Least Squares Estimator for Ornstein-Uhlenbeck Processes Driven by $\alpha$-Stable Motions

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Abstract

We study the problem of parameter estimation for generalized Ornstein-Uhlenbeck processes driven by $\alpha$-stable noises, observed at discrete time instants. Least squares method is used to obtain an asymptotically consistent estimator. The strong consistency and the rate of convergence of the estimator have been studied. It is observed that the estimator has, surprisingly, a higher order of convergence in the general stable, non-Gaussian case than in the classical Gaussian case.

Keywords: Parameter estimation; generalized Ornstein-Uhlenbeck processes; $\alpha$-stable processes; least squares method; consistency of LSE; asymptotic distribution of LSE

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\textsuperscript{2}Long is supported by FAU Start-up funding at the C. E. Schmidt College of Science.
1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a basic probability space equipped with a right continuous and increasing family of $\sigma$-algebras $\{\mathcal{F}_t, t \geq 0\}$. Let $\{Z_t, t \geq 0\}$ be a standard symmetric $\alpha$-stable Lévy motion. For technical reason, we assume that $1 < \alpha < 2$. The generalized Ornstein-Uhlenbeck process $\{X_t, t \geq 0\}$, starting from $x \in \mathbb{R}$ is defined as the unique solution to the following linear stochastic differential equation (SDE)

$$
dX_t = -\theta_0 X_t dt + dZ_t, \quad X_0 = x. \tag{1}\n$$

Assume that this process is observed at some discrete time instants $\{t_i = ih, i = 0, 1, 2, \cdots\}$ but the valued of $\theta_0$ is unknown. The purpose of this paper is to study the least squares estimator (LSE) for the true value $\theta_0$ based on the sampling data $(X_{t_i})_{i=0}^n$.

In the case of diffusion processes driven by Brownian motions, a popular method is to use the maximum likelihood estimator (MLE) based on the Girsanov density which is asymptotically equivalent to the least squares estimator. Dorogovcev [5] and Le Breton [13] proved that the LSE converges to the true parameter in probability and Kasonga [11] studied the strong consistency of the LSE. The asymptotic distribution of the LSE was studied in Prakasa Rao [17]. For a comprehensive discussion on LSE for diffusion processes driven by Brownian motions, we refer to Prakasa Rao [18], Kutoyants [12] and in particular the references therein. For MLE based on discrete observations, we also refer to Aït-Sahalia [1].

Recently there has been a growing interest in parameter estimation for stochastic processes with jumps (generated by Lévy processes) due to its promising applications for example in finance. Substantial progress has been made. Shimizu [21] and Shimizu and Yoshida [22] studied asymptotic normality of the LSE and MLE for pure jump cases when the underlying jumps are of compound Poisson type. Masuda [15] dealt with the consistency and asymptotic normality of LSE for stochastic processes driven by a zero-mean adapted process (including Lévy process) with finite moments. However, the parameter estimation for stochastic processes (O-U processes) driven by $\alpha$-stable Lévy
motions discussed in this paper are beyond the aforementioned scope due to the infinite variance property of \( \alpha \)-stable processes.

The main focus of this paper is the study of the strong consistency and asymptotic distributions of the LSE for generalized O-U processes satisfying the SDE (1). Our results are analogues of the LSE and Yule-Walker estimator for stationary autoregressive, moving average (ARMA) models driven by a sequence of i.i.d. random variables in the domain of attraction of a stable law (see Davis and Resnick [4]). For other related estimators like M-estimator and the Whittle estimator for AR or ARMA models in the stable setting, we refer to Davis, Knight and Liu [3] and Mikosch, Gadrich, Klüppelberg and Adler [16] as well as references therein. Our results indicate that the asymptotic behavior of the LSE in the stable setting is completely different from that for diffusion or finite variance jump-diffusion processes. The rate of convergence in the stable setting is considerably faster than that of the classical Gaussian setting.

To obtain the LSE, we introduce the following contrast function

\[
\rho_n(\theta) = \rho_n(\theta; (X_t)_{t=0}^n) = \sum_{i=1}^n |X_{t_i} - X_{t_{i-1}} + \theta X_{t_{i-1}} : \Delta t_{i-1}|^2.
\]  

(2)

Then the LSE \( \hat{\theta}_n \) is defined as \( \hat{\theta}_n = \arg \min_{\theta \geq 0} \rho_n(\theta) \), which can be explicitly represented as

\[
\hat{\theta}_n = -\frac{\sum_{i=1}^n (X_{t_i} - X_{t_{i-1}}) X_{t_{i-1}}}{h \sum_{i=1}^n X_{t_{i-1}}^2}.
\]

(3)

Note that the solution \( X_t \) of the SDE (1) can be represented as

\[
X_t = e^{-\theta_0 t} X_0 + \int_0^t e^{-\theta_0(t-s)} dZ_s.
\]

(4)

Some basic calculation yields

\[
X_{t_i} - X_{t_{i-1}} = (e^{-\theta_0 h} - 1)X_{t_{i-1}} + \int_{t_{i-1}}^{t_i} e^{-\theta_0(t_i-s)} dZ_s.
\]

(5)

Then, we can represent the LSE \( \hat{\theta}_n \) as

\[
\hat{\theta}_n = \frac{1 - e^{-\theta_0 h}}{h} - \frac{\sum_{i=1}^n X_{t_{i-1}} \cdot \int_{t_{i-1}}^{t_i} e^{-\theta_0(t_i-s)} dZ_s}{h \sum_{i=1}^n X_{t_{i-1}}^2}.
\]

(6)
Our goal is to prove that $\hat{\theta}_n \to \theta_0$ almost surely and to establish the rate of convergence with an order of convergence $(T/ \log T)^{1/\alpha}$.

If the Ornstein-Uhlenbeck processes can be observed continuously, a trajectory fitting method combined with weighted least squares technique is discussed in Hu and Long [8].

The paper is organized as follows. In Section 2 we establish the strong consistency of the LSE $\hat{\theta}_n$. In Section 3, we study the rate of convergence for the LSE. Finally, some simulation results are provided in Section 4.

## 2 Strong Consistency of the LSE

In this section we show the strong consistency of the LSE $\hat{\theta}_n$. We assume $\theta_0 > 0$.

