Least squares estimator for Ornstein–Uhlenbeck processes driven by $\alpha$-stable motions

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Received 6 March 2007; received in revised form 19 September 2008; accepted 15 December 2008
Available online 5 January 2009

Abstract

We study the problem of parameter estimation for generalized Ornstein–Uhlenbeck processes driven by $\alpha$-stable noises, observed at discrete time instants. Least squares method is used to obtain an asymptotically consistent estimator. The strong consistency and the rate of convergence of the estimator have been studied. The estimator has a higher order of convergence in the general stable, non-Gaussian case than in the classical Gaussian case.

MSC: 60G52; 62M05; 65C30; 93E24

Keywords: Asymptotic distribution of LSE; Consistency of LSE; Discrete observation; Least squares method; Generalized Ornstein–Uhlenbeck processes; Parameter estimation; $\alpha$-stable processes

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a basic probability space equipped with a right continuous and increasing family of $\sigma$-algebras $\{\mathcal{F}_t, t \geq 0\}$. Let $\{Z_t, t \geq 0\}$ be a standard symmetric $\alpha$-stable Lévy motion. For technical reasons, we assume that $1 < \alpha < 2$. The generalized Ornstein–Uhlenbeck process $\{X_t, t \geq 0\}$, starting from $x \in \mathbb{R}$ is defined as the unique solution to the following linear stochastic differential equation (SDE)

$$dX_t = -\theta_0 X_t dt + dZ_t, \quad X_0 = x.$$  

(1.1)
Assume that this process is observed at some discrete time instants \( \{ t_i = i h, i = 0, 1, 2, \ldots \} \), but the value of \( \theta_0 \) is unknown. The purpose of this paper is to study the least squares estimator (LSE) for the true value \( \theta_0 \) based on the sampling data \((X_t)^n_{i=0}\).

In the case of diffusion processes driven by Brownian motions, a popular method is the maximum likelihood estimator (MLE) based on the Girsanov density (see [1]). It is asymptotically equivalent to the least squares estimator. For the LSE the convergence in probability is proved in [2,3], the strong consistency is studied in [4], and the asymptotic distribution was studied in [5]. For a more recent comprehensive discussion, we refer to [6,7] and the references therein. For MLE based on discrete observations, see for example [8].

Recently there has been a growing interest in parameter estimation for stochastic processes driven by Lévy processes with finite moments due to its promising applications for example to finance. Substantial progress has been made. The asymptotic normality of the LSE and MLE for pure jump process is studied in [9,10]. The paper [11] dealt with the consistency and asymptotic normality when the driving process is a zero-mean adapted process (including Lévy process) with finite moments. However, when the driving processes are \( \alpha \)-stable Lévy motions there has been no study yet due to the infinite variance property of \( \alpha \)-stable processes.

The main focus of this paper is the study of the strong consistency and asymptotic distributions of the LSE for generalized O–U processes satisfying the SDE (1.1). Our results are analogues of the LSE and the Yule–Walker estimator for ARMA models driven by a sequence of i.i.d. random variables in the domain of attraction of a stable law (see Davis and Resnick [12]). Other related estimators such as M-estimator and the Whittle estimator can be found in [13,14].

To obtain the LSE, we introduce the following contrast function

\[
\rho_n(\theta) = \rho_n(\theta; (X_t)^n_{i=0}) = \sum_{i=1}^n |X_{t_i} - X_{t_{i-1}} + \theta X_{t_{i-1}} \cdot \Delta t_{i-1}|^2. \tag{1.2}
\]

Then the LSE \( \hat{\theta}_n \) is defined as \( \hat{\theta}_n = \arg\min_{\theta > 0} \rho_n(\theta) \), which can be explicitly represented as

\[
\hat{\theta}_n = -\frac{\sum_{i=1}^n (X_{t_i} - X_{t_{i-1}}) X_{t_{i-1}}}{h \sum_{i=1}^n X_{t_{i-1}}^2}. \tag{1.3}
\]

The equation can be solved explicitly so that we can represent the LSE \( \hat{\theta}_n \) as

\[
\hat{\theta}_n = \frac{1 - e^{-\theta_0 h}}{h} - \frac{\sum_{i=1}^n X_{t_{i-1}} \cdot \int_{t_{i-1}}^{t_i} e^{-\theta_0 (t_i - s)} dZ_s}{h \sum_{i=1}^n X_{t_{i-1}}^2}. \tag{1.4}
\]

In this paper, high frequency \((h \to 0)\) asymptotic of the LSE \( \hat{\theta}_n \) is considered in the ergodic case \((\theta_0 > 0)\). Our goal is to prove that \( \hat{\theta}_n \to \theta_0 \) almost surely and to establish the rate of convergence \((\log n)^{1/\alpha} \). This rate is considerably faster than in the Brownian motion case.

If the processes can be observed continuously, a trajectory fitting method combined with weighted least squares technique is discussed in [15].
The paper is organized as follows. In Section 2 we establish the strong consistency of the LSE \( \hat{\theta}_n \). In Section 3, we study the rate of convergence for the LSE. Finally, some simulation results are provided in Section 4.

