}(M, \mu) \) and \( \mathcal{D} \subseteq C^{1}(M, \mathbb{R}) \cap L^{2}(M, \mu) \). Now we define a second order differential operator \( L \) with domain \( \mathcal{D} \) by

\[
(2.6) \quad Lf(x) = \sum_{k=1}^{\infty} \partial_{X_{h}}^2 f(x) + \langle df, b(x) \rangle, \quad f \in \mathcal{D},
\]

where \( b : M \to M \) (or \( TM \)) is a measurable vector field and \( \{ e_{k} \}_{k=1}^{\infty} \) is an orthonormal basis of \( H \). Note that in the expression (2.6), we don’t write out explicitly the precise pairing for \( \langle \cdot, \cdot \rangle \). If \( M \) is a locally convex space (e.g. separable Banach space) \( E \), then \( \langle \cdot, \cdot \rangle \) means \( E \langle \cdot, \cdot \rangle_{E} \). If \( M \) is a separable metrizable Banach manifold, then \( \langle \cdot, \cdot \rangle \) means \( T_{X}M \langle \cdot, \cdot \rangle_{T_{X}M} \). Generally speaking, there are different ways to define our second-order differential operators on \( M \) according to different given conditions on \( M \). If we assume that \( M \) is a separable metrizable \( C^{2} \)-Banach manifold with Finsler structure modelled on abstract Wiener space \( (i, H, E) \) and endowed with a linear connection which induces a covariant derivative \( \nabla \), and let \( \mathcal{D} \) be an algebra on \( (M, \mathcal{B}(M)) \) which is dense in \( L^{2}(M, \mu) \), then we can define the differential operator \( L \) with domain \( \mathcal{D} \) by

\[
(2.7) \quad Lf(x) = \text{Tr}_{H} i^{*} \sigma^{\ast}(x) \nabla(df) \sigma(x)i + \tau, \langle b(x), df(x) \rangle_{T_{X}M},
\]

where \( \sigma(\cdot) : M \to \mathcal{L}(E, TM) \) and \( b : M \to TM \) is a measurable vector field.

In the flat case \( M = E \), let \( H \) be an separable Hilbert space. As in Brzeźniak and Elworthy [17], we put

\[
\mathcal{M}(\tilde{H}, E) := \{ Q : \tilde{H} \to E, Q \text{ is linear bounded and } \mu \text{-radonifying} \}. \]
Let $v_Q$ be the $\sigma$-additive measure induced by $Q$ from the canonical cylindrical Gaussian measure $\gamma_H$ on $H$. For $Q \in \mathcal{M}(\tilde{H}, E)$ we put

$$
\|Q\|_{\mathcal{M}(\tilde{H}, E)} := \left\{ \int_E |x|^2 dv_Q(x) \right\}^{1/2},
$$

which, in view of Fernique's Theorem, is a finite number. For a given $Q \in \mathcal{M}(\tilde{H}, E)$ and a bilinear bounded map $G : E \times E \to F$, we define

$$
\text{Tr}_Q G := \int_E G(x, x) dv_Q(x).
$$

It is well known and easy to prove that

$$
\text{Tr}_Q G = \sum_{k=1}^\infty G(Q\tilde{v}_k, Q\tilde{v}_k) := \text{Tr}_\tilde{H} Q^* G Q,
$$

where $\{\tilde{v}_k\}_{k=1}^\infty$ is an orthonormal basis of $\tilde{H}$.

Let $\mathcal{D}$ be an algebra on $E$ which is dense in $L^2(E, \mu)$. Now we define the differential operator $L$ with domain $\mathcal{D}$ as follows

$$
Lf(x) = \text{Tr}_\tilde{H} \sigma^*(x) f''(x) \sigma(x) + E\langle b(x), f'(x) \rangle_E,
$$

where $\sigma : E \to \mathcal{M}(\tilde{H}, E)$ and $b : E \to E$ are measurable mappings and satisfy the following assumption

(C1) \quad \|\sigma(\cdot)\|_{M(\tilde{H}, E)}^2 \in L^2(E, \mu) \quad \text{and} \quad \|b(\cdot)\|_E \in L^2(E, \mu).

From the argument in section 2 of Albeverio and Hoegh-Krohn [3], we know that $\mathcal{S} C^\infty_{b,0}$ is dense in $L^p(E, \mu)$ for any $p \geq 1$. Obviously we have the inclusion relation

$$
\mathcal{S} C^\infty_{b,0} \subset C^2_b \subset L^p(E, \mu)(p \geq 1),
$$

which implies that $C^2_b$ is dense in $L^p(E, \mu)$ for any $p \geq 1$.

More concretely, we can assume that $(\tilde{i}, \tilde{H}, \tilde{E})$ and $(i, H, E)$ are two abstract Wiener spaces (AWS). Further we suppose that $\mathcal{A}(x) \in \mathcal{L}(E)$ leaves $H$ invariant for all $x \in E$ and satisfies the following assumptions

(A1) \quad $A \in C_b^1(\mathcal{L}(E))$ and there exists $B(x) \in \mathcal{L}(\tilde{H}, H)$ for each $x \in E$ such that the restriction $\mathcal{A}_H(x)$ of $\mathcal{A}(x)$ from $E$ to $H$ satisfies

$$
\mathcal{A}_H(x) = B(x) B^*(x)
$$

and $\mathcal{A}_H(x)$ is uniformly symmetric positive definite operator in $\mathcal{L}(H)$, i.e. there exists a constant $\lambda > 0$ such that $\langle \mathcal{A}_H(x) h, h \rangle \geq \lambda \|h\|^2_H$ for any $h \in H$ and $x \in E$. 

(A2) There exists \(a(x) \in \mathcal{L}(E, E)\) for each \(x \in E\) such that \(a(x)^* = iB(x)\).

(A3) For each \(x \in E\), the divergence of \(A_H(x)\) defined by

\[
\text{Tr} \, DA_H(x) = \sum_{k=1}^{\infty} DA_H(x)(e_k, e_k)
\]

exists in \(H\) in the sense that the series is absolutely convergent, where \(\{e_k\}_{k=1}^{\infty}\) is an orthonormal basis of \(H\).

Now we define the second order differential operator \(L\) for \(f \in C^2_0(E, \mathbb{R})\) by

\[
(Lf)(x) = \text{Tr}_{a(x)} f''(x) + \frac{d}{dx} \langle A(x)b(x) + \text{Tr} \, DA_H(x), df(x) \rangle_{E^*},
\]

where \(b : E \rightarrow E\) is a measurable mapping.

From the fact that for any \(\Phi \in \mathcal{L}(E, E^*)\), the restriction of \(\Phi\) to \(H\) is a trace class operator of \(\mathcal{H}\) (cf. Theorem 4.6 (Goodman) of Chapter I in Kuo [42]), we know that \(\text{Tr}_{a(x)} f''(x)\) is well-defined.

**Definition 2.6.** Let \(L\) be a second-order differential operator with domain \(\mathcal{D}\) which is dense in \(L^2(M, \mu)\). Then we say that \((L, \mathcal{D})\) is **symmetric** on \(L^2(M, \mu)\) if the following equality holds

\[
\int_M Lf(x) \cdot g(x) \mu(dx) = \int_M f(x)Lg(x) \mu(dx) \quad \text{for all } f, g \in \mathcal{D}.
\]

**Definition 2.7.** We say that \(\mu\) is a **generalized \((L, \mathcal{D})\)-invariant measure** on \((E, \mathcal{B}(E))\) if

\[
\int_E L f(x) \mu(dx) = 0 \quad \text{for all } f \in \mathcal{D}.
\]

**Remark 2.8.** (1) Generally speaking, the symmetrizability of \((L, \mathcal{D})\) should not imply that \(\mu\) is a generalized \((L, \mathcal{D})\)-invariant measure on \((M, \mathcal{B}(M))\). However if \(1 \in \mathcal{D}\), then obviously the implication is true.