**Theorem 2.1.** Assume that $h \to 0$ and $t_n = nh \to \infty$ as $n \to \infty$. Then, the following strong consistency holds:

$$\hat{\theta}_n \to \theta_0 \text{ almost surely as } n \to \infty.$$  \hfill (7)

**Proof.** Let $\phi_n(t) = \sum_{i=1}^{n} X_{t_{i-1}} e^{-\theta_0(t_{i-1})} 1_{[t_{i-1}, t_i)}(t)$. It is clear that

$$\sum_{i=1}^{n} X_{t_{i-1}} \int_{t_{i-1}}^{t_i} e^{-\theta_0(t_i-s)} dZ_s = \int_{0}^{t_n} \phi_n(s) dZ_s.$$  \hfill (8)

Let $\tau_n(t_n) = \int_{0}^{t_n} |\phi_n(t)|^\alpha dt$. Then, it is easy to find

$$\tau_n(t_n) = \int_{0}^{t_n} \sum_{i=1}^{n} |X_{t_{i-1}}|^\alpha e^{-\alpha \theta_0 (t_{i-1})} 1_{[t_{i-1}, t_i)}(t) dt \hfill (9)$$

From (6), we have

$$\hat{\theta}_n = \frac{1 - e^{-\theta_0 h}}{h} \left( \frac{\int_{0}^{t_n} \phi_n(t) dZ_t}{h \sum_{i=1}^{n} X_{t_{i-1}}^2} \right) \frac{\tau_n(t_n)}{h \sum_{i=1}^{n} X_{t_{i-1}}^2}.$$  \hfill (10)
It is well-known that if $\theta_0 > 0$, $X_t$ is ergodic and $X_t \Rightarrow X_\infty$ as $t \to \infty$, where $X_\infty = \int_0^\infty e^{-\theta_0 t} dZ_t$ is a $\alpha$-stable random variable (see Sato [20], Hu and Long [8]). Thus, it follows by the ergodic theorem,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n |X_{t_i-1}|^\alpha = E[X_\infty^\alpha] = \infty \quad \text{a.s.}$$

which implies that $\tau_n(t_n) \to \infty$. Note also that $\int_1^\infty t^{-\alpha} dt = 1/(\alpha - 1) < \infty$. By Corollary 3.1 of Rosinski and Woyczynski [19], we have

$$\limsup_{t_n \to \infty} \left| \frac{\int_0^{t_n} \phi_n(t) dZ_t}{\tau_n(t_n)} \right| = 0 \quad \text{a.s.} \quad (11)$$

By Hölder inequality, we have

$$\frac{\tau_n(t_n)}{h \sum_{i=1}^n X_{t_i-1}^2} = \frac{1 - e^{-\theta_0 h}}{\alpha \theta_0 h} \cdot \frac{\sum_{i=1}^n |X_{t_i-1}|^\alpha}{\sum_{i=1}^n X_{t_i-1}^2} \leq \frac{1 - e^{-\theta_0 h}}{\alpha \theta_0 h} \cdot \frac{\left( \sum_{i=1}^n |X_{t_i-1}|^2 \right)^{\alpha/2} n^{(2-\alpha)/2}}{\sum_{i=1}^n X_{t_i-1}^2} \leq \frac{1 - e^{-\theta_0 h}}{\alpha \theta_0 h} \cdot \left( \frac{1}{n} \sum_{i=1}^n |X_{t_i-1}|^2 \right)^{-2-\alpha/2}, \quad (12)$$

which converges to zero almost surely as $n \to \infty$, since $\lim_{h \to 0} \frac{1 - e^{-\theta_0 h}}{\alpha \theta_0 h} = 1$ and by the ergodic theorem again,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n |X_{t_i-1}|^2 = E[X_\infty^2] = \infty \quad \text{a.s.}$$

It is easy to see that

$$\lim_{h \to 0} \frac{1 - e^{-\theta_0 h}}{h} = \theta_0. \quad (13)$$

Combining (10)-(13), we conclude that $\hat{\theta}_n \to \theta_0$ almost surely as $n \to \infty$. □

3 Asymptotic Properties of The LSE

To obtain the rate of convergence we need to make the following assumption.
(A1): \( nh^{1+\alpha} = O(1) \) and
\[
\begin{aligned}
\text{if } 0 < p < 1 + \alpha & \Rightarrow \infty \\
\text{if } p > 1 + \alpha & \Rightarrow 0
\end{aligned}
\]

It is easy to observe that under (A1), we have the strong consistency result.

Denote \( C_\alpha = (\int_0^\infty x^{-\alpha} \sin(x) dx)^{-1} = [\Gamma(1-\alpha) \cos(\pi\alpha/2)]^{-1} \). Our main result about the asymptotic distribution of \( \hat{\theta}_n - \theta_0 \) is stated in the following:

**Theorem 3.1** Under the condition (A1), we have
\[
\left( \frac{n}{\log n} \right)^{1/\alpha} nh^{1/\alpha}(\hat{\theta}_n - \theta_0) \Rightarrow \frac{2\theta_0(\alpha\theta_0)^{-1/\alpha}\tilde{Y}}{Y_0},
\]
where \( Y_0 \) and \( \tilde{Y} \) are independent stable random variables, \( Y_0 \) is positive \( \alpha/2 \)-stable with distribution \( S_{\alpha/2}(\sigma_1, 1, 0) \), \( \tilde{Y} \) is symmetric \( \alpha \)-stable with distribution \( S_\alpha(\sigma_2, 0, 0) \), and the two scaling constants \( \sigma_1 \) and \( \sigma_2 \) are given by
\[
\sigma_1 = [\Gamma(1-\alpha/2) \cos(\pi\alpha/4)]^{2/\alpha} = C_{\alpha/2}^{-2/\alpha},
\]
and
\[
\sigma_2 = [\Gamma(1-\alpha) \cos(\pi\alpha/2)]^{1/\alpha} = C_{\alpha}^{-1/\alpha}.
\]

**Remark 3.2** When \( Z_t \) is replaced by a standard Brownian motion, it is well known that \( \hat{\theta}_n \) is asymptotically normal with rate of convergence of order \( \sqrt{nh} \). Theorem 3.1 states that the LSE \( \hat{\theta}_n \) converges to \( \theta_0 \) with a considerably faster rate of convergence of order \( (n/\log n)^{1/\alpha}h^{1/\alpha} \). The leading coefficient is also identified.

Theorem 3.1 will be proved by establishing several preliminary lemmas and propositions. We first give an explicit expression for \( \left( \frac{n}{\log n} \right)^{1/\alpha} nh^{1/\alpha}(\hat{\theta}_n - \theta_0) \). By using (6), we find
\[
\left( \frac{n}{\log n} \right)^{1/\alpha} nh^{1/\alpha}(\hat{\theta}_n - \theta_0) = \left( \frac{n}{\log n} \right)^{1/\alpha} h^{1/\alpha}[h^{-1}(1 - e^{-\theta_0 h}) - \theta_0] \\
- \frac{(n \log n)^{-1/\alpha} h^{-1/\alpha} \sum_{i=1}^{n} X_{i-1} \int_{t_i}^{t_i} e^{-\theta_0(s-t_i)} dZ_s}{n^{-2/\alpha} h^{1-2/\alpha} \sum_{i=1}^{n} X_{i-1}^2} \\
:= \Lambda_n - \frac{\Phi_1(n)}{\Phi_2(n)}. \tag{15}
\]
So, the asymptotic behavior of \( \left( \frac{n}{\log n} \right)^{1/\alpha} h^{1/\alpha} (\hat{\theta}_n - \theta_0) \) will be determined by the asymptotic behavior of \( \Lambda_n, \Phi_1(n) \) and \( \Phi_2(n) \).