2. Strong consistency of the LSE

Recall that a random variable \( \eta \) is said to have a stable distribution with index of stability \( \alpha \in (0, 2] \), scale parameter \( \sigma \in (0, \infty) \), skewness parameter \( \beta \in [-1, 1] \), and location parameter \( \mu \in (-\infty, \infty) \) if it has a characteristic function of the following form:

\[
\phi_\eta(u) = \mathbb{E}\exp[iu\eta] = \begin{cases} 
\exp \left\{ -\sigma |u|^\alpha \left( 1 - i \beta \text{sgn}(u) \tan \frac{\alpha \pi}{2} \right) + i \mu u \right\}, & \text{if } \alpha \neq 1, \\
\exp \left\{ -\sigma |u| \left( 1 + i \beta \frac{2}{\pi} \text{sgn}(u) \log |u| \right) + i \mu u \right\}, & \text{if } \alpha = 1.
\end{cases}
\]

We denote \( \eta \sim S_\alpha(\sigma, \beta, \mu) \). When \( \mu = 0 \), we say \( \eta \) is strictly \( \alpha \)-stable. If in addition \( \beta = 0 \), we call \( \eta \) symmetric \( \alpha \)-stable. We refer to [16,17] for more details on stable distributions. Throughout this paper, it is assumed that \( Z_1 \sim S_\alpha(1,0,0) \). We use the notation “\( \Rightarrow \)” to denote “convergence in distribution”.

**Theorem 2.1.** Assume that \( h \to 0 \) and \( t_n = nh \to \infty \) as \( n \to \infty \). Then, the following strong consistency holds:

\[
\hat{\theta}_n \to \theta_0 \quad \text{almost surely as } n \to \infty.
\] (2.1)

**Proof.** Let \( \phi_n(t) = \sum_{i=1}^n X_{t_{i-1}} e^{-\theta(t_{i-1} - t)} 1_{(t_{i-1}, t_i)}(t) \). It is clear that

\[
\sum_{i=1}^n X_{t_{i-1}} \int_{t_{i-1}}^{t_i} e^{-\theta_0(t_s - t)} dZ_s = \int_0^{t_n} \phi_n(s) dZ_s.
\] (2.2)

Let \( \tau_n(t_n) = \int_0^{t_n} |\phi_n(t)|^\alpha dt \). Then, it is easy to find

\[
\tau_n(t_n) = \int_0^{t_n} \sum_{i=1}^n |X_{t_{i-1}}|^\alpha e^{-\alpha \theta_0(t_{i-1} - t)} 1_{(t_{i-1}, t_i)}(t) dt
\]

\[
= \sum_{i=1}^n |X_{t_{i-1}}|^\alpha \left( \frac{1 - e^{-\alpha \theta_0 h}}{\alpha \theta_0} \right).
\] (2.3)

From (1.4), we have

\[
\hat{\theta}_n = 1 - \frac{e^{-\theta_0 h}}{h} - \frac{\int_0^{t_n} \phi_n(t) dZ_t}{\tau_n(t_n)} \cdot \frac{\tau_n(t_n)}{h \sum_{i=1}^n X_{t_{i-1}}^2}.
\] (2.4)

It is well-known that if \( \theta_0 > 0 \), \( X_t \) is ergodic and \( X_t \Rightarrow X_\infty \) as \( t \to \infty \), where \( X_\infty = \int_0^\infty e^{-\theta_0 t} dZ_t \) is a \( \alpha \)-stable random variable (see [17,15]). Thus, it follows by the ergodic theorem,

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n |X_{t_{i-1}}|^\alpha = \mathbb{E}[X_\infty^\alpha] = \infty \quad \text{a.s.}
\]

which implies that \( \tau_n(t_n) \to \infty \). Note also that \( \int_1^\infty t^{-\alpha} dt = 1/(\alpha - 1) < \infty \). By Corollary 3.1 of [18], we have
\[
\limsup_{t_n \to \infty} \frac{\int_0^{t_n} \phi_n(t) \, dZ_t}{\tau_n(t_n)} = 0 \quad \text{a.s.} \quad (2.5)
\]

By the Hölder inequality, we have
\[
\tau_n(t_n) \leq \frac{1 - e^{-\alpha \theta_0 h}}{\alpha \theta_0 h} \cdot \frac{\sum_{i=1}^{n} |X_{t_{i-1}}|^{\alpha}}{\sum_{i=1}^{n} X_{t_{i-1}}^2} \leq \frac{1 - e^{-\alpha \theta_0 h}}{\alpha \theta_0 h} \cdot \left( \frac{\sum_{i=1}^{n} |X_{t_{i-1}}|^{2}}{\sum_{i=1}^{n} X_{t_{i-1}}^2} \right)^{\alpha/2} n^{(2-\alpha)/2}
\]

which converges to zero almost surely as \( n \to \infty \), since by the ergodic theorem again,
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} |X_{t_{i-1}}|^{2} = \mathbb{E}[X_{t}^2] = \infty \quad \text{a.s.}
\]

Combining (2.4)–(2.6), we conclude that \( \hat{\theta}_n \to \theta_0 \) almost surely as \( n \to \infty \). \( \square \)

3. Asymptotic properties of the LSE

To obtain the rate of convergence we need to make the following assumption.

(A1): As \( n \to \infty \), \( h \to 0 \), \( nh^{1+\alpha}/\log n \to 0 \), \( nh^{2\alpha-1} \log n \to \infty \), and \( nh^{2-\alpha/2+\rho} \to \infty \) for some \( \rho > 0 \) small enough such that all the convergence conditions are compatible (we omit the dependence of \( h \) on \( n \)).