(2) Suppose that \(M\) is a d-dimensional complete manifold and the differential operator \(L\) with domain \(C^\infty_0(M, \mathbb{R})\) is well-defined on \(L^2(M, \mu)\) for suitable Radon measure \(\mu\) on \(M\). Then the symmetrizability of \((L, \mathcal{D})\) on \(L^2(M, \mu)\) implies that \(\mu\) is a generalized \((L, \mathcal{D})\)-invariant measure on \(M\). In fact, it's well-known that there exists a sequence of functions \(\{h_n\}_{n \in \mathbb{N}} \in C^\infty_0(\mathbb{R})\) such that \(0 \leq h_n \leq 1\) and

\[
h_n(t) = \begin{cases} 1 & \text{if } t \in \{t \in \mathbb{R} : \lvert t \rvert \leq n\} \\ 0 & \text{if } t \in \{t \in \mathbb{R} : \lvert t \rvert > n + 1\} \end{cases}
\]

Since \(M\) is complete, there exists a function \(\varphi \in C^\infty(M, \mathbb{R})\) such that \(\{x \in M : \)
\(|\varphi(x)| \leq K\) is compact for any \(K > 0\). Now we set \(f_n = h_n \circ \varphi\). Then for any \(g \in C_0^\infty(M)\) we have

\[
\int_M f_n(x)Lg(x)\mu(dx) = \int_M g(x)Lf_n(x)\mu(dx).
\]

For any \(g \in C_0^\infty(M)\), we can always find an \(n_0 \in \mathbb{N}\) such that \(\text{supp} g \subseteq \text{supp} f_{n_0}\). Therefore, obviously when \(n \geq n_0 + 1\) one has

\[
\int_M f_n(x)Lg(x)\mu(dx) = \int_M Lg(x)\mu(dx)
\]

and

\[
\int_M g(x)Lf_n(x)\mu(dx) = 0.
\]

It follows that

\[
\int_M Lg(x)\mu(dx) = 0
\]

for any \(g \in C_0^\infty(M)\). This means that \(\mu\) is a generalized \((L, C_0^\infty(M))\)-invariant measure on \(M\).

3 Necessary and sufficient conditions for the symmetrizability of differential operators over infinite dimensional state spaces

In this section, we shall provide some necessary and sufficient conditions for the symmetrizability of differential operator with non-constant diffusion part on infinite dimensional state spaces. First we have the following necessary and sufficient condition for the symmetrizability of differential operators defined as in (2.6).

**Theorem 3.1.** (1) Let \(L\) be well-defined with domain \(\mathcal{D}\) on \(L^2(M, \mu)\). If \((L, \mathcal{D})\) is symmetric and \(\mu\) is a generalized \((L, \mathcal{D})\)-invariant measure, then \(\mu\) is \(\mathcal{D}\)-differentiable along the vector field \(\xi = \alpha XD_X f\) for any \(\alpha, f \in \mathcal{D}\) and has logarithmic derivative

\[
(3.1) \quad \rho_\mu(\xi; x) = \sum_{k=1}^{\infty} \hat{\delta}_{X_k}(\alpha(x)\hat{\delta}_{X_k} f(x)) + \langle b(x), \alpha(x)df(x) \rangle.
\]

(2) If \(\mu\) is \(\mathcal{D}\)-differentiable along the vector fields \(\eta = XD_X f\) and \(\xi = \alpha XD_X f\) for any \(\alpha, f \in \mathcal{D}\) and has logarithmic derivative \(\rho_\mu(\xi; x)\) represented as in (3.1) which satisfies (2.5), i.e. the divergence theorem holds for the vector field \(\xi\), then \((L, \mathcal{D})\) is symmetric on \(L^2(E, \mu)\). In addition, if there exists a sequence of functions \(\{h_n\}_{n \in \mathbb{N}} \subset \mathcal{D}\) converg-
ing to \( 1 \) such that \( \int_{\mathcal{M}} (D_X h_n, D_X f)_H \mu(dx) \) tends to zero as \( n \to \infty \), then \( \mu \) is also a generalized \((L, \mathcal{D})\)-invariant measure.

**Proof.** (1) It is easy to calculate that
\[
\partial^2_{X_k}(f g) = \partial_{X_k} (\partial_{X_k} f \cdot g + f \cdot \partial_{X_k} g) = \partial^2_{X_k} f \cdot g + 2 \partial_{X_k} f \cdot \partial_{X_k} g + f \cdot \partial^2_{X_k} g.
\]

Then for \( f, g \in \mathcal{D} \)

\[
L(f g) =gLf + fLg + 2 \sum_{i=1}^{\infty} \partial_{X_k} f \cdot \partial_{X_k} g =gLf + fLg + 2(D_X f, D_X g)_H.
\]

Since \((L, \mathcal{D})\) is symmetrizable on \( L^2(M, \mu) \) and \( \mu \) is a generalized \((L, \mathcal{D})\)-invariant measure, thus for any \( f, g \in \mathcal{D} \)

\[
\int_M Lf \cdot g \mu(dx) = \int_M Lg \mu(dx)
\]

and

\[
\int_M Lf \mu(dx) = 0.
\]

Therefore we have

\[
0 = \int_M L(f g) \mu(dx) = 2 \int_M gLf \mu(dx) + 2 \int_M (D_X f, D_X g)_H \mu(dx),
\]

which implies that

\[
(3.2) \quad \int_M gLf \mu(dx) = - \int_M (D_X f, D_X g)_H \mu(dx).
\]

This shows that \( \mu \) has logarithmic derivative along vector field \( \xi = XD_X f \) and

\[
(3.3) \quad \rho_{\mu}(\xi; x) = \sum_{k=1}^{\infty} \partial^2_{X_k} f + \langle b(x), df(x) \rangle.
\]
In (3.2) by using $\alpha g$ instead of $g$ ($x \in \mathcal{Q}$) we obtain
\[
\int_M \alpha g L f \mu(dx) = -\int_M \left[ (D_X f, x(x)D_X g(x))_H + (D_X f(x), g(x)D_X x(x))_H \right] \mu(dx),
\]
which is equivalent to
\[
\int_M g(x) \left[ \sum_{k=1}^N \partial_{x_k} (\alpha(x)\partial_{x_k} f(x)) + \langle b(x), \alpha(x)df(x) \rangle \right] \mu(dx)
\]
\[
= -\int_M \langle \alpha(x)D_X f(x), D_X g(x) \rangle_H \mu(dx).
\]
Hence $\mu$ is $\mathcal{Q}$-differentiable along the vector field $\xi = \alpha X D_X f$ and has logarithmic derivative
\[
(3.5) \quad \rho_\mu(\xi; x) = \sum_{k=1}^N \partial_{x_k} (\alpha(x)\partial_{x_k} f(x)) + \langle b(x), \alpha(x)df(x) \rangle.
\]
This completes the proof.
(2) Since $\rho_\mu(\xi; \cdot)$ satisfies (2.5), by using Lemma 2.4, we have for the vector field $\xi = g(x)X(x)D_X f(x)$ ($f, g \in \mathcal{Q}$)
\[
0 = \int_M \rho_\mu(\xi; x) \mu(dx)
\]
\[
= \int_M g(x) \rho_\mu(XD_X f; x) \mu(dx) + \int_M \partial_{XD_X f} g(x) \mu(dx)
\]
\[
= \int_M g(x) \rho_\mu(XD_X f; x) \mu(dx) + \int_M \langle D_X g(x), D_X f(x) \rangle_H \mu(dx).
\]
It follows that
\[
\int_M g(x) \rho_\mu(XD_X f; x) \mu(dx) = -\int_M \langle D_X f(x), D_X g(x) \rangle_H \mu(dx),
\]
which is equivalent to
\[
(3.6) \quad \int_M g(x)L f(x) \mu(dx) = -\int_M \langle D_X f(x), D_X g(x) \rangle_H \mu(dx).
\]
Therefore \((L, \mathcal{D})\) is symmetric on \(L^2(M, \mu)\). Since \(h_n \in \mathcal{D}\) for each \(n \in \mathbb{N}\), from (3.6) we have

\[
(3.7) \quad \int_M h_n(x)Lf(x)\mu(dx) = -\int_M (D_X h_n(x), D_X f(x))_\mu(dx).
\]