**Lemma 3.3** Suppose that (A1) is satisfied. Then, we have

\[
\Lambda_n \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{16}
\]

**Proof.** It is easy to see that

\[
|\Lambda_n| = \left| \left( \frac{n}{\log n} \right)^{1/\alpha} h^{1/\alpha} \left[ h^{-1} (1 - e^{-\theta_0 h}) - \theta_0 \right] \right| \\
= \left( \frac{n}{\log n} \right)^{1/\alpha} h^{1+1/\alpha} \left| 1 - e^{-\theta_0 h} - \theta_0 h \right| \leq \frac{\theta_0^2 n^{1/\alpha} h^{1+1/\alpha}}{2 (\log n)^{1/\alpha}}, \tag{17}
\]

which tends to zero as \( n \rightarrow \infty \) under the condition (A1) (when \( nh^{1+\alpha} = O(1) \), or equivalently \( h = O(n^{-1/\alpha}) \)). \( \Box \)

From now on, we shall denote \( X_t \) by \( X_i \). Note that

\[
X_i = e^{-\theta_0 i h} X_0 + \int_{t_k}^{t_i} e^{-\theta_0 (t_i - s)} dZ_s = e^{-\theta_0 i h} X_0 + \sum_{k=1}^{i} \int_{t_{k-1}}^{t_k} e^{-\theta_0 (t_i - s)} dZ_s
\]

\[
= e^{-\theta_0 i h} X_0 + \sum_{k=1}^{i} e^{-\theta_0 i h} \int_{t_{k-1}}^{t_k} e^{\theta_0 s} dZ_s. \tag{18}
\]

Let \( V_{k-1} = \int_{t_{k-1}}^{t_k} e^{\theta_0 s} dZ_s \). By the inner clock property for the \( \alpha \)-stable stochastic integral (see Rosinski and Woyczynski [19], Kallenberg [10], and Zanzotto [23]), we know that \( \int_{t_{k-1}}^{t_k} e^{\theta_0 s} dZ_s \) has the same distribution as \( Z'_{\tau_{k-1}} \), where \( Z' \) has the same law as \( Z \) and

\[
\tau_{k-1} = \int_{t_{k-1}}^{t_k} |e^{\theta_0 s}|^\alpha d\tau = e^{\alpha \theta_0 i h} \left( \frac{e^{\alpha \theta_0 i h} - 1}{\alpha \theta_0} \right) \tag{19}
\]

Let \( U_{k-1} = V_{k-1}/\tau_{k-1}^{1/\alpha} \). Then, by the scaling property of stable distribution, we know that \( U_0, U_1, U_2, \ldots \) are i.i.d. with the same stable distribution \( S_{\alpha}(1, 0, 0) \). Let \( c_{i,h} = e^{-\theta_0 i h} \) and \( \gamma_h = \left( \frac{e^{\alpha \theta_0 h} - 1}{\alpha \theta_0} \right)^{1/\alpha} \). Then, \( X_i \) can be represented as

\[
X_i = e^{-\theta_0 i h} X_0 + \left( \frac{e^{\alpha \theta_0 h} - 1}{\alpha \theta_0} \right)^{1/\alpha} \sum_{k=1}^{i} e^{-\theta_0 (i-k+1) h} U_{k-1}
\]

\[
= c_{i,h} X_0 + \gamma_h \sum_{j=1}^{i} c_{j,h} U_{i-j}. \tag{19}
\]
Remark 3.4 From Janicki and Weron [9], for symmetric $\alpha$-stable random variable $U_1 \sim S_\alpha(1, 0, 0)$, we have

$$\lim_{x \to \infty} x^\alpha P(U_1 > x) = C_\alpha/2 \text{ and } \lim_{x \to \infty} x^\alpha P(U_1 < -x) = C_\alpha/2.$$ 

So, the tail distribution of $|U_1|$ is asymptotically equivalent to a Pareto, i.e. $P(|U_1| > x) \sim C_\alpha x^{-\alpha}$. Following Davis and Resnick [4], we define

$$a_n = \inf \{ x : P(|U_1| > x) \leq n^{-1} \}$$
and

$$\tilde{a}_n = \inf \{ x : P(|U_0 U_1| > x) \leq n^{-1} \}.$$

Thanks to the asymptotic Pareto tail distribution of $U_1$, we may take

$$a_n = (C_\alpha n)^{\frac{1}{\alpha}} \text{ and } \tilde{a}_n = C_\alpha^2 (n \log n)^{\frac{1}{\alpha}}.$$ 

Note that $\left(\frac{n}{\log n}\right)^{1/\alpha} = \tilde{a}_n^{-1} a_n^2$.

The following lemma, which is a special case of Theorem 3.3 in Davis and Resnick [4], will be crucial in studying the asymptotic properties of $\Phi_1(n)$ and $\Phi_2(n)$:

**Lemma 3.5** Let $\{U_i\}_{i=0}^\infty$ be i.i.d. with the same stable distribution $S_\alpha(1, 0, 0)$. Then, for $a_n$ and $\tilde{a}_n$ defined as above, we have for $m \in \mathbb{N}$

$$\left( a_n^{-2} \sum_{i=1}^n U_i^2, \tilde{a}_n^{-1} \sum_{i=1}^n U_i U_{i+1}, \cdots, \tilde{a}_n^{-1} \sum_{i=1}^n U_i U_{i+m} \right) \Rightarrow (Y_0, Y_1, \cdots, Y_m),$$

where $Y_0, Y_1, \cdots, Y_m$ are independent stable random variables, $Y_0$ is positive $\alpha/2$-stable with distribution $S_{\alpha/2}(\sigma_1, 1, 0)$, $Y_1, \cdots, Y_m$ are i.i.d. symmetric $\alpha$-stable with distribution $S_\alpha(\sigma_2, 0, 0)$.

**Remark 3.6** In Davis and Resnick [4], the precise values of $\sigma_1$ and $\sigma_2$ are not provided explicitly. However, it is not hard to determine their values as given in Theorem 3.1 (see Mikosch, Gadrich, Klüppelberg, and Adler [16] as well). Why the value of $\mu$ (location
parameter) is equal to zero in the distribution of \( Y_0 \)? From the proof of the stable law (see Durrett [6]), we know that

\[
\bar{a}_n^{-1} \left( \sum_{i=1}^{n} U_i^2 - \bar{b}_n \right) \Rightarrow \overline{Y}_0, 
\]

where \( \bar{a}_n = a_n^2 \) and \( \bar{b}_n = n \mathbb{E}[U_1^2 1_{|U_1| \leq a_n}] \). and \( \overline{Y}_0 \) has a non-degenerate stable distribution \( S_{\alpha/2}(\sigma_1, 1, \mu_1) \) with \( \mu_1 = -\alpha/(2 - \alpha) \). By Karamata’s theorem (see Feller [7], Theorem 9.2 in Chapter VIII and Problem 9.30 in the same chapter), it follows that

\[
\bar{b}_n/\bar{a}_n = na_n^{-2} \mathbb{E}[U_1^2 1_{|U_1| \leq a_n}] \to \alpha/(2 - \alpha)
\]
as \( n \to \infty \). Therefore, we can conclude that

\[
a_n^{-2} \sum_{i=1}^{n} U_i^2 = \bar{a}_n^{-1} \left( \sum_{i=1}^{n} U_i^2 - \bar{b}_n \right) + \bar{b}_n/\bar{a}_n \Rightarrow \overline{Y}_0 + \alpha/(2 - \alpha) \sim S_{\alpha/2}(\sigma_1, 1, 0).
\]

All these clarifications are important for the further development of the asymptotic theory for LSE in the stable setting.