It is easy to see that under (A1), the convergence (2.1) holds.

Denote \( C_{\alpha} = \left( \int_{0}^{\infty} x^{-\alpha} \sin(x) \, dx \right)^{-1} = \left[ \Gamma(1 - \alpha) \cos(\pi \alpha/2) \right]^{-1} \), \( \sigma_1 = C_{\alpha/2}^{-2/\alpha} \), and \( \sigma_2 = C_{\alpha}^{-1/\alpha} \). Our main result is as follows.

**Theorem 3.1.** Under condition (A1), we have
\[
\left( \frac{n}{\log n} \right)^{1/\alpha} h^{1/\alpha}(\hat{\theta}_n - \theta_0) \Rightarrow \frac{2\theta_0(\alpha \theta_0)^{-1/\alpha} \tilde{Y}}{Y_0} \quad (3.1)
\]
where \( Y_0 \) and \( \tilde{Y} \) are independent stable random variables, \( Y_0 \) is positive \( \alpha/2 \)-stable with distribution \( S_{\alpha/2}(\sigma_1, 1, 0) \), and \( \tilde{Y} \) is symmetric \( \alpha \)-stable with distribution \( S_\alpha(\sigma_2, 0, 0) \).

**Remark 3.2.** Theorem 3.1 states that the rate at which \( \hat{\theta}_n \) converges to \( \theta_0 \) is \( \left( \frac{\log n}{nh} \right)^{1/\alpha} \), which is considerably faster than the rate \( (nh)^{-1/2} \) in the classical Brownian motion case. This is not surprising due to the works [19,13,12].
Remark 3.3. The $h$ in (A1) can be $h = cn^{-\lambda}$, where $c > 0$ and
\[
\lambda \in \begin{cases} 
\frac{1}{1+\alpha}, & \text{if } \alpha \in \left(\frac{1}{2}, \frac{2}{\alpha} \right), \\
\frac{1}{1+\alpha}, & \text{if } \alpha \in \left(\frac{1}{5}, \frac{3}{5} \right). 
\end{cases}
\]

The choice $\lambda = \frac{1}{1+\alpha}$ leads to the optimal convergence rate in Theorem 3.1.

Theorem 3.1 will be proved by establishing several preliminary lemmas and propositions. We first give an explicit expression for $\left(\frac{n}{\log n}\right)^{1/\alpha} H^{1/\alpha}(\hat{\Theta}_n - \theta_0)$. From now on, we shall denote $X_i$ by $X_i$. By using (1.4), we find
\[
\left(\frac{n}{\log n}\right)^{1/\alpha} H^{1/\alpha}(\hat{\Theta}_n - \theta_0) = \left(\frac{n}{\log n}\right)^{1/\alpha} H^{1/\alpha}\left[\frac{1}{h^{-1}} \left(1 - e^{-\theta_0 h}\right) - \theta_0 \right]
\]
\[
= (n \log n)^{-1/\alpha} H^{-1/\alpha} \sum_{i=1}^{n} \int_{\tau_{i-1}}^{\tau_i} e^{-\theta_0 (t_i - s)} dZ_s
\]
\[
= A_n - \frac{\phi_1(n)}{\phi_2(n)}. \tag{3.2}
\]
So, the asymptotic behavior of $\left(\frac{n}{\log n}\right)^{1/\alpha} H^{1/\alpha}(\hat{\Theta}_n - \theta_0)$ will be determined by the asymptotic behavior of $A_n$, $\phi_1(n)$ and $\phi_2(n)$.

Lemma 3.4. Suppose that (A1) is satisfied. Then, we have $A_n \to 0$ as $n \to \infty$.

Proof. It is easy to see that
\[
|A_n| = \left|\left(\frac{n}{\log n}\right)^{1/\alpha} H^{1/\alpha}\left[\frac{1}{h^{-1}} \left(1 - e^{-\theta_0 h}\right) - \theta_0 \right]\right| \leq \frac{\theta_0^2}{2} n^{1/\alpha} h^{1+1/\alpha} (\log n)^{1/\alpha}, \tag{3.3}
\]
which tends to zero as $n \to \infty$ under condition (A1). \hfill \square

Note that
\[
X_i = e^{-\theta_0 h} X_0 + \sum_{k=1}^{i} e^{-\theta_0 h} \int_{\tau_{k-1}}^{\tau_k} e^{\theta_0 s} dZ_s. \tag{3.4}
\]
Let $V_{k-1} = \int_{\tau_{k-1}}^{\tau_k} e^{\theta_0 s} dZ_s$. By the inner clock property for the $\alpha$-stable stochastic integral (see [18, 20, 21]), we know that $\int_{\tau_{k-1}}^{\tau_k} e^{\theta_0 s} dZ_s$ has the same distribution as $Z_{\tau_{k-1}}$, where
\[
\tau_{k-1} = \int_{\tau_{k-1}}^{\tau_k} |e^{\theta_0 s}|^\alpha ds = e^{\alpha \theta_0 \tau_{k-1}} \left(\frac{e^{\theta_0 \tau_{k-1}} - 1}{\alpha \theta_0}\right).
\]
Let $U_{k-1} = V_{k-1}/\tau_{k-1}^{1/\alpha}$. Then, by the scaling property of stable distribution, we know that $U_0, U_1, U_2, \ldots$ are i.i.d. with the same stable distribution $S_{\alpha}(1, 0, 0)$. Let $c_{i,h} = e^{-\theta_0 h}$ and
\[ y_h = \left(\frac{\epsilon^\alpha - h}{\alpha \theta_0} \right)^{1/\alpha}. \] Then, \( X_i \) can be represented as

\[ X_i = e^{-\theta_0 h} X_0 + \left(\frac{\epsilon^\alpha - h}{\alpha \theta_0} \right)^{1/\alpha} \sum_{k=1}^{i} e^{-\theta_0 (i-k+1) h} U_{k-1} \]

\[ = c_{i,h} X_0 + \gamma_h \sum_{j=1}^{i} c_{j,h} U_{i-j}. \] (3.5)