Since the right-hand side of (3.7) tends to zero and \(h_n \to 1\) when \(n \to \infty\), by using Lebesgue dominated convergence theorem we can easily conclude that

\[
\int_M Lf(x)\mu(dx) = 0.
\]

This completes the proof. \(\Box\)

Combining (1) and (2) in Theorem 3.1 we have the following immediate result:

**Corollary 3.2.** Let \((L, \mathcal{D})\) be well-defined on \(L^2(M, \mu)\). If \(1 \in \mathcal{D}\), then \((L, \mathcal{D})\) is symmetric on \(L^2(M, \mu)\) if and only if \(\mu\) is \(\mathcal{D}\)-differentiable along the vector field \(\xi(x) = \sigma(x)X(x)\) for any \(x, f \in \mathcal{D}\) and has logarithmic derivative

\[
\rho_\mu(\xi, x) = \sum_{k=1}^\infty \langle \hat{\sigma}_k(x), \partial_{X_k} f(x) \rangle + \langle b(x), \sigma(x)df(x) \rangle.
\]

Similarly we can prove the following results:

**Theorem 3.3.** Let \(M\) be a separable metrizable Banach manifold with Finsler structure and endowed with a connection \(\nabla\) and \(\mathcal{D}\) be an algebra on \((M, \mathcal{B}(M))\) which is dense in \(L^2(M, \mu)\). Assume that the differential operator \(L\) is defined as in (2.7).

(1) If \((L, \mathcal{D})\) is symmetric on \(L^2(M, \mu)\) and \(\mu\) is a generalized \((L, \mathcal{D})\)-invariant measure, then \(\mu\) is \(\mathcal{D}\)-differentiable along the vector field

\[
\xi(x) = \sigma(x)\sigma(x)ii^*\sigma^*(x)df(x)
\]

for any \(\sigma, f \in \mathcal{D}\) and has logarithmic derivative

\[
(3.8) \quad \rho_\mu(\xi, x) = \nabla_1i^*\sigma^*(x)V(\sigma(x)df(x))\sigma(x)i + \nabla_\mu(b(x), \sigma(x)df(x))_{T\mathcal{M}}.
\]

(2) If \(\mu\) is \(\mathcal{D}\)-differentiable along the vector field \(\eta(x) = \sigma(x)ii^*\sigma^*(x)df(x)\) and \(\xi = \sigma\eta\) for any \(\sigma, f \in \mathcal{D}\) and has logarithmic derivative \(\rho_\mu(\xi, x)\) defined as in (3.8) which satisfies (2.5), then \((L, \mathcal{D})\) is symmetric on \(L^2(E, \mu)\).

(3) If \(1 \in \mathcal{D}\), then \((L, \mathcal{D})\) is symmetric on \(L^2(M, \mu)\) if and only if \(\mu\) is \(\mathcal{D}\)-differentiable along the vector field

\[
\xi(x) = \sigma(x)\sigma(x)ii^*\sigma^*(x)df(x)
\]

for any \(\sigma, f \in \mathcal{D}\) and has logarithmic derivative \(\rho_\mu(\xi, x)\) as defined in (3.8).
**Theorem 3.4.** Let $E$ be a separable Banach space and $L$ be the differential operator defined as in (2.8) with domain $\mathcal{D}$.

(1) If $(L, \mathcal{D})$ is symmetric on $L^2(E, \mu)$ and $\mu$ is a generalized $(L, \mathcal{D})$-invariant measure, then $\mu$ is $\mathcal{D}$-differentiable along the vector field $\xi = az^{\ast}df$ for any $z$ and $f \in \mathcal{D}$ and has logarithmic derivative

$$
\rho_{\mu}(\xi; x) = \text{Tr}_H \sigma^*(x)(\sigma(x)f'(x))'\sigma(x) + E \langle \sigma(x)df(x), b(x) \rangle_E.
$$

(2) If $\mu$ is $\mathcal{D}$-differentiable along the vector fields $\eta = \sigma^*df$ and $\xi = az^{\ast}df$ for any $z$ and $f \in \mathcal{D}$ and has logarithmic derivative $\rho_{\mu}(\xi; x)$ defined as in (3.9) which satisfies (2.5), then $(L, \mathcal{D})$ is symmetric on $L^2(E, \mu)$.

(3) If $1 \in \mathcal{D}$, then $(L, \mathcal{D})$ is symmetric on $L^2(E, \mu)$ if and only if $\mu$ is $\mathcal{D}$-differentiable along the vector fields $\xi = az^{\ast}df$ for any $z$ and $f \in \mathcal{D}$ and has logarithmic derivative $\rho_{\mu}(\xi; x)$ defined as in (3.9).

**Remark 3.5.** In Theorem 3.1, we can regard the symmetric operator $L$ as $D_X^*D_X$, where $D_X^*$ is the adjoint operator of $D_X$ with respect to the measure $\mu$ via the integration by parts formula (3.2). This explanation is valid for Theorem 3.3 and Theorem 3.4 by taking $D_Xf = i^{\ast}\sigma^*df$ and $D_Xf = \sigma^*df$ respectively.

Next we want to discuss the relationship between the symmetrizability of the differential operator $L$ defined as in (2.9) and the differentiable measure $\mu$. However, in order to ensure the operator $(L, C_0^\infty)$ is well-defined, we need two additional assumptions on the coefficients of $L$:

(A4) $\|\text{Tr}DA_H(\cdot)\|_H \in L^2(E, \mu)$ and $\|a(\cdot)\|_{L^2(E, \mu)} \in L^2(E, \mu)$.

(b1) $\|b(\cdot)\|_E \in L^2(E, \mu)$.

In the following we will use the notation $\text{div} \xi(x) = \text{Tr}_H i^{\ast}\xi^\prime(x) i$ which is well-defined for any $\xi \in C_0^\infty(E, E^\ast)$ (cf. Efimova and Uglanov [25]). Then we can rewrite the operator $L$ as follows:

$$
L^\prime f(x) = \text{Tr}_H f''(x) + E \langle A(x)b(x), df(x) \rangle_E.
$$

$$
= \text{Tr}_H A_H(x)D^2f(x) + (\text{Tr}_H D^2A_H(x), df(x))_H
$$

$$
+ E \langle A(x)b(x), df(x) \rangle_E.
$$

By using the same arguments as in the proof of Theorem 3.1, we can prove the following result.
Theorem 3.6. Let the assumptions (A1)–(A4) and (b1) be fulfilled. Then \((L, C^2_b)\) is symmetric on \(L^2(E, \mu)\) if and only if \(\mu\) is \(C^1_b\)-differentiable along the vector field \(\xi = A^*(\cdot)df\) for any \(f \in C^1_b\) and has logarithmic derivative

\[
\rho_\mu(\xi; x) = \text{div}(A^*(x)f'(x)) + E \langle A^*(x)df(x), b(x) \rangle_E.
\]

Now we intend to consider some special case e.g. \(E\) is a locally convex space and the diffusion part is the identity i.e. \(X = i\) and \(H = H\) which can be continuously and densely embedded into \(E\). After identifying \(E^*\) and \(H\) with their images in \(E\), we have the Gelfand triple

\[
E^* \subset H \subset E.
\]

Let \(\mu\) be a finite Radon measure on \((E, \mathcal{B}(E))\). Let \(\{e_k\}_{k=1}^\infty\) be an orthonormal basis of \(H\). Now we define the differential operator \(L\) with domain \(\mathcal{F}C^\infty_b = \mathcal{F}C^\infty_b(E, \mathbb{R})\) by

\[
Lf = \Delta_H f(x) + E \langle b(x), df(x) \rangle_{E^*},
\]

where \(\Delta_H\) is Gross's Laplacian and \(b : E \to E\) is a measurable mapping. Then we have the following basic proposition concerning the vector logarithmic derivative of the measure \(\mu\).