We first deal with the asymptotic behavior of \( \Phi_2(n) \). We have the following result:

**Proposition 3.7** Assume that (A1) is satisfied. Then, we have

\[
\Phi_2(n) - n^{-2/\alpha} h_1^{1-2/\alpha} \sum_{i=1}^{n} c_{i,h} X_{i-1}^2 \to P_0. \tag{21}
\]

**Proof.** We have

\[
\Phi_2(n) = n^{-2/\alpha} h_1^{1-2/\alpha} \sum_{i=1}^{n} X_{i-1}^2
\]

\[
= n^{-2/\alpha} h_1^{1-2/\alpha} X_0^2 + n^{-2/\alpha} h_1^{1-2/\alpha} \sum_{i=1}^{n-1} X_i^2
\]

\[
:= \Phi_{2,1}(n) + \Phi_{2,2}(n). \tag{22}
\]

It is clear that \( \Phi_{2,1}(n) \to 0 \) in probability as \( n \to \infty \). For \( \Phi_{2,2}(n) \), by the expression of \( X_i \) in (19), we find

\[
\Phi_{2,2}(n) = n^{-2/\alpha} h_1^{1-2/\alpha} \left[ \sum_{i=1}^{n-1} c_{i,h} X_0 + \gamma h \sum_{j=1}^{i} c_{j,h} U_{i-j} \right]^2
\]
\begin{align*}
\Phi_{2,2}(n) & - n^{-2/\alpha} h^{1-2/\alpha} \sum_{i=1}^{n-1} \left[ c_{i,h} X_0^2 + 2 \gamma_h c_{i,h} X_0 \sum_{j=1}^{i} c_{j,h} U_{i-j} + \frac{1}{\alpha} \left( \sum_{j=1}^{i} c_{j,h} U_{i-j} \right)^2 \right] \\
& = n^{-2/\alpha} h^{1-2/\alpha} \sum_{i=1}^{n-1} c_{i,h} X_0^2 + 2 n^{-2/\alpha} h^{1-2/\alpha} \gamma_h \sum_{i=1}^{n-1} c_{i,h} X_0 \sum_{j=1}^{i} c_{j,h} U_{i-j} \\
& \quad + n^{-2/\alpha} h^{1-2/\alpha} \gamma_h \sum_{i=1}^{n-1} \sum_{j=1}^{i} c_{j,h} U_{i-j}^2 \\
& \quad + n^{-2/\alpha} h^{1-2/\alpha} \gamma_h \sum_{i=1}^{n-1} \sum_{j=1}^{i} \sum_{k=1, k \neq j} c_{j,h} c_{k,h} U_{i-j} U_{i-k}.
\end{align*}

Hence,

\begin{align*}
\Phi_{2,2}(n) & - n^{-2/\alpha} h^{1-2/\alpha} \sum_{i=1}^{n-1} \sum_{j=1}^{c_{j,h} U_{i-j}} \\
& = n^{-2/\alpha} h^{1-2/\alpha} \sum_{i=1}^{n-1} c_{i,h} X_0^2 + 2 n^{-2/\alpha} h^{1-2/\alpha} \gamma_h \sum_{i=1}^{n-1} c_{i,h} X_0 \sum_{j=1}^{i} c_{j,h} U_{i-j} \\
& \quad + n^{-2/\alpha} h^{1-2/\alpha} \gamma_h \sum_{i=1}^{n-1} \sum_{j=1}^{i} \sum_{k=1, k \neq j} c_{j,h} c_{k,h} U_{i-j} U_{i-k} \\
& \quad + n^{-2/\alpha} h^{1-2/\alpha} \gamma_h \sum_{i=1}^{n-1} \sum_{j=1}^{i} c_{j,h} U_{i-j}^2 \\
& = \Sigma_1 + \Sigma_2 + \Sigma_3.
\end{align*}

For \( \Sigma_1 \), it is easy to see that

\begin{align*}
\Sigma_1 & = n^{-2/\alpha} h^{1-2/\alpha} \sum_{i=1}^{n-1} \left( e^{-2 \theta_0 h} X_0^2 \right) \\
& \leq X_0^2 n^{-2/\alpha} h^{1-2/\alpha} \frac{e^{-2 \theta_0 h}}{1 - e^{-2 \theta_0 h}} \leq \theta_0^{-1} (nh)^{-2/\alpha} X_0^2,
\end{align*}

which converges to zero in probability as \( n \to \infty \). For \( \Sigma_2 \), by Markov inequality, we find for any given \( \varepsilon > 0 \)

\begin{align*}
P(\| \Sigma_2 \| > \varepsilon) & \leq \varepsilon^{-1} \mathbb{E} \left[ 2 n^{-2/\alpha} h^{1-2/\alpha} \gamma_h \sum_{i=1}^{n-1} c_{i,h} c_{j,h} X_0 U_{i-j} \right] \\
& \leq 2 \varepsilon^{-1} n^{-2/\alpha} h^{1-2/\alpha} \gamma_h \sum_{i=1}^{n-1} \sum_{j=1}^{i} c_{i,h} c_{j,h} \mathbb{E} |X_0| \cdot \mathbb{E} |U_{i-j}| \\
& \leq C \varepsilon^{-1} h^{1-2/\alpha} \gamma_h (1 - e^{-\theta_0 h})^{-2} \leq C \varepsilon^{-1} \left( nh^{1+\alpha} \right)^{-2/\alpha}.
\end{align*}
By the hypothesis (A1), we see that this tends to zero as $n \to \infty$ (subject to $nh^{(1+\alpha)/2} \to \infty$). We are going to use some techniques in Davis and Resnick [4] to show that $\Sigma_3$ converges to zero in probability. By truncation technique, for any given $\varepsilon > 0$, we have

$$P \left( n^{-2/\alpha} h^{1-2/\alpha} \xi_h^2 \sum_{i=1}^{n-1} \sum_{j=1}^{i} \sum_{k \neq j, k=1}^{i} c_{j,h} c_{k,h} U_{i-j} U_{i-k} \geq \varepsilon \right) \leq P \left( n^{-2/\alpha} h^{1-2/\alpha} \xi_h^2 \sum_{i=1}^{n-1} \sum_{j=1}^{i} \sum_{k \neq j, k=1}^{i} c_{j,h} c_{k,h} U_{i-j} U_{i-k} 1(\{ U_{i-j} U_{i-k} \leq \hat{a}_n \}) \geq \frac{\varepsilon}{2} \right) + P \left( n^{-2/\alpha} h^{1-2/\alpha} \xi_h^2 \sum_{i=1}^{n-1} \sum_{j=1}^{i} \sum_{k \neq j, k=1}^{i} c_{j,h} c_{k,h} U_{i-j} U_{i-k} 1(\{ U_{i-j} U_{i-k} > \hat{a}_n \}) \geq \frac{\varepsilon}{2} \right) =: B_1 + B_2. \quad (27)$$

Note that $h^{-2/\alpha} \xi_h^2 = O(1)$ which can always be dominated by some universal constant $C$. By Chebyshev’s inequality, we find

$$B_1 \leq C \left( \frac{\varepsilon}{2} \right)^2 n^{-4/\alpha} h^2 \mathbb{E} \left[ \sum_{i=1}^{n-1} \sum_{j=1}^{i} \sum_{k \neq j, k=1}^{i} c_{j,h} c_{k,h} U_{i-j} U_{i-k} 1(\{ U_{i-j} U_{i-k} \leq \hat{a}_n \}) \right]^2 \leq 4 \varepsilon^{-2} n^{-4/\alpha} h^2 \sum_{i=1}^{n-1} \sum_{j=1}^{i} \sum_{k \neq j, k=1}^{i} \sum_{r=1}^{i} c_{j,h} c_{k,h} c_{j',h} c_{k',h} \times \mathbb{E} \left[ U_{i-j} U_{i-k} 1(\{ U_{i-j} U_{i-k} \leq \hat{a}_n \}) U_{r-j} U_{r-k} 1(\{ U_{r-j} U_{r-k} \leq \hat{a}_n \}) \right]. \quad (28)$$