**Remark 3.5.** From [16], for the symmetric \( \alpha \)-stable random variable \( U_1 \sim S_\alpha(1, 0, 0) \), we have

\[ \lim_{x \to \infty} x^\alpha P(U_1 > x) = C_\alpha/2 \quad \text{and} \quad \lim_{x \to \infty} x^\alpha P(U_1 < -x) = C_\alpha/2. \]

So, the tail distribution of \( \{U_1\} \) is asymptotically equivalent to a Pareto, i.e. \( P(|U_1| > x) \sim C_\alpha x^{\alpha - \alpha}. \) Following [12], we define

\[ a_n = \inf \{ x : P(|U_1| > x) \leq n^{-1} \} \quad \text{and} \quad \bar{a}_n = \inf \{ x : P(U_0 U_1 > x) \leq n^{-1} \} \]

Thanks to the asymptotic Pareto tail distribution of \( U_1 \), we may take

\[ a_n = (C_\alpha n)^{\frac{1}{\alpha}} \quad \text{and} \quad \bar{a}_n = C_\alpha^{\frac{2}{\alpha}} (n \log n)^{\frac{1}{\alpha}}. \]

Note that \( \left( \frac{n \log n}{n} \right)^{1/\alpha} = \bar{a}_n^{-1} a_n \).

The following lemma, which is a special case of Theorem 3.3 in [12], will be crucial in studying the asymptotic properties of \( \Phi_1(n) \) and \( \Phi_2(n) \):

**Lemma 3.6.** Let \( \{U_i\}_{i=0}^\infty \) be i.i.d. with the same stable distribution \( S_\alpha(1, 0, 0) \). Then, for \( a_n \) and \( \bar{a}_n \) defined as above, we have for \( m \in \mathbb{N} \)

\[ \left( a_n^{-2} \sum_{i=1}^{n} U_i^2, \bar{a}_n^{-1} \sum_{i=1}^{n} U_i U_{i+1}, \ldots, \bar{a}_n^{-1} \sum_{i=1}^{n} U_i U_{i+m} \right) \Rightarrow (Y_0, Y_1, \ldots, Y_m), \] (3.6)

where \( Y_0, Y_1, \ldots, Y_m \) are independent stable random variables, \( Y_0 \) is positive \( \alpha/2 \)-stable with distribution \( S_{\alpha/2}(\sigma_1, 1, 0) \), and \( Y_1, \ldots, Y_m \) are i.i.d. symmetric \( \alpha \)-stable with distribution \( S_{\alpha}(\sigma_2, 0, 0) \).

**Remark 3.7.** In [12], the precise values of \( \sigma_1 \) and \( \sigma_2 \) are not provided explicitly. However, it is not hard to determine their values as given in Theorem 3.1 (see [14] as well). Why is the value of \( \mu \) (location parameter) equal to zero in the distribution of \( Y_0 \)? From the proof of the stable law (see [22]), we know that

\[ \bar{a}_n^{-1} \left( \sum_{i=1}^{n} U_i^2 - \tilde{b}_n \right) \Rightarrow \bar{Y}_0, \]

where \( \bar{a}_n = a_n^2 \) and \( \tilde{b}_n = n \mathbb{E}[U_1^2 1_{(U_1^2 \leq \bar{a}_n)}] = n \mathbb{E}[U_1^2 1_{(|U_1| \leq a_n)}] \), and \( \bar{Y}_0 \) has a non-degenerate stable distribution \( S_{\alpha/2}(\sigma_1, 1, \mu_1) \) with \( \mu_1 = -\alpha/(2 - \alpha) \). By Karamata’s theorem (see [23], Theorem 9.2 in Chapter VIII and Problem 9.30 in the same chapter), it follows that

\[ \tilde{b}_n/\bar{a}_n = na_n^{-2} \mathbb{E}[U_1^2 1_{(|U_1| \leq a_n)}] \to \alpha/(2 - \alpha) \]
as \( n \to \infty \). Therefore, we can conclude that
\[
a_n^{-2} \sum_{i=1}^{n} U_i^2 = \tilde{a}_n^{-1} \left( \sum_{i=1}^{n} U_i^2 - \tilde{b}_n \right) + \tilde{b}_n / \tilde{a}_n \Rightarrow \overline{Y}_0 + \alpha / (2 - \alpha) \sim S_{\alpha/2}(\sigma_1, 1, 0).
\]
All these clarifications are important for the further development of the asymptotic theory for LSE in the stable setting.