Proposition 3.7. Assume that \(E \langle l, b(\cdot) \rangle_E \in L^2(E, \mu)\) for each \(l \in E^*\). Then \(\mu\) is \(\mathcal{F}C^\infty_b\)-differentiable along the vector field \(\xi = \alpha df\) for any \(\alpha, f \in \mathcal{F}C^\infty_b\) and has logarithmic derivative

\[
\rho_\mu(\xi; x) = \sum_{k=1}^\infty \delta_{e_k}(\alpha(x)\delta_{e_k}f(x)) + E \langle b(x), \alpha(x)df(x) \rangle_E.
\]

if and only if the vector logarithmic derivative \(\beta : E \to E\) exists and \(\beta = b\).

Proof. Necessary part. Taking \(\xi = df\) and from the Definition 2.2 of \(\mathcal{F}C^\infty_b\)-differentiability of measure \(\mu\) along the vector field \(\xi\), we have for any \(g \in \mathcal{F}C^\infty_b\)

\[
\int_E \delta_{df}gd\mu = -\int_E gp_\mu(df; x)d\mu.
\]

As in Bogachev and Röckner [15], we can choose a sequence of functions \(\varphi_n \in \mathcal{C}^\infty_c(\mathbb{R})\), \(n \in \mathbb{N}\) such that \(\varphi_n(t) = t\) on \([-n, n]\) and \(\sup\{|\varphi_n'(t)| + |\varphi_n''(t)|, n \in \mathbb{N}, t \in \mathbb{R}\} < +\infty\). Let \(l \in E^*\) and \(h = i^*(l)\). Then by applying (3.12) to \(f = \varphi_n(l)\), we get for any \(g \in \mathcal{F}C^\infty_b\)
(3.13) \[ \int_E \varphi_0'(t)(h, D_0(x))_H d\mu = - \int_E g(x)[\varphi_0''(t)(h, h)_H + \varphi_0'(t)E \cdot \langle l, b(x) \rangle_E] d\mu. \]

According to Lebesgue's dominated convergence theorem, by taking \( n \to \infty \) in (3.13), we obtain

(3.14) \[ \int_E \hat{c}_n g(x) d\mu = - \int_E g(x)E \cdot \langle l, b(x) \rangle_E d\mu. \]

This implies that the vector logarithmic derivative \( \beta : E \to E \) exists and \( \beta = b \).

Sufficient part. If the vector logarithmic derivative \( \beta : E \to E \) exists and \( \beta = b \), then for any \( l \in E^* \) and \( g \in \mathcal{F} C_b^\infty \)

(3.15) \[ \int_E \hat{c}_{l^{-1}} g(x) d\mu = - \int_E g(x)E \langle \beta(x), l \rangle_E d\mu. \]

By using \( \varphi g \) instead of \( g \) for any \( \varphi, g \in \mathcal{F} C_b^\infty \) in (3.15), we obtain

(3.16) \[ \int_E \hat{c}_{\varphi(x)l^{-1}} g(x) d\mu = - \int_E g(x)(\varphi(x)E \langle \beta(x), l \rangle_E + \hat{c}_{l^{-1}} \varphi(x)) d\mu. \]

Now for any \( f \in \mathcal{F} C_b^\infty \), we can find \( \varphi \in C_b^\infty (\mathbb{R}^n, \mathbb{R}) \) and \( l_1, \ldots, l_n \in E^* \) such that \( f(x) = \varphi(l_1(x), \ldots, l_n(x)) \). Then obviously we have

\[ df(x) = \sum_{p=1}^n \frac{\partial \varphi}{\partial x_p} (l_1(x), \ldots, l_n(x)) l_p := \sum_{p=1}^n \varphi_p(x) l_p. \]

From (3.16) and taking finite summation from \( p = 1 \) to \( n \), we get

(3.17) \[ \int_E \hat{c}_{\sum_{p=1}^n \varphi_p(x)l^{-1}_p} g(x) d\mu \]

\[ = - \int_E g(x) \left[ E \langle \beta(x), \sum_{p=1}^n \varphi_p(x) l_p \rangle_E + \sum_{p=1}^n \hat{c}_{l^{-1}_p} \varphi_p(x) \right] d\mu. \]

Note that
\[ A_H f(x) = \sum_{p,q=1}^{n} \frac{\partial^{2} \varphi}{\partial x_p \partial x_q} (I_1(x), \ldots, I_n(x))(i^*(l_p), i^*(l_q))_H \]
\[ = \sum_{p=1}^{n} \partial_i (l_p) \varphi_p(x) \]
\[ = \sum_{k=1}^{\infty} \partial_{\xi_k}^2 f(x). \]

Therefore we have proved that \( \mu \) is \( \mathcal{F} \mathcal{C}_b^\infty \)-differentiable along vector field \( \xi = df \) and
\[ \rho_{\mu}(\xi; x) = \sum_{k=1}^{\infty} \partial_{\xi_k}^2 f(x) + \mathbb{E} \langle \beta(x), df(x) \rangle_E. \]

Similarly by using \( z \varphi \) instead of \( \varphi \) in (3.17), we can get
\[ \int_{\mathbb{E}} \partial_{z(x)} \varphi(x) g(x) d\mu \]
\[ = - \int_{\mathbb{E}} g(x) \left[ \mathbb{E} \langle \beta(x), f(x) \rangle_E, + \mathbb{E} A_H f(x) + (Dz(x), Df(x))_H \right] d\mu \]
i.e.

(3.18) \[ \int_{\mathbb{E}} \partial_{z(x)} \varphi(x) g(x) d\mu \]
\[ = - \int_{\mathbb{E}} g(x) \left[ \mathbb{E} \langle \beta(x), f(x) \rangle_E, + \sum_{k=1}^{\infty} \partial_{\xi_k} (z(x) \partial_{\xi_k} f(x)) \right] d\mu. \]

This shows that \( \mu \) is \( \mathcal{F} \mathcal{C}_b^\infty \)-differentiable along vector field \( \xi = zdf \) for any \( z, f \in \mathcal{F} \mathcal{C}_b^\infty \) and has logarithmic derivative
\[ \rho_{\mu}(\xi; x) = \sum_{k=1}^{\infty} \partial_{\xi_k} (z(x) \partial_{\xi_k} f(x)) + \mathbb{E} \langle \beta(x), z(x) df(x) \rangle_E. \]

This completes the proof. \( \Box \)

Combining Corollary 3.2 and Proposition 3.7, we conclude the following important corollary which was due to Bogachev and Röckner [15].