We consider the expectation of the right hand side of (28) according to two different sets of indices: (i) all the indices $i-j, i-k, r-j', r-k'$ are different; (ii) one of the two indices $\{ i-j, i-k \}$ is equal to one of the two indexes $\{ r-j', r-k' \}$. In the case (i), it is easy to see that the expectation for these terms is equal to zero. Next, we deal with the expectation in the case (ii). For convenience, we put $\sigma_n^2 = \mathbb{E} \left[ (U_1 U_2)^2 1(\{ U_1 U_2 \leq \hat{a}_n \}) \right]$. We just calculate the expectation for one of the four subcases, namely, $i-j = r-j'$, or equivalently $j' = r - i + j$ (the expectations for all other subcases are the same):

$$B_1 \leq 16 C \varepsilon^{-2} n^{-4/\alpha} h^2 \sum_{i=1}^{n-1} \sum_{j=1}^{i} \sum_{k \neq j, k=1}^{i} \sum_{r=1}^{i} c_{j,h} c_{k,h} c_{r-i+j,h} c_{r-i+j,k} \sigma_n^2 \leq 16 C \varepsilon^{-2} n^{-4/\alpha} h^2 \left( \sum_{j=1}^{n-1} c_{j,h} \right)^4 \cdot n \sigma_n^2 \leq 16 C \varepsilon^{-2} n^{-4/\alpha} h^2 \left( \frac{1}{1 - e^{-\theta_0 k}} \right)^4 \cdot n \sigma_n^2 \leq C' \varepsilon^{-2} n^{-4/\alpha} h^{-2} n \sigma_n^2. \quad (29)$$
By Karamata’s theorem, \( a_n^{-2}n\sigma_n^2 \to \alpha/(2-\alpha) \). Thus, it follows that

\[
B_1 \leq C' \varepsilon^{-2} n^{-4/\alpha} - 2 a_n^2 \tilde{\alpha}_n^{-2} n\sigma_n^2,
\]

which tends to zero as \( n \to \infty \) subject to the condition \( \frac{\log n}{nh^\alpha} \to 0 \) (which holds under condition (A1) since \( nh^\alpha = O(n^{1/\alpha}) \)).

Next, we turn to \( B_2 \). By Markov inequality, we have

\[
B_2 \leq 2\varepsilon^{-1} n^{-2/\alpha} h^{1-\alpha/2} \gamma_h^2 \mathbb{E} \left[ \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} c_{j,h} c_{k,h} U_{i-j} U_{i-k} 1_{\{|U_{i-j} U_{i-k}| > \tilde{a}_n\}} \right]
\]

\[
\leq 2\varepsilon^{-1} n^{-2/\alpha} h^{1-\alpha/2} \gamma_h^2 \sum_{i=1}^{n-1} \mathbb{E} \left[ U_{i} U_{2} 1_{\{|U_{i} U_{2}| > \tilde{a}_n\}} \right]
\]

\[
\leq 2\varepsilon^{-1} n^{-2/\alpha} h^{1-\alpha/2} \gamma_h^2 \mathbb{E} \left[ U_{i} U_{2} 1_{\{|U_{i} U_{2}| > \tilde{a}_n\}} \right]
\]

\[
\leq C\varepsilon^{-1} \left( \frac{\log n}{nh^\alpha} \right)^{1/\alpha} n\tilde{a}_n^{-1} \mathbb{E} \left[ U_{i} U_{2} 1_{\{|U_{i} U_{2}| > \tilde{a}_n\}} \right],
\]

which converges to zero as \( n \to \infty \) under condition (A1), since \( n\tilde{a}_n^{-1} \mathbb{E} \left[ U_{i} U_{2} 1_{\{|U_{i} U_{2}| > \tilde{a}_n\}} \right] \) converges to \( \alpha/(\alpha - 1) \) by Karamata’s theorem. Thus, the proof is complete. \( \square \)

The next proposition gives the limit of \( n^{-2/\alpha} h^{(\alpha-2)/\alpha} \gamma_h^2 \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} c_{j,h} U_{i-j}^2 \).

**Proposition 3.8** If the condition (A1) is satisfied, then we have

\[
n^{-2/\alpha} h^{(\alpha-2)/\alpha} \gamma_h^2 \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} c_{j,h} U_{i-j}^2 \Rightarrow \frac{C_\alpha^{2/\alpha} Y_0}{2\theta_0},
\]

where \( Y_0 \) is a random variable with positively skewed stable distribution \( S_{\alpha/2}(\sigma_1, 1, 0) \) as specified in Theorem 3.1.

**Proof.** By interchanging the order of summation and regrouping terms, we have

\[
n^{-2/\alpha} h^{(\alpha-2)/\alpha} \gamma_h^2 \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} c_{j,h} U_{i-j}^2 = C_\alpha^{2/\alpha} a_n^{-2} h^{(\alpha-2)/\alpha} \gamma_h^2 \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} c_{j,h} U_{i-j}^2
\]

\[
= C_\alpha^{2/\alpha} a_n^{-2} h^{(\alpha-2)/\gamma_h^2} \sum_{j=1}^{n-1} \sum_{i=j}^{n-1-1} c_{j,h} U_{i-j}^2 = C_\alpha^{2/\alpha} a_n^{-2} h^{(\alpha-2)/\gamma_h^2} \sum_{j=1}^{n-1} \sum_{k=0}^{n-1-j} c_{j,h} U_k^2
\]

\[
= C_\alpha^{2/\alpha} a_n^{-2} h^{(\alpha-2)/\gamma_h^2} \sum_{j=1}^{n-1} c_{j,h} \left( \sum_{k=0}^{n-1-j} U_k - \sum_{k=n-j}^{n-1} U_k \right)
\]

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\[
C^2_\alpha a_n^{-2}h^{(\alpha-2)/\alpha \gamma_h^2} \sum_{j=1}^{n-1} c_{j,h}^2 \sum_{k=0}^{n-1} U_k^2 - C^2_\alpha a_n^{-2}h^{(\alpha-2)/\alpha \gamma_h^2} \sum_{j=1}^{n-1} c_{j,h}^2 \sum_{k=0}^{n-1} U_k^2
:= D_1 - D_2. \tag{33}
\]