We first deal with the asymptotic behavior of \( \Phi_2(n) \). We have the following result:

**Proposition 3.8.** Assume that (A1) is satisfied. Then, we have

\[
\Phi_2(n) = n^{-2/\alpha} h^{1-2/\alpha} \gamma_h^2 \sum_{i=1}^{n-1} \sum_{j=1}^{i} c_{i,h}^2 U_{i-j}^2 \to 0.
\]  
(3.7)

**Proof.** We have
\[
\Phi_2(n) = n^{-2/\alpha} h^{1-2/\alpha} \sum_{i=1}^{n} X_i^2
\]
\[
= n^{-2/\alpha} h^{1-2/\alpha} X_0^2 + n^{-2/\alpha} h^{1-2/\alpha} \sum_{i=1}^{n-1} X_i^2
\]
\[
:= \Phi_{2,1}(n) + \Phi_{2,2}(n).
\]  
(3.8)

It is clear that \( \Phi_{2,1}(n) \to 0 \) in probability as \( n \to \infty \) under (A1). For \( \Phi_{2,2}(n) \), by the expression of \( X_i \) in (3.5), we find
\[
\Phi_{2,2}(n) = n^{-2/\alpha} h^{1-2/\alpha} \sum_{i=1}^{n-1} \left[ c_{i,h} X_0 + \gamma_h \sum_{j=1}^{i} c_{j,h} U_{i-j} \right]^2
\]
\[
= n^{-2/\alpha} h^{1-2/\alpha} \sum_{i=1}^{n-1} \left[ c_{i,h}^2 X_0^2 + 2 \gamma_h c_{i,h} X_0 \sum_{j=1}^{i} c_{j,h} U_{i-j} + \gamma_h^2 \left( \sum_{j=1}^{i} c_{j,h} U_{i-j} \right)^2 \right]
\]
\[
= n^{-2/\alpha} h^{1-2/\alpha} \sum_{i=1}^{n-1} c_{i,h}^2 X_0^2 + 2 n^{-2/\alpha} h^{1-2/\alpha} \gamma_h \sum_{i=1}^{n-1} c_{i,h} X_0 \sum_{j=1}^{i} c_{j,h} U_{i-j}
\]
\[
+ n^{-2/\alpha} h^{1-2/\alpha} \gamma_h \sum_{i=1}^{n-1} \sum_{j=1}^{i} c_{j,h}^2 U_{i-j}^2
\]
\[
+ n^{-2/\alpha} h^{1-2/\alpha} \gamma_h \sum_{i=1}^{n-1} \sum_{j=1}^{i} \sum_{k=1}^{i} c_{j,h} c_{k,h} U_{i-j} U_{i-k}
\]  
(3.9)

Hence,
\[
\Phi_{2,2}(n) \to n^{-2/\alpha} h^{1-2/\alpha} \sum_{i=1}^{n-1} \sum_{j=1}^{i} c_{j,h}^2 U_{i-j}^2
\]
\[
= n^{-2/\alpha} h^{1-2/\alpha} \sum_{i=1}^{n-1} c_{i,h}^2 X_0^2 + 2 n^{-2/\alpha} h^{1-2/\alpha} \gamma_h \sum_{i=1}^{n-1} c_{i,h} X_0 \sum_{j=1}^{i} c_{j,h} U_{i-j}
\]
\[ + n^{-2/\alpha} h^{1-2/\alpha} \gamma_h^2 \sum_{i}^{n-1} \sum_{j=1}^{i} \sum_{k=1, k \neq j}^{i} c_{j,h} c_{k,h} U_{i-j}U_{i-k} \]

\[ := \Sigma_1 + \Sigma_2 + \Sigma_3. \]  

(3.10)

For \( \Sigma_1 \), it is easy to see that

\[ \Sigma_1 = n^{-2/\alpha} h^{1-2/\alpha} \sum_{i=1}^{n} \exp^{-2\theta_0 i h} X_0^2 \leq \theta_0^{-1}(nh)^{-2/\alpha} X_0^2, \]

(3.11)

which converges to zero in probability as \( n \to \infty \). For \( \Sigma_2 \), by the Markov inequality, we find for any given \( \varepsilon > 0 \)

\[ P(|\Sigma_2| > \varepsilon) \leq \varepsilon^{-1} \mathbb{E} \left[ 2n^{-2/\alpha} h^{1-2/\alpha} \gamma_h \sum_{i=1}^{n} \sum_{j=1}^{i} c_{i,h} c_{j,h} X_0 \right] \leq 2\varepsilon^{-1} n^{-2/\alpha} h^{1-2/\alpha} \gamma_h \sum_{i=1}^{n} \sum_{j=1}^{i} c_{i,h} c_{j,h} \mathbb{E}[X_0] \cdot \mathbb{E}[U_{i-j}] \leq C\varepsilon^{-1} n^{-2/\alpha} h^{1-2/\alpha} \gamma_h(1 - \varepsilon^{\theta_0 h})^{-2} \leq C\varepsilon^{-1} \left( nh^{1+\alpha} \right)^{-2/\alpha}, \]