**Corollary 3.8.** Suppose that the conditions of Proposition 3.7 are fulfilled. Then \( (L, \mathcal{F} \mathcal{C}_b^\infty) \) is symmetric on \( L^2(E, \mu) \) if and only if vector logarithmic derivative \( \beta \) of measure \( \mu \) exists and \( \beta = b \).
Next we would like to provide some sufficient conditions which ensure our differential operator $L$ defined as in (2.9) with certain domain is symmetric on $L^2(E, \mu)$. Let $(i, H, E)$ be an AWS. Since $i^*$ is a one to one continuous map from $E^*$ into $H^* = H$ with dense range, thus there exists a subset $\{l_j\}_{j \geq 1}$ of $E^*$ such that $\{e_j\}_{j \geq 1} = \{i^*(l_j)\}_{j \geq 1}$ is a complete orthonormal basis of $H$. Let us set

$$P_n x = \sum_{j=1}^{n} E \cdot \langle l_j, x \rangle_{E} i^*(l_j), x \in E, n = 1, 2, \ldots$$

We first prove a useful property for differentiable measures on separable Banach spaces. For convenience, we shall denote by $\|u\|_2$ the $L^2$-norm of real-valued function $u$ in $L^2(E, \mu)$.

**Proposition 3.9.** If $\mu$ is $C^1_b$-differentiable and has vector logarithmic derivative $\beta : E \to E$ which satisfies

\[(\beta)_1: \|\beta\|_E \in L^2(E, \mu) \text{ and } \|P_n \beta - \beta\|_E \text{ tends to zero as } n \to \infty \text{ in } L^2(E, \mu).
\]

Then $\mu$ is $C^1_b$-differentiable along the vector field $\xi \in C^1_b(E, E^*)$ and has logarithmic derivative

$$\rho_{\mu}(\xi; x) = \langle \beta(x), \xi(x) \rangle + \text{div} \xi(x).$$

**Proof.** For $z(x) = x(x)^l$, $l \in E^*$ and $x \in C^1_b(E, \mathbb{R})$, we have

$$\text{div}z(x) = Tr_{H} i^* z^l(x) i$$

$$= \sum_{k=1}^{\infty} \langle i e_k, z'(x) \rangle(l^* l, e_k)_H$$

$$= (i^* z'(x), i^* l)_H$$

$$= D_{n^*} z(x).$$

From Lemma 2.4, it follows that

$$\rho_{\mu}(z(x); x) = \langle z(x), \beta(x) \rangle + \text{div} z(x).$$

For $\xi \in C^1_b(E, E^*)$, we have

$$i^* \xi(x) = \sum_{k=1}^{\infty} (i^* \xi(x), e_k) e_k =: \sum_{k=1}^{\infty} a_k(x) e_k,$$

where $a_k \in C^1_b$ for any $k \in \mathbb{N}$. Set $\xi_n(x) = \sum_{k=1}^{n} a_k(x) e_k$. Since the “div” operator is
linear, from (3.20) we can easily obtain for \( g \in C^1_0(E, \mathbb{R}) \)

\[
(3.21) \quad \left( \int_E g(x) \left[ \langle \beta(x), \sum_{k=1}^n \varphi_k(x)l_k \rangle + \text{div} \left( \sum_{k=1}^n \varphi_k(x)l_k \right) \right] \mu(dx) \right)
\]

\[
= - \int_E \left( Dg(x), \sum_{k=1}^n \varphi_k(x)\xi_k \right) \mu(dx),
\]

which implies that

\[
(3.22) \quad \rho_\mu(\xi_n; x) = \langle \beta(x), \xi_n(x) \rangle + \text{div} \xi_n(x).
\]

Since

\[
\int_E \|i^*\xi(x)\|_{H}^2 \mu(dx) = \sum_{k=1}^\infty \int_E \varphi_k^2(x) \mu(dx) < +\infty,
\]

it follows that as \( n \to \infty \)

\[
\sum_{k=n+1}^\infty \int_E \varphi_k^2(x) \mu(dx) \to 0.
\]

By using Cauchy-Schwartz inequality, we have

\[
(3.23) \quad \left| \int_E \left( Dg(x), i^*\xi_n(x) \right) \mu(dx) \right| \leq \left( \int_E \|Dg(x)\|^2_{H} \mu(dx) \right)^{1/2} \cdot \left( \sum_{k=n+1}^\infty \int_E \varphi_k^2(x) \mu(dx) \right)^{1/2},
\]

which tends to zero as \( n \to \infty \). For the divergence term, we have

\[
\left| \int_E g(x) [\text{div}(\xi_n(x)) - \text{div}(\xi(x))] \mu(dx) \right|
\]

\[
= \left| \int_E g(x) [\text{Tr} i^*\xi_n^0(x)i - \text{Tr} i^*\xi^0(x)i] \mu(dx) \right|
\]

Since

\[
\text{Tr} i^*\xi^0(x)i = \lim_{n \to \infty} \sum_{k=1}^n \xi_k^0(x)(ie_k, ie_k)
\]

\[
= \lim_{n \to \infty} \text{Tr} i^*\xi_n^0(x)i
\]
and

\[ |\text{Tr} i^* \xi^f(x)\| \leq \sup_{x \in E} \|\xi^f(x)\|_{L^2(E, \mathbb{R})} \cdot \int_E \|x\|^2 g(dx), \]

where \( \gamma \) is the standard Wiener measure on \( E \), thus by using Lebesgue's dominated convergence theorem we get

(3.24) \[ \lim_{n \to \infty} \left| \int_E g(x) \text{div} \xi_n(x) \mu(dx) - \int_E g(x) \text{div} \xi(x) \mu(dx) \right| = 0. \]

Finally, by using the assumption \((\beta1)\) and Cauchy-Schwartz inequality, we have

(3.25) \[
\left| \int_E g(x) \langle \beta(x), \xi_n(x) \rangle \mu(dx) - \int_E g(x) \langle \beta(x), \xi(x) \rangle \mu(dx) \right| \\
= \left| \int_E g(x) \langle P_n \beta(x) - \beta(x), \xi(x) \rangle \mu(dx) \right| \\
\leq \sup_{x \in E} \|\xi(x)\| \cdot \|g\|_2 \cdot \|\|P_n \beta - \beta\|\|_2,
\]

which tends to zero as \( n \to \infty \). By letting \( n \to \infty \) in (3.21) and combining the above arguments from (3.23)–(3.25), we conclude that for any \( g \in C^1_b(E, \mathbb{R}) \)

(3.26) \[ \int_E g(x) \langle \beta(x), \xi(x) \rangle + \text{div} \xi(x) \mu(dx) = - \int_E (Dg(x), i^* \xi(x))_{H} \mu(dx). \]

This completes the proof. \( \square \)

**Remark 3.10.** (1) The above Proposition 3.9 was first announced in [22] and then proved in a sketch form in [11]. But our assumptions on the space structure of \( E \) are weaker than those in [11].

(2) The assumption \((\beta1)\) in Proposition 3.9 is satisfied for Gaussian measures and certain measures which are absolutely continuous with respect to Gaussian measures on \((i, H, E)\) (cf. Proposition 3 in Carmona [18], Theorem 3.5 in Bogachev and Röckner [15] and Corollary 2.1 in Goodman [35]).

**Corollary 3.11.** Suppose that the assumptions \((A1)-(A4)\) and \((\beta1)\) are fulfilled. Then the differential operator \( L \) with domain \( C^2_b(E, \mathbb{R}) \) defined by (2.9) with \( b = \beta \) is symmetric on \( L^2(E, \mu) \).

**Proof.** This is the immediate consequence of Proposition 3.9 and Theorem 3.6.

If we want to consider the symmetrizability of second-order differential operators with larger domain \( \mathcal{D} \supset C^2_b(E, \mathbb{R}) \) (e.g. certain Sobolev spaces), then we need some
stronger conditions on the vector logarithmic derivatives as in section 3 of Chapter 1 of [11]. We assume that \( \mu \) is a finite Radon measure which has vector logarithmic derivative \( \beta : E \to E \) satisfying (\( \beta 1 \)) and

(\( \beta 2 \)) \( \beta'(x) \in L'(H) \) for each \( x \in E \) and \( \sup_{x \in E} \| \beta'(x) \|_{L'(H)} < \infty \).