We first show that \(D_2 \to 0\) in probability. By Markov inequality and the fact that \(h^{-2/\alpha \gamma_h^2} = O(1)\), we have for \(\delta < \alpha (\delta/2 < \alpha/2 < 1)\)
\[
P(|D_2| > \varepsilon) \leq \varepsilon^{-\delta/2} \mathbb{E} \left| n^{-2/\alpha} h^{(\alpha-2)/\alpha \gamma_h^2} \sum_{j=1}^{n-1} c_{j,h}^2 \sum_{k=0}^{n-1} U_k^2 \right|^{\delta/2}
\leq C \varepsilon^{-\delta/2} n^{-\delta/\alpha} h^{\delta/2} \sum_{j=1}^{n-1} e^{-\delta \theta_0 j h} \sum_{k=0}^{n-1} \mathbb{E} |U_k|^\delta
\leq C \varepsilon^{-\delta/2} n^{-\delta/\alpha} h^{\delta/2} \sum_{j=1}^{n-1} j e^{-\delta \theta_0 j h}
\leq C \varepsilon^{-\delta/2} n^{-\delta/\alpha} h^{\delta/2} h^{-2} [h(\delta \theta_0)^{-1} + (\delta \theta_0)^{-2}]
\leq C \varepsilon^{-\delta/2} (\delta \theta_0)^{-1} [n h^{\alpha(4-\delta)}]^{-\delta/\alpha} + C \varepsilon^{-\delta/2} \theta_0^{-2} \delta^2 [n h^{\alpha(4-\delta)}]^{-\delta/\alpha}
\to 0, \tag{34}
\]

as \(n \to \infty\) subject to \(n h^{\alpha(4-\delta)} \to \infty\) which is satisfied when \(4\alpha/(3\alpha + 2) < \delta < \alpha\) or \(\alpha > 2/3\). Now, for \(D_1\), it is clear that \(h^{-2/\alpha \gamma_h^2} \to 1\) and
\[
\sum_{j=1}^{n-1} c_{j,h}^2 h = h \sum_{j=1}^{n-1} e^{-2\theta_0 j h} = \frac{h e^{-2\theta_0 h} - e^{-2\theta_0 nh}}{1 - e^{-2\theta_0 h}} \to \frac{1}{2\theta_0}. \tag{35}
\]

Also, by Lemma 3.5, we have
\[
a_n^{-2} \sum_{k=0}^{n-1} U_k^2 \Rightarrow Y_0, \tag{36}
\]
where \(Y_0\) is a positive random variable with stable distribution \(S_{\alpha/2}(\sigma_1, 1, 0)\). Therefore, we have
\[
D_1 = C^2_\alpha a_n^{-2} h^{-2/\alpha \gamma_h^2} \sum_{j=1}^{n-1} c_{j,h}^2 h [a_n^{-2} \sum_{k=0}^{n-1} U_k^2] \Rightarrow \frac{C^2_\alpha Y_0}{2\theta_0}. \tag{37}
\]
This completes the proof. \(\square\)

Now, we turn to study the asymptotic behavior of \(\Phi_1(n)\). We first have the following
Proposition 3.9 Suppose that condition \((A1)\) is satisfied. Then, we have

\[
\left| \Phi_1(n) - (n \log n)^{-1/\alpha} h^{-1/\alpha} \gamma_h^2 \sum_{k=1}^{n-1} \sum_{l=0}^{n-1-k} c_{k,h} U_l U_{l+k} \right| \to p. 0. \quad (38)
\]

Proof. Note that

\[
\Phi_1(n) = (n \log n)^{-1/\alpha} h^{-1/\alpha} \sum_{i=1}^{n} X_{i-1} \cdot \int_{t_{i-1}}^{t_i} e^{-\theta_h(t-s)} dZ_s
\]

\[
= (n \log n)^{-1/\alpha} h^{-1/\alpha} \sum_{i=1}^{n} X_{i-1} \cdot e^{-\theta a_t_i V_{i-1}}
\]

\[
= (n \log n)^{-1/\alpha} h^{-1/\alpha} \sum_{i=1}^{n} X_{i-1} \cdot e^{-\theta b \gamma_h U_{i-1}}
\]

\[
e^{-\theta b}(n \log n)^{-1/\alpha} h^{-1/\alpha} X_0 \gamma_h U_0 + e^{-\theta b}(n \log n)^{-1/\alpha} h^{-1/\alpha} \sum_{i=2}^{n} X_{i-1} \cdot \gamma_h U_{i-1}
\]

\[
e^{-\theta b}(n \log n)^{-1/\alpha} h^{-1/\alpha} X_0 \gamma_h U_0 + e^{-\theta b}(n \log n)^{-1/\alpha} h^{-1/\alpha} \sum_{i=1}^{n-1} X_i \cdot \gamma_h U_i
\]

\[
e^{-\theta b}(n \log n)^{-1/\alpha} h^{-1/\alpha} X_0 \gamma_h U_0 + e^{-\theta b}(n \log n)^{-1/\alpha} h^{-1/\alpha} \sum_{i=1}^{n-1} \left( c_{i,h} X_0 + \gamma_h \sum_{j=1}^{i} c_{j,h} U_{i-j} \right) \cdot \gamma_h U_i
\]

\[
e^{-\theta b}(n \log n)^{-1/\alpha} h^{-1/\alpha} \gamma_h \sum_{i=0}^{n-1} c_{i,h} X_0 U_i
\]

\[
e^{-\theta b}(n \log n)^{-1/\alpha} h^{-1/\alpha} \gamma_h \sum_{i=0}^{n-1} c_{i,h} X_0 U_i + e^{-\theta b}(n \log n)^{-1/\alpha} h^{-1/\alpha} \gamma_h^2 \sum_{i=1}^{n-1} \sum_{j=1}^{i} c_{j,h} U_{i-j} U_i
\]

\[
e^{-\theta b}(n \log n)^{-1/\alpha} h^{-1/\alpha} \gamma_h \sum_{i=0}^{n-1} c_{i,h} X_0 U_i + e^{-\theta b}(n \log n)^{-1/\alpha} h^{-1/\alpha} \gamma_h^2 \sum_{j=1}^{n-1} \sum_{i=j}^{n-1} c_{j,h} U_{i-j} U_i
\]

\[
e^{-\theta b}(n \log n)^{-1/\alpha} h^{-1/\alpha} \gamma_h \sum_{i=0}^{n-1} c_{i,h} X_0 U_i + e^{-\theta b}(n \log n)^{-1/\alpha} h^{-1/\alpha} \gamma_h^2 \sum_{j=1}^{n-1} \sum_{i=j}^{n-1} U_{i-j} U_i
\]

\[
e^{-\theta b}(n \log n)^{-1/\alpha} h^{-1/\alpha} \gamma_h \sum_{i=0}^{n-1} c_{i,h} X_0 U_i + e^{-\theta b}(n \log n)^{-1/\alpha} h^{-1/\alpha} \gamma_h^2 \sum_{j=1}^{n-1} \sum_{i=j}^{n-1} U_{i-j} U_i
\]
We rewrite

\[ + e^{-\theta_{ih}} (n \log n)^{-1/\alpha} h^{-1/\alpha} \gamma_h^2 \sum_{j=1}^{n-1} c_{j,h} \sum_{l=0}^{n-1-j} U_l U_{l+j}. \]  

(39)

It follows that

\[ \Phi_1(n) - e^{-\theta_{ih}} (n \log n)^{-1/\alpha} h^{-1/\alpha} \gamma_h^2 \sum_{j=1}^{n-1} c_{j,h} \sum_{l=0}^{n-1-j} U_l U_{l+j} \]

\[ = e^{-\theta_{ih}} (n \log n)^{-1/\alpha} h^{-1/\alpha} \gamma_h \sum_{i=0}^{n-1} c_{i,h} X_0 U_i. \]

(40)