(3.12)

which tends to zero as \( n \to \infty \) under (A1). We are going to use some techniques in [12] to show that \( \Sigma_3 \) converges to zero in probability. By truncation technique, for any given \( \varepsilon > 0 \), we have

\[ P \left( n^{-2/\alpha} h^{1-2/\alpha} \gamma_h^2 \sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1, k \neq j}^{i} c_{j,h} c_{k,h} U_{i-j}U_{i-k} > \varepsilon \right) \leq P \left( n^{-2/\alpha} h^{1-2/\alpha} \gamma_h^2 \sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1, k \neq j}^{i} c_{j,h} c_{k,h} U_{i-j}U_{i-k} 1(|U_{i-j}U_{i-k}| \leq \tilde{a}_n) > \varepsilon/2 \right) \]

\[ + P \left( n^{-2/\alpha} h^{1-2/\alpha} \gamma_h^2 \sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1, k \neq j}^{i} c_{j,h} c_{k,h} U_{i-j}U_{i-k} 1(|U_{i-j}U_{i-k}| > \tilde{a}_n) > \varepsilon/2 \right) \]

\[ := B_1 + B_2. \]  

(3.13)

Note that \( h^{-2/\alpha} \gamma_h^2 = O(1) \) which can always be dominated by some universal constant \( C \). By Chebyshev’s inequality, we find

\[ B_1 \leq C \left( \frac{\varepsilon}{2} \right)^{-2} n^{-4/\alpha} h^2 \mathbb{E} \left[ \sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1, k \neq j}^{i} c_{j,h} c_{k,h} U_{i-j}U_{i-k} 1(|U_{i-j}U_{i-k}| \leq \tilde{a}_n) \right]^2 \]

\[ \leq C4\varepsilon^{-2} n^{-4/\alpha} h^2 \sum_{i=1}^{n-1} \sum_{j=1}^{i} \sum_{k=1, k \neq j}^{i} c_{j,h} c_{k,h} c_{j',h} c_{k',h} \sum_{j'=1}^{j} \sum_{j'=1}^{j} c_{j,h} c_{k,h} c_{j',h} c_{k',h} \mathbb{E} \left[ U_{i-j}U_{i-k} 1(|U_{i-j}U_{i-k}| \leq \tilde{a}_n) U_{r-j'}U_{r-k'} 1(|U_{r-j'}U_{r-k'}| \leq \tilde{a}_n) \right]. \]

(3.14)

We consider the expectation of the right hand side of (3.14) according to two different sets of indices: (i) all the indices \( i-j, i-k, r-j', r-k' \) are different; (ii) one of the two indices \( i-j, i-k \) is equal to one of the two indices \( r-j', r-k' \). In case (i), it is easy to see that the
expectation for these terms is equal to zero. Next, we deal with the expectation in case (ii). For convenience, we put \( \sigma_n^2 = \mathbb{E}\left[|U_1 U_2|^2 1_{(|U_1 U_2| > \tilde{a}_n)}\right] \). We just calculate the expectation for one of the four sub-cases, namely, \( i - j = r - j' \), or equivalently \( j' = r - i + j \) (the expectations for all other sub-cases are the same):

\[
B_1 \leq 16C \varepsilon^{-2} n^{-4/\alpha} h^4 \left( \sum_{j=1}^{n-1} c_{j,h}^2 \right) \cdot n \sigma_n^2 \leq 16C \varepsilon^{-2} n^{-4/\alpha} h^4 \left( \frac{1}{1 - e^{-\theta_0 h}} \right)^4 \cdot n \sigma_n^2
\]

By Karamata’s theorem, \( \tilde{a}_n^{-2} n \sigma_n^2 \to \alpha/(2 - \alpha) \). Thus, it follows that

\[
B_1 \leq C' \varepsilon^{-2} n^{-4/\alpha} h^2 \tilde{a}_n^{-2} n \sigma_n^2,
\]

which tends to zero as \( n \to \infty \) under (A1).

Next, we turn to \( B_2 \). By the Markov inequality, we have

\[
B_2 \leq 2 \varepsilon^{-1} n^{-2/\alpha} h^{1-\alpha/2} \gamma_h^2 \mathbb{E} \left[ \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} c_{j,h} c_{k,h} U_{i-j} U_{i-k} 1_{(|U_{i-j} U_{i-k}| > \tilde{a}_n)} \right]
\]

\[
\leq 2 \varepsilon^{-1} n^{-2/\alpha} h^{1-\alpha/2} \gamma_h^2 \sum_{j=1}^{n-1} \sum_{j=1}^{n-1} \sum_{j=1}^{n-1} c_{j,h} c_{k,h} \mathbb{E}\left[ |U_1 U_2|^2 1_{(|U_1 U_2| > \tilde{a}_n)} \right]
\]

\[
\leq 2 \varepsilon^{-1} n^{-2/\alpha} h^{1-\alpha/2} \gamma_h^2 (1 - e^{-\theta_0 h})^{-2} n \mathbb{E}\left[ |U_1 U_2|^2 1_{(|U_1 U_2| > \tilde{a}_n)} \right]
\]

\[
\leq C \varepsilon^{-1} \left( \frac{\log n}{n h^\alpha} \right)^{1/\alpha} \tilde{a}_n^{-1} \mathbb{E}\left[ |U_1 U_2|^2 1_{(|U_1 U_2| > \tilde{a}_n)} \right],
\]

which converges to zero as \( n \to \infty \) under condition (A1), since \( n \tilde{a}_n^{-1} \mathbb{E}\left[ |U_1 U_2|^2 1_{(|U_1 U_2| > \tilde{a}_n)} \right] \) converges to \( \alpha/\alpha - 1 \) by Karamata’s theorem. Thus, the proof is complete.