Consider the Sobolev space \( W^{1,2}(E, H, \mu) \) of vector functions \( \xi : E \to H \) which is the completion of \( C^1_b(E, H, H) \) (this notation was introduced in section 2) with respect to the norm

\[
\| \xi \|_{1,2} = \left\{ \int_E \left[ \| \xi(x) \|^2_H + \text{Tr}(\xi'(x))^*(\xi'(x)) \right] \mu(dx) \right\}^{1/2}.
\]

Then according to Theorem 3.2 of Chapter 1 in [11], we know that \( \mu \) is differentiable along vector field \( \xi \in W^{1,2}(E, H, \mu) \) and has logarithmic derivative

\[
\rho_\mu(\xi, x) = \xi(\beta(x), \xi(x))_H + \text{div} \xi(x)
\]

which satisfies the following formula

\[
\int_E \| \rho_\mu(\xi, x) \|^2 \mu(dx) = \int_E \left\{ \text{Tr}(\xi'(x))^*\xi'(x) - \langle \beta'(x)\xi(x), \xi(x) \rangle \right\} \mu(dx).
\]

For our purpose, we define a Sobolev space \( W^{2,2}(E, \mu) \) which is the completion of \( C^2(E, \mathbb{R}) \) with respect the norm

\[
\| f \|_{2,2} = \left\{ \int_E \left[ \| f(x) \|^2 + \| Df(x) \|^2_H + \| D^2f \|_{L'(H)}^2 \right] \mu(dx) \right\}^{1/2} < \infty.
\]

In addition, we assume that \( A(\cdot) : E \to L'(H) \) satisfies the following two assumptions

(\( \tilde{A}1 \)) \( A \in C^1_b(E, H, L'(H)) \) and \( A(x) = A^*(x) \) for each \( x \in E \), and there exists a constant \( \lambda > 0 \) such that \( (A(x)h, h)_H \geq \lambda \| h \|_H^2 \) for any \( x \in E \) and \( h \in H \).

(\( \tilde{A}2 \)) \( \int_E \sum_{k=1}^{\infty} \| D(A(x)Df(x))e_k \|_H^2 \mu(dx) < \infty \) for any \( f \in W^{2,2}(E, \mu) \).

It can be easily verified that if \( A(\cdot) \) satisfies

\[
\sup_{x \in E} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \| DA(x)(e_k, e_j) \|_H^2 < \infty,
\]

then the assumption (\( \tilde{A}2 \)) is fulfilled. Now we define the differential operator \( \hat{L} \) with domain \( W^{2,2}(E, \mu) \) by

\[
(3.27) \quad \hat{L}f(x) = \rho_\mu(A(x)Df(x); x)
\]

\[
= \xi(\beta(x), A(x)Df(x))_H + \text{div}(A(x)Df(x)).
\]
Obviously, under the assumptions (A1)–(A2) and (B1)–(B2), our differential operator \( \tilde{L} \) with domain \( W^{2,2}(E, \mu) \) is well defined on \( L^2(E, \mu) \).

Now we have the following result:

**Theorem 3.12.** If the assumptions (A1)–(A2) and (B1)–(B2) are fulfilled, then the differential operator \( (\tilde{L}, W^{2,2}(E, \mu)) \) is symmetric on \( L^2(E, \mu) \).

**Proof.** Set \( \xi = gADf \) for \( f, g \in W^{2,2}(E, \mu) \). Then, from Lemma 2.4, we have

\[
0 = \int_E \rho_\mu(\xi; x) \mu(dx) \\
= \int_E [g(x)\rho_\mu(A(x)Df(x); x) + \partial_{A(x)Df(x)}g(x)] \mu(dx) \\
= \int_E g(x)\tilde{L}f(x) \mu(dx) + \int_E (Dg(x), A(x)Df(x))_H \mu(dx),
\]

i.e.

\[
(3.28) \quad \int_E g(x)\tilde{L}f(x) \mu(dx) = -\int_E (Dg(x), A(x)Df(x))_H \mu(dx).
\]

This means that \( (\tilde{L}, W^{2,2}(E, \mu)) \) is symmetric on \( L^2(E, \mu) \). This completes the proof.

Finally we shall provide an abstract sufficient condition for the symmetrizability of a special class of differential operators over infinite dimensional state spaces. Let \( M \) be a locally convex space or infinite dimensional Banach manifold and \( \mathcal{D} \) be an algebra on \( (M, \mathcal{B}(M)) \) which is dense in \( L^2(M, \mu) \) for any positive Radon measure \( \mu \) on \( (M, \mathcal{B}(M)) \). In addition, we assume that there exists a collection of real-valued Borel functions \( A \) on \( M \) such that \( \mathcal{D} \subseteq A \subseteq L^2(M, \mu) \) and \( A \times \mathcal{D} \subseteq A \). Motivated from the notion of strongly \( \mathcal{D} \)-admissible vector fields introduced in Elworthy and Ma [29] and the notion of well-\( \mu \)-admissible vectors on locally convex spaces introduced by Albeverio, Kusuoka and Röckner [5], we introduce a new class of admissible vector fields.

**Definition 3.13.** We call vector field \( \xi \) a strongly \( (A, \mathcal{D}) \)-admissible vector field on \( M \) if

(i) \( \partial_{\xi} f \in A \) for all \( f \in \mathcal{D} \).

(ii) there exists an element \( \rho_\mu(\xi) \in L^2(M, \mu) \) satisfying

\[
(3.29) \quad \int_M \partial_{\xi} f d\mu = -\int_M f \rho_\mu(\xi) d\mu
\]

for all \( f \in A \).
If $\xi$ is a strongly $(A, \mathcal{D})$-admissible vector field, then the adjoint operator $\partial^*_\xi$ of $\partial_\xi$ is given by

$$\partial^*_\xi f = -\partial_\xi f - f \rho_\mu(\xi) \quad \text{for any } f \in A.$$ 

Now we can define a second-order differential operator $L$ with domain $\mathcal{D}$ by

$$(3.30) \quad Lf = - \sum_{\xi \in \mathcal{D}} \partial^*_\xi \partial_\xi f,$$

where $\mathcal{A}$ is a countable or finite family of strongly $(A, \mathcal{D})$-admissible vector fields on $M$ which satisfies the following conditions

(1)

$$\int_M \sum_{\xi \in \mathcal{A}} |\partial_\xi f(x)|^2 \mu(dx) < +\infty \quad \text{for any } f \in \mathcal{D}$$

(2) There exists a sequence of $\{\mathcal{A}_n\}_{n \geq 1}$, $\mathcal{A}_n \subset \mathcal{A}$ and $\mathcal{A}_n \uparrow \mathcal{A}$ such that

$$- \sum_{\xi \in \mathcal{A}_n} \partial^*_\xi \partial_\xi f$$

converges as $n \to \infty$ in $L^2(M, \mu)$ for all $f \in \mathcal{D}$. Then $Lf$ will denote the limit.

We have the following general result:

**Theorem 3.14.** We assume that $\partial_\xi (fg) = \partial_\xi f \cdot g + f \cdot \partial_\xi g$ holds for any $\xi \in \mathcal{A}$, $f \in A$ and $g \in \mathcal{D}$. Then $(L, \mathcal{D})$ defined as above is symmetric on $L^2(M, \mu)$.