By Chebyshev’s inequality, for any given \( \varepsilon > 0 \), we find

\[ P \left( \left| e^{-\theta_{ih}} (n \log n)^{-1/\alpha} h^{-1/\alpha} \gamma_h \sum_{i=0}^{n-1} c_{i,h} X_0 U_i \right| > \varepsilon \right) \]

\[ \leq \frac{\varepsilon^{-1} E \left[ \left| e^{-\theta_{ih}} (n \log n)^{-1/\alpha} h^{-1/\alpha} \gamma_h \sum_{i=0}^{n-1} c_{i,h} X_0 U_i \right| \right]}{\varepsilon^{-1} E \left( e^{-\theta_{ih}} (n \log n)^{-1/\alpha} h^{-1/\alpha} \gamma_h \sum_{i=0}^{n-1} c_{i,h} X_0 U_i \right)} \]

\[ \leq \varepsilon^{-1} e^{-\theta_{ih}} (n \log n)^{-1/\alpha} h^{-1/\alpha} \gamma_h \sum_{i=0}^{n-1} c_{i,h} |X_0| : E|U_i| \leq C \varepsilon^{-1} (n \log n)^{-1/\alpha} \sum_{i=0}^{n-1} e^{-\theta_{ih}} \]

\[ \leq C \varepsilon^{-1} (n \log n)^{-1/\alpha} \cdot \frac{1}{1 - e^{-\theta_{ih}}} \leq C \varepsilon^{-1} (n \log n)^{-1/\alpha} (nh^\alpha)^{-1/\alpha}, \]

(41)

which tends to zero as \( n \to \infty \) under condition (A1). This completes the proof. \( \square \)

Next, we shall deal with the convergence of

\[ e^{-\theta_{ih}} (n \log n)^{-1/\alpha} h^{-1/\alpha} \gamma_h^2 \sum_{j=1}^{n-1} c_{j,h} \sum_{l=0}^{n-1-j} U_l U_{l+j} := F_1(n). \]

We rewrite \( F_1(n) \) as follows:

\[ F_1(n) = e^{-\theta_{ih}} (n \log n)^{-1/\alpha} h^{-1/\alpha} \gamma_h^2 \sum_{j=1}^{n-1} c_{j,h} \left[ \sum_{l=0}^{n-1} U_l U_{l+j} - \sum_{l=n-j}^{n-1} U_l U_{l+j} \right] \]

\[ = e^{-\theta_{ih}} (n \log n)^{-1/\alpha} h^{-1/\alpha} \gamma_h^2 \sum_{j=1}^{n-1} c_{j,h} \sum_{l=0}^{n-1} U_l U_{l+j} \]

\[ - e^{-\theta_{ih}} (n \log n)^{-1/\alpha} h^{-1/\alpha} \gamma_h^2 \sum_{j=1}^{n-1} c_{j,h} \sum_{l=n-j}^{n-1} U_l U_{l+j} \]

\[ := F_{1,1}(n) - F_{1,2}(n). \]

(42)
Proposition 3.10 Under the condition (A1), we have

\[ F_{1,2}(n) \to_p 0 \text{ as } n \to \infty. \]  

(43)

Proof. By Markov inequality, for any given \( \varepsilon > 0 \), we have

\[
P(|F_{1,2}(n)| > \varepsilon) \leq \varepsilon^{-1}e^{-\theta_0 h (n \log n)^{-1/\alpha}h^{-1/\alpha}h^2 \gamma_h^2 \sum_{j=1}^{n-1} c_{j,h} \sum_{l=n-j}^{n-1} U_l U_{l+j}}
\]

\[
\leq \varepsilon^{-1}(n \log n)^{-1/\alpha}h^{-1/\alpha}h^2 \sum_{j=1}^{n-1} c_{j,h} \sum_{l=n-j}^{n-1} \mathbb{E}|U_l|\mathbb{E}|U_{l+j}|
\]

\[
\leq C\varepsilon^{-1}(n \log n)^{-1/\alpha}h^{1/\alpha} \sum_{j=1}^{n-1} j \cdot c_{j,h}
\]

\[
\leq C\varepsilon^{-1}(n \log n)^{-1/\alpha}h^{1/\alpha}e^{\theta_0 h}h^{-2} \int_0^\infty e^{-\theta_0 x}dx
\]

\[
\leq C\varepsilon^{-1}(n \log n)^{-1/\alpha}h^{1/\alpha}e^{\theta_0 h}h^{-2} e^{-\theta_0 h} (h\theta_0^{-1} + \theta_0^{-2})
\]

\[
= C(\theta_0 \varepsilon)^{-1}(n \log n)^{-1/\alpha}h^{(1-\alpha)/\alpha} + C\theta_0^{-2} \varepsilon^{-1}(n \log n)^{-1/\alpha}h^{(1-2\alpha)/\alpha}
\]

(44)

which tends to zero as \( n \to \infty \) since \( nh^{\alpha-1} = O(n^{2/(1+\alpha)}) \) and \( nh^{2\alpha-1} = O(n^{(2-\alpha)/(1+\alpha)}) \) under condition (A1).

Now, we turn to consider the asymptotic behavior of \( F_{1,1}(n) \). For convenience, we rewrite \( F_{1,1}(n) \) as

\[
F_{1,1}(n) = e^{-\theta_0 h (n \log n)^{-1/\alpha}h^{-1/\alpha}h^2 \gamma_h^2 \sum_{j=1}^{n-1} c_{j,h} \sum_{l=0}^{n-1} U_l U_{l+j}} 
\]

\[
= e^{-\theta_0 h (n \log n)^{-1/\alpha}h^{-1/\alpha}h^2 \sum_{j=1}^{n-1} c_{j,h} \sum_{l=0}^{n-1} U_l U_{l+j}}
\]

\[
- e^{-\theta_0 h (n \log n)^{-1/\alpha}h^{-1/\alpha}h^2 \sum_{j=n}^{\infty} c_{j,h} \sum_{l=0}^{n-1} U_l U_{l+j}}
\]

\[
:= G_1(n) - G_2(n).
\]

(45)
Proposition 3.11 Under the condition (A1), we have $G_2(n) \to 0$ in probability as $n \to \infty$.

**Proof.** By Markov inequality, for given $\varepsilon > 0$, we have

$$P(|G_2(n)| > \varepsilon) \leq \varepsilon^{-1} e^{-\theta_0 h(n \log n)^{-1/\alpha} h^{-1/\alpha} \gamma_h^2} \sum_{j=n}^{\infty} \sum_{l=0}^{n-1} E[U_l | E[U_{l+j}]] \leq C \varepsilon^{-1} (n \log n)^{-1/\alpha} h^{-1/\alpha} \gamma_h^2 \sum_{j=n}^{\infty} e^{-\theta_0 h} \leq C \varepsilon^{-1} n^{(\alpha-1)/\alpha} (\log n)^{-1/\alpha} h^{-1/\alpha} \gamma_h^2 \sum_{j=n}^{\infty} e^{-\theta_0 h},$$

which tends to zero as $n \to \infty$ under the condition (A1). This completes the proof. □

Proposition 3.12 Assume that (A1) is satisfied. Then, $G_1(n) \Rightarrow C^2/\alpha Y$ as $n \to \infty$, where $Y$ is a random variable with stable distribution $S_{\alpha}(\alpha \theta_0)^{-1/\alpha} \sigma_2, 0, 0)$.