**Proposition 3.9.** If condition (A1) is satisfied, then we have

\[
n^{-2/\alpha} h^{(\alpha-2)/\alpha} \gamma_h^2 \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} c_{j,h}^2 U_{i-j}^2 \Rightarrow \frac{C_{\alpha}^2 Y_0}{2 \theta_0},
\]

where \( Y_0 \) is a random variable with positively skewed stable distribution \( S_{\alpha/2}(\sigma_1, 1, 0) \) as specified in Theorem 3.1.

**Proof.** By interchanging the order of summation and regrouping terms, we find

\[
n^{-2/\alpha} h^{(\alpha-2)/\alpha} \gamma_h^2 \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} c_{j,h}^2 U_{i-j}^2 = C_{\alpha}^2 \tilde{a}_n^{-2} h^{(\alpha-2)/\alpha} \gamma_h^2 \sum_{j=1}^{n-1} \left( \sum_{k=0}^{n-1} U_k^2 - \sum_{k=n-j}^{n-1} U_k^2 \right)
\]
Lemma 3.6 and the fact that

By some basic calculations, we find

\[
\Phi_1(n) = \left| \frac{C^{2/\alpha}a_n^{-2}h^{(\alpha-2)/\alpha}}{\gamma_h} \sum_{j=1}^{n-1} c_{j,h}^{2} \sum_{k=0}^{n-1} U_k^2 - C^{2/\alpha}a_n^{-2}h^{(\alpha-2)/\alpha} \sum_{j=1}^{n-1} c_{j,h}^{2} \sum_{k=n-j}^{n-1} U_k^2 \right|
\]

\[:= D_1 - D_2. \tag{3.19}\]

We first show that \(D_2 \to 0\) in probability. By the Markov inequality, we have for \(\delta = \frac{2\alpha}{2+\rho} < \alpha\) with \(\rho > 0\) (\(\delta/2 < \alpha/2 < 1\))

\[
P(\|D_2\| > \epsilon) \leq e^{-\delta/2} \left| \frac{n^{-2/\alpha}h^{(\alpha-2)/\alpha}}{\gamma_h} \sum_{j=1}^{n-1} \sum_{k=n-j}^{n-1} U_k^2 \right| \delta/2
\]

\[\leq C e^{-\delta/2} n^{-\delta/\alpha} h^{\delta/2} \sum_{j=1}^{n-1} e^{-\delta \theta_0 j} h \sum_{k=n-j}^{n-1} \mathbb{E}[|U_k|^\delta]
\]

\[\leq C e^{-\delta/2} n^{-\delta/\alpha} h^{\delta/2} \sum_{j=1}^{n-1} \delta^{-\delta} \sum_{j=1}^{n-1} e^{-\delta \theta_0 j} h
\]

\[\leq C e^{-\delta/2} \frac{(\delta \theta_0)^{-1} [nh^{-\alpha/2+\rho/2}]}{-\delta/\alpha} + C e^{-\delta/2} \frac{\theta_0^{-2} e^{2 [nh^{-\alpha/2+\rho}]^{-\delta/\alpha}}}{\delta/\alpha} \to 0. \tag{3.20}\]

as \(n \to \infty\) under (A1). By Lemma 3.6 and the fact that \(h^{-2/\alpha} \gamma_h^2 \sum_{j=1}^{n-1} c_{j,h}^{2} \to \frac{1}{2\theta_0}\), we have

\[D_1 = C^{2/\alpha} h^{-2/\alpha} \frac{\gamma_h^2}{\gamma_h} \sum_{j=1}^{n-1} c_{j,h}^{2} a_n^{-2} \sum_{k=0}^{n-1} U_k^2 \Rightarrow \frac{C^{2/\alpha} Y_0}{2\theta_0}. \tag{3.21}\]

This completes the proof. \(\square\)

Now, we turn to study the asymptotic behavior of \(\Phi_1(n)\). We first have the following proposition.

**Proposition 3.10.** Suppose that condition (A1) is satisfied. Then, we have

\[
\left| \Phi_1(n) - e^{-\theta_0 h} (n \log n)^{-1/\alpha} h^{-1/\alpha} \gamma_h^2 \sum_{k=1}^{n-1} c_{k,h} \sum_{l=0}^{n-1-k} U_l U_{l+k} \right| \to 0. \tag{3.22}\]

**Proof.** By some basic calculations, we find

\[
\Phi_1(n) = (n \log n)^{-1/\alpha} h^{-1/\alpha} \sum_{i=1}^{n} X_{i-1} \cdot \left. \int_{t_{i-1}}^{t_i} e^{-\theta_0(t_i-s)} dZ_s \right| 
\]

\[= (n \log n)^{-1/\alpha} h^{-1/\alpha} \sum_{i=1}^{n} X_{i-1} \cdot e^{-\theta_0 h} V_{i-1}
\]

\[= (n \log n)^{-1/\alpha} h^{-1/\alpha} \sum_{i=1}^{n} X_{i-1} \cdot e^{-\theta_0 h} \gamma_h U_{i-1}
\]

\[= e^{-\theta_0 h} (n \log n)^{-1/\alpha} h^{-1/\alpha} \gamma_h \sum_{i=0}^{n-1} c_{i,h} X_0 U_i
\]
By the Markov inequality, for any given $\varepsilon > 0$, we have

$$P\left(\left|e^{-\theta_0 h}(n \log n)^{-1/\alpha} h^{-1/\alpha} y_h \sum_{j=1}^{n-1} c_{j,h} \sum_{l=0}^{n-1-j} U_l U_{l+j}\right| > \varepsilon\right) \leq e^{-\varepsilon^{-1} \log n} \leq e^{-\varepsilon^{-1} (n h^{\alpha})^{-1/\alpha},}

which tends to zero as $n \to \infty$ under condition (A1). This completes the proof. \hfill \Box