**Proof.** From the definition of strongly $(A, \mathcal{D})$-admissible vector field and the expression for the operator $\partial^*_\xi$, we have for any $f, g \in \mathcal{D}$

$$(3.31) \quad \int_M Lf \cdot gd\mu = - \int_M \sum_{\xi \in \mathcal{D}} (\partial^*_\xi \partial_\xi f) \cdot gd\mu$$

$$= - \sum_{\xi \in \mathcal{D}} \int_M \partial^*_\xi \partial_\xi f \cdot gd\mu$$

$$= - \sum_{\xi \in \mathcal{D}} \int_M [-\partial^2_\xi f - \partial_\xi f \rho_\mu(\xi)] gd\mu$$

$$= \sum_{\xi \in \mathcal{D}} \left[ \int_M \partial^2_\xi f \cdot gd\mu + \int_M (\partial_\xi f \cdot g) \rho_\mu(\xi) d\mu \right]$$
\[
\begin{align*}
&= \sum_{\xi \in \mathcal{D}} \left( \int_M \partial_\xi f \cdot g d\mu - \int_M \partial_\xi (\partial_\xi f \cdot g) d\mu \right) \\
&= \sum_{\xi \in \mathcal{D}} \left( \int_M \partial_\xi^2 f \cdot g d\mu - \left( \int_M \partial_\xi^2 f \cdot gd\mu + \int_M \partial_\xi f \cdot \partial_\xi gd\mu \right) \right) \\
&= -\sum_{\xi \in \mathcal{D}} \int_M \partial_\xi f \cdot \partial_\xi gd\mu.
\end{align*}
\]

Therefore \((L, \mathcal{D})\) is symmetric on \(L^2(M, \mu)\). This completes the proof. \qed

**Remark 3.15.** Theorem 3.14 is simple and well-known in many special cases. The only contribution in here is to introduce the notion of strongly \((A, \mathcal{D})\)-admissible vector fields on infinite dimensional state spaces which generalized the notion of well-\(\mu\)-admissible vectors on locally convex spaces introduced by Albeverio, Kusuoka and Röckner [5], and consider the problem in a general and abstract setting. Consequently this result can be applied to verify more examples as shown in next section.

**4 Examples**

In this section we would like to provide some important examples which can be verified by using general criteria given in section 3. In some examples, we shall use and keep some definitions and notation from the cited paper. Concerning the theory of stochastic analysis over path spaces, we refer to Driver [23], Hsu [37] and Fang and Malliavin [31].

**Example 4.1 (Elworthy [27]).** Let \(M\) be the abstract Wiener manifold modeled on the abstract Wiener space \((i, H, E)\). For example the space of continuous paths \(\theta : [0, 1] \to \mathbb{R}^n\) with \(\theta(0) \in P\) and \(\theta(1) \in Q\), where \(P\) and \(Q\) are closed submanifolds of \(\mathbb{R}^n\) (cf. Elworthy [28]). Consider the bundle embedding \(i_x : H_x \to T_x M\), where \(H_x\) is the fiber of Hilbert bundle \(H(M)\). Let \(X(x) = i_x Y(x)\), where \(Y(x) : H \to H_x\) is a linear operator from \(H\) onto \(H_x\) which determines the Riemannian metric \(<.,.>_x\) on the bundle \(H(M)\) over \(M\). If \(\{e_k\}_{k=1}^\infty\) is an orthonormal basis of \(H\), then \(\{e_k(x) = Y(x)e_k\}_{k=1}^\infty\) forms a local orthonormal basis of \(H_x\). For the abstract Wiener space \((i, H, E)\) and an open set \(U \subset E\), a map \(g : U \to E\) is called a \(C^r W(1)\)-map \((r \geq 1)\) if it has the form \(g(x) = x + \alpha(x)\), where \(\alpha : U \to E^\ast\) is \(C^r\). The Wiener densities on \(M\) are determined by the so-called Wiener data \((G, Z)\), where \(G\) is the Riemannian metric and \(Z\) is a position field on \(M\) such that in a local chart \((U_i, Q_i)\) the principal part of \(Z, Z_i : Q_i(U_i) \to E\) is a \(C^r W(1)\)-map \((r \geq 1)\). For such data and each chart \((U_i, Q_i)\) of \(M\) define \(\sigma_i = \sigma_i^r(G, Z) : Q_i(U_i) \to \mathbb{R}\) by

\[
\sigma_i(x) = |\det G_i^r| \exp\left\{ -\frac{1}{2s} \left[ 2 \langle G_i^r Z_i^r(x) - x, x \rangle + |x - G_i^r Z_i^r(x)|^2 \right] \right\}
\]

This is the local expression of a positive density \(\sigma^r(G, Z)\) on \(M\) with variance
parameter $s$. Thus Wiener data on $M$ determines a measure $\mu^s = \mu^s(G, Z)$ on $M$ for all $0 < s < \infty$. Now we assume that the Wiener data $(G, Z)$ is $C^1$. From the Divergence Theorem of [27], we know that $\mu^s$ is differentiable along any vector field $\xi$ which factors through a section of $T^*(M) \to M$ and has logarithmic derivative $\rho_{\mu^s}(\xi; x)$ as follows

$$\rho_{\mu^s}(\xi; x) = \text{Tr}_{H} \nabla \xi(x) - \frac{1}{s} \langle \nabla Z(\xi), Z \rangle.$$

For standard Wiener data (inner product on $H$ and the vector field $Z$ with $Z(x) = x$) on abstract Wiener space $(\mu^s, H, E)$, the above formula reduces to

$$\rho_{\mu^s}(\xi; x) = \text{Tr}_H D\xi(x) - \frac{1}{s} \langle \xi(x), x \rangle.$$

Note that for a $C^1$ map $f : M \to \mathbb{R}$ on an AWM with a Wiener-Riemannian metric we can define the section $df$ of $T^*(M)$ to be the admissible vector field. If $f$ is $C^2$ we define the Laplacian $A_s$ of $f$ by

$$A_s f(x) = \text{Tr} V^2 f - \frac{1}{s} \langle VZ(df), Z \rangle.$$

Then $A_s$ is a negative definite symmetric operator. When $M = E$ with standard Wiener data it reduces to the Ornstein-Uhlenbeck operator

$$A_s f = \text{Tr} D^2 f(x) - \frac{1}{s} \langle Df(x), x \rangle.$$

This is an example of the type of operators considered in Theorem 3.3 and Theorem 3.4.

**Example 4.2** (Daletskii [20] or Daletskii and Fomin [21]). Let $H_+ \subset H \subset H_-$ be a rigged Hilbert space with Hilbert-Schmidt embedding and $\mu$ be the standard Gaussian measure on $H_-$. Then $\mu$ is $C^1_1(H_-, \mathbb{R})$-differentiable and has vector logarithmic derivative $\beta(x) = -x$ which obviously satisfies the assumptions ($\beta_1$) and ($\beta_2$) as in section 3 [cf. Proposition 3.2 in Chapter II of [21]]. Now if we assume that $A(\cdot) \in C^1_1(H_-, H, \mathcal{L}(H))$ satisfies assumptions (A1)–(A2) for Gaussian measure $\mu$, then according to Theorem 3.12 the differential operator $\tilde{L}$ defined by

$$\tilde{L} f(x) = \text{Tr} D(A(x)Df(x)) - \langle x, A(x)Df(x) \rangle$$

is symmetric on $L^2(H_-, \mu)$ with domain $W^{2,2}(H_-, \mu)$ (cf. Theorem 1.2 in Chapter IV of [21]). This is an example of the type of operators considered in Theorem 3.12.