**Proof.** Let $H_{n,j} = \tilde{a}_n^{-1} \sum_{l=0}^{n-1} U_l U_{l+j}, \forall j \in \mathbb{N}$. Then, we have

$$G_1(n) = e^{-\theta_0 h} (n \log n)^{-1/\alpha} h^{-1/\alpha} \gamma_h^2 \sum_{j=1}^{\infty} \sum_{l=0}^{n-1} U_l U_{l+j}$$

$$= e^{-\theta_0 h} C^2/\alpha \tilde{a}_n^{-1} h^{-1/\alpha} \gamma_h^2 \sum_{j=1}^{\infty} \sum_{l=0}^{n-1} U_l U_{l+j}$$

$$= C^2/\alpha e^{-\theta_0 h} h^{-2/\alpha} \gamma_h^2 \sum_{j=1}^{\infty} h^{1/\alpha} c_{j,h} H_{n,j}. \quad (47)$$

Note that $e^{-\theta_0 h} h^{-2/\alpha} \gamma_h^2 \to 1$ as $n \to \infty$. Thus, the asymptotic distribution of $G_1(n)$ is completely determined by that of $\sum_{j=1}^{\infty} h^{1/\alpha} c_{j,h} H_{n,j}$. For fixed $m \in \mathbb{N}$, we consider the partial sum $\sum_{j=1}^{m} h^{1/\alpha} c_{j,h} H_{n,j}$. By Lemma 3.5, we know that

$$(H_{n,1}, H_{n,2}, \cdots, H_{n,m}) \Rightarrow (Y_1, Y_2, \cdots, Y_m).$$

By Skorohod’s representation theorem, there exists two sequences of random variables \{I_{n,j}\}_{j=1}^{m} and \{S_j\}_{j=1}^{m} defined on some new probability space $(\mathcal{G}, \mathcal{F}, P)$ such that

$$\mathcal{L}(H_{n,1}, \cdots, H_{n,m}) = \mathcal{L}(I_{n,1}, \cdots, I_{n,m}),$$

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\[ \mathcal{L}(S_1, \ldots, S_m) = \mathcal{L}(Y_1, \ldots, Y_m), \]

and \( I_{n,j} \to S_j, \overline{P} \text{-a.s.}, j = 1, \ldots, m. \) (Here \( m \) can be \( \infty \)).

**Claim 1:**
\[ \sum_{j=1}^{\infty} h^{1/\alpha} c_{j,h} I_{n,j} - \sum_{j=1}^{\infty} h^{1/\alpha} c_{j,h} S_j \to 0 \text{ as } n \to \infty. \]

**Proof.** Note that \( \sup_{j \in \mathbb{N}} (h^{1/\alpha} c_{j,h}) < 1. \) Then, we have
\[
\mathbb{P} \left( \left| \sum_{j=1}^{\infty} h^{1/\alpha} c_{j,h} (I_{n,j} - S_j) \right| > \varepsilon \right) \leq \mathbb{P} \left( \sum_{j=1}^{\infty} h^{1/\alpha} c_{j,h} |I_{n,j} - S_j| > \varepsilon \right) \\
\leq \sum_{j=1}^{\infty} \mathbb{P} (h^{1/\alpha} c_{j,h} |I_{n,j} - S_j| > 2^{-j} \varepsilon) \leq \sum_{j=1}^{\infty} \mathbb{P} (|I_{n,j} - S_j| > 2^{-j} \varepsilon). \quad (48)
\]

Letting \( n \to \infty, \) we find
\[
\lim_{n \to \infty} \mathbb{P} \left( \left| \sum_{j=1}^{\infty} h^{1/\alpha} c_{j,h} (I_{n,j} - S_j) \right| > \varepsilon \right) = 0. \quad (49)
\]

So, the claim 1 is true.

**Claim 2:**
\[ \sum_{j=1}^{\infty} h^{1/\alpha} c_{j,h} S_j \Rightarrow Y, \]
where \( Y \) has a stable distribution \( S_\alpha((\alpha \theta_0)^{-1/\alpha} \sigma_2, 0, 0). \)

**Proof.** Since \( \{S_j\}_{j=1}^{\infty} \) is a sequence of independent random variables with the same distribution \( S_\alpha(\sigma_2, 0, 0), \) it follows that
\[
\sum_{j=1}^{\infty} h^{1/\alpha} c_{j,h} S_j \sim S_\alpha((\sum_{j=1}^{\infty} e^{-\alpha \theta_0 h} h)^{1/\alpha} \sigma_2, 0, 0).
\]

By L’Hospital’s rule, we find that
\[
\lim_{n \to \infty} \sum_{j=1}^{\infty} e^{-\alpha \theta_0 j h} h = \lim_{n \to \infty} \frac{h e^{-\alpha \theta_0 h}}{1 - e^{-\alpha \theta_0 h}} = \lim_{n \to \infty} \frac{e^{-\alpha \theta_0 h} + h(-\alpha \theta_0) e^{-\alpha \theta_0 h}}{\alpha \theta_0 e^{-\alpha \theta_0 h}} = \frac{1}{\alpha \theta_0}. \quad (50)
\]

The desired claim follows immediately.
By Theorem 25.4 of Billingsley [2] and using Claims 1 and 2, we find that
\[ \sum_{j=1}^{\infty} h^{1/\alpha} c_{j,h} I_{n,j} \Rightarrow Y \]
and consequently
\[ \sum_{j=1}^{\infty} h^{1/\alpha} c_{j,h} H_{n,j} \Rightarrow Y', \]
where \( Y' \) is a random variable with same distribution as \( Y \) and \( Y' \) is independent of \( Y_0 \).
Hence, \( G_1(n) \Rightarrow C_2^{2/\alpha} Y' \).

Finally, we are in a position to prove Theorem 3.1.

**Proof of Theorem 3.1.** By Lemma 3.3 and Propositions 3.7-3.12, we conclude that
\[
\left( \frac{n}{\log n} \right)^{1/\alpha} h^{1/\alpha}(\hat{\theta}_n - \theta_0) = \Lambda_n - \frac{\Phi_1(n)}{\Phi_2(n)} \Rightarrow \frac{-C_2^{2/\alpha} Y'}{C_2^{2/\alpha}(2\theta_0)^{-1}Y_0} \sim \frac{2\theta_0(\alpha \theta_0)^{-1/\alpha} \tilde{Y}}{Y_0}, \tag{51}
\]
where \( \tilde{Y} \) has a distribution \( S_\alpha(\sigma_2, 0, 0) \) independent of \( Y_0 \). This completes the proof. \( \square \)

**4 Simulation**

The left figure describes the least squares estimator \( \hat{\theta}_T \) for the following equation
\[ dX_t = -\theta_0 X_t dt + dZ_t, \quad X_0 = 1 \]
for \( \theta_0 = 2 \). In this simulation, we choose \( \alpha = 1.8, \ h = 0.05, \) and \( n = 2000 \). For comparison we plot the simulation result (right figure) for \( \theta_0 = -2 \) (non ergodic case).

**Acknowledgment:** We thank Qinghua Zhang (University of Kansas) for the simulations.
Figure 1: Plot of $\hat{\theta}_T$ as functions of $T$

References