**Example 4.3** (Berezanski and Samoilenko [12]). Let $H$ be a separable Hilbert space and $A$ a negative type of self-adjoint operator on $H$ with the inverse $A^{-1}$ nuclear. Let
\( \gamma_A \) be a Gaussian measure with mean zero and correlation operator \(-\frac{1}{2} A^{-1}\). For any \( U \in C_b^1(H, \mathbb{R}) \), we set \( \mu = e^{2U(x)} \gamma_A \). Let \( \{e_k\}_{k=1}^\infty \) be an orthonormal basis in \( H \) consisting of eigenvectors of \( A : Ae_k = \lambda_k e_k \) with \( \lambda_k < 0 \) for each \( 1 \leq k < \infty \). Now let us define a second-order differential operator \( L \) by
\[
Lf(x) = \text{Tr}[f''(x)] + 2\langle x, Af'(x) \rangle + 2\langle U'(x), f'(x) \rangle
\]
with domain \( \mathcal{D} \) given by
\[
\mathcal{D} = \{ f \in C_b^2(H, \mathbb{R}) | f'(x) \in D(A) \text{ for each } x \in H \text{ and } \sup_{x \in H} \| f''(x) \|_{L^2(H)} < \infty \}.
\]
If we set \( A = C_b^1(H, \mathbb{R}) \), then obviously \( \mathcal{D} \subset A \) which are dense in \( L^2(H, \mu) \) and \( \partial_h f \in A \) for any \( h \in D(A) \) and \( f \in \mathcal{D} \). From the integration by parts formula as proven in Lemma 2.1 of [12], we know that for any \( f, g \in A \)
\[
(4.1) \quad \int_H (f'(x), h)g(x)\gamma_A(dx) = -\int_H f(x)\langle [g'(x), h] + 2(x, Ah) \rangle \gamma_A(dx).
\]
It follows that
\[
(4.2) \quad \int_H (f'(x), h)\mu(dx) = -\int_H f(x)\langle [2(x, Ah) + 2(U'(x), h)] \rangle \mu(dx).
\]
This implies that all \( h \) in \( D(A) \) are strongly \((A, \mathcal{D})\)-admissible vectors and \( \mu \) has logarithmic derivative along \( h \)
\[
(4.3) \quad \rho_\mu(h; x) = 2[(x, Ah) + (U'(x), h)].
\]
By a simple calculation, we have
\[
Lf(x) = \text{Tr}_H f''(x) + 2\langle x, Af'(x) \rangle + 2\langle U'(x), f'(x) \rangle
\]
\[
= \sum_{k=1}^\infty [\partial_{e_k}^2 f(x) + \partial_{e_k} f(x) \cdot \rho_\mu(e_k; x)].
\]
Thus according to our general criterion Theorem 3.14, we conclude that \((L, \mathcal{D})\) is symmetric on \( L^2(H, \mu) \). This example can be considered to be the analytical counterpart of Theorem 2 in Zabczyk [52].

**Example 4.4** (Albeverio, Kusuoka and Röckner [5] or Albeverio, Röckner and Zhang [7]). Let \( M = E \) be a locally convex space (e.g. Sausin or Polish space) and \( \mathcal{D} \) be the smooth cylindrical function space \( \mathcal{F}C_0^\infty(E, \mathbb{R}) \) which is dense in \( L^2(H, \mu) \) for any finite positive Radon measure \( \mu \) on \((E, \mathcal{B}(E))\). Further we assume that there exists a
separable Hilbert space $H$ such that $H \subset E$ continuously and densely. Thus after identifying $H$ with its dual $H^*$ we have

$$E^* \subset H \subset E$$

and $E, \langle \cdot, \cdot \rangle_E$ coincides with the inner product $(\cdot, \cdot)_H$ on $H$ when restricted to $E^* \times H$. Then the so-called well-$\mu$-admissible vector in $E$ introduced in [5] coincides with our strongly $(\mathcal{F} C^\infty_b, \mathcal{F} C^\infty_B)$-admissible constant vector field on $E$. Let $K$ be a countable or finite family of well-$\mu$-admissible vectors in $E^*$. We define the differential operator $L_{\mu, H, K}$ with dense domain $\mathcal{F} C^\infty_B$ by

$$L_{\mu, H, K}f = \sum_{k \in K} \left( \frac{\partial}{\partial k} \left( \frac{\partial f}{\partial k} \right) + \beta_k \frac{\partial f}{\partial k} \right),$$

where $\beta_k$ is the logarithmic derivative of measure $\mu$ along vector $k$. Then from our Theorem 3.14 or Corollary 3.8, we know that $(L_{\mu, H, K}, \mathcal{F} C^\infty_B)$ is symmetric on $L^2(E, \mu)$.

**Example 4.5** (Hsu [37], Fang [30], Acosta [1], Aida [2]). Let $M$ be a $d$-dimensional compact Riemannian manifold equipped with a connection compatible with the Riemannian metric (the torsion of the connection is antisymmetric). We use $O(M)$ to denote the bundle of orthonormal frames over $M$. Let $o \in M$ be a fixed point on $M$ and $u_o \in O(M)$ a fixed orthonormal frame over $o$. We will use $W_o(M)$ and $W_o(O(M))$ to denote the path spaces based on $M$ and $O(M)$, i.e. the spaces of continuous functions from the unit interval $[0, 1]$ to $M$ and $O(M)$ starting from $o$ and $u_o$, respectively. We denote by $W_0(\mathbb{R}^d)$ and $\mathcal{H}$ the usual classical Wiener space and Cameron-Martin space. The stochastic development map $J : W_0(\mathbb{R}^d) \to W_o(M)$ maps a euclidean Brownian motion to a Riemannian Brownian motion on $M$, i.e., the Wiener measure $\nu$ over path space $W_o(M)$ is given by $\nu = \mu \circ J^{-1}$, where $\mu$ is the Wiener measure on $W_0(\mathbb{R}^d)$. Let $F$ be real-valued function on $W_o(M)$. For any $h \in \mathcal{H}$, we denote by $\partial_h F$ the derivative of function $F$ along the vector field $X_h = U(\gamma)h$ on $W_o(M)$ obtained by stochastic parallel translation of $h$ in the usual way (see section 4 in Hsu [37] for detail). Set $A = \text{Dom}(D) \cap L^2(E^\infty)$ and $\mathcal{D} = \mathcal{C}$ the smooth cylindrical function space on $W_o(M)$. Then from Theorem 5.2 and Theorem 6.6 in [37], we know that $X_h = U(\gamma)h$ is a strongly $(A, \mathcal{D})$-admissible vector field for any $h \in \mathcal{H}$. Moreover by using Hölder or Cauchy-Schwartz inequality and Lebesgue's dominated convergence theorem, we can easily verify that

$$\partial_h (fg) = \partial_h f \cdot g + f \cdot \partial_h g$$

for any $f \in A$ and $g \in \mathcal{D}$. For any $h \in \mathcal{H}$, the adjoint operator $\partial^*_h$ of $\partial_h$ on $A$ is given by

$$\partial^*_h = -\partial_h + h,$$
where \( h_0(\gamma) = \int_0^1 h_0 - \frac{1}{2} H_1 \partial \nu_i (H_i, H_h) - \frac{1}{2} \text{Ric}_{\nu_i} (H_h), \) provided that \( \omega = J^{-1} \gamma \) is the stochastic anti-development of \( \gamma \) in \( \mathbb{R}^d \) and \( U = U(\gamma) \) is the horizontal lift of \( \gamma \) to \( O(M) \). Now we define the differential operator \( L \) with domain \( \mathcal{D} \) by

\[
L f = - \sum_{i=1}^{\infty} \partial^*_e \partial_e f := -D^* D f,
\]

where \( \{ e_i \}_{i=1}^{\infty} \) is an orthonormal basis of \( H \) and \( D \) is the \( H \)-gradient operator. Then the operator \( (L, \mathcal{D}) \) satisfies all the assumptions in Theorem 3.14, and consequently it is symmetric on \( L^2 (W^q \mu(M), \gamma) \).

One can also verify a similar example from Fang [30] by setting \( A = W^q \mathcal{D} \) (cf. Definition 2.2.3 in Fang [30]) for \( q > 2 \) and \( \mathcal{D} = \mathcal{D} \) via the same procedure as above.

Some examples on group-valued path space or loop space from Acosta [1] and Aida [2] can be easily verified by using our Theorem 3.14. We omit the detail here.

References

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