Next, we shall deal with the convergence of

$$e^{-\theta_0 h}(n \log n)^{-1/\alpha} h^{-1/\alpha} y_h \sum_{j=1}^{n-1} c_{j,h} \sum_{l=0}^{n-1-j} U_l U_{l+j} := F_1(n).$$

We rewrite $F_1(n)$ as follows:

$$F_1(n) = e^{-\theta_0 h}(n \log n)^{-1/\alpha} h^{-1/\alpha} y_h \sum_{j=1}^{n-1} c_{j,h} \left[ \sum_{l=0}^{n-1} U_l U_{l+j} - \sum_{l=n-j}^{n-1} U_l U_{l+j} \right]$$

$$= e^{-\theta_0 h}(n \log n)^{-1/\alpha} h^{-1/\alpha} y_h \sum_{j=1}^{n-1} c_{j,h} \sum_{l=0}^{n-1-j} U_l U_{l+j}$$

$$- e^{-\theta_0 h}(n \log n)^{-1/\alpha} h^{-1/\alpha} y_h \sum_{j=1}^{n-1} c_{j,h} \sum_{l=n-j}^{n-1} U_l U_{l+j}$$

$$:= F_{1,1}(n) - F_{1,2}(n).$$

**Proposition 3.11.** Under condition (A1), we have

$$F_{1,2}(n) \to 0 \quad \text{as} \quad n \to \infty.$$

**Proof.** By the Markov inequality, for any given $\varepsilon > 0$, we have

$$P(|F_{1,2}(n)| > \varepsilon) \leq e^{-\varepsilon^{-1} e^{-\theta_0 h}(n \log n)^{-1/\alpha} h^{-1/\alpha} y_h \sum_{j=1}^{n-1} c_{j,h} \sum_{l=n-j}^{n-1} U_l U_{l+j}|.$$
Let $\theta > 0$ be given. By the Markov inequality, for a random variable with stable distribution $S$, it follows that

$$
\Pr[|G_2(n)| > \varepsilon] \leq \frac{\varepsilon}{\alpha} \left( n \log n \right)^{-1/\alpha} h^{-1/\alpha} \gamma_h^2 \sum_{j=1}^{n-1} c_{j,h} \sum_{l=0}^{n-1} \mathbb{E}|U_l| \mathbb{E}|U_{l+j}|
$$

which tends to zero as $n \to \infty$ under condition (A1). \qed

Now, we turn to consider the asymptotic behavior of $F_{1,1}(n)$. For convenience, we rewrite $F_{1,1}(n)$ as

$$
F_{1,1}(n) = e^{-\theta h} (n \log n)^{-1/\alpha} h^{-1/\alpha} \gamma_h^2 \sum_{j=1}^{n-1} c_{j,h} \sum_{l=0}^{n-1} U_l U_{l+j}
$$

$$
= e^{-\theta h} (n \log n)^{-1/\alpha} h^{-1/\alpha} \gamma_h^2 \sum_{j=1}^{\infty} c_{j,h} \sum_{l=0}^{n-1} U_l U_{l+j}
$$

$$
- e^{-\theta h} (n \log n)^{-1/\alpha} h^{-1/\alpha} \gamma_h^2 \sum_{j=n}^{\infty} c_{j,h} \sum_{l=0}^{n-1} U_l U_{l+j}
$$

$$
:= G_1(n) - G_2(n).
$$

**Proposition 3.12.** Under condition (A1), we have $G_2(n) \to 0$ in probability as $n \to \infty$.

**Proof.** By the Markov inequality, for given $\varepsilon > 0$, we have

$$
\Pr[|G_2(n)| > \varepsilon] \leq \varepsilon \left( n \log n \right)^{-1/\alpha} h^{-1/\alpha} \gamma_h^2 \sum_{j=n}^{\infty} c_{j,h} \sum_{l=0}^{n-1} \mathbb{E}|U_l| \mathbb{E}|U_{l+j}|
$$

$$
\leq C \varepsilon \left( n \log n \right)^{-1/\alpha} h^{-1/\alpha} \gamma_h^2 \sum_{j=1}^{\infty} c_{j,h} \sum_{l=0}^{n-1} U_l U_{l+j}
$$

$$
\leq C \left( \theta_0 \varepsilon \right)^{-1} n h^{-1} \log n \left( n h \right)^{-1/\alpha} + C \theta_0^{-2} \varepsilon^{-1} (n h^2)^{-1/\alpha},
$$

(3.28)

which tends to zero as $n \to \infty$ under condition (A1). \qed

**Proposition 3.13.** Assume that (A1) is satisfied. Then, $G_1(n) \Rightarrow C_2^{2/\alpha} Y$ as $n \to \infty$, where $Y$ is a random variable with stable distribution $S_\alpha((\alpha \theta_0)^{-1/\alpha} \sigma_2, 0, 0)$.

**Proof.** Let $H_{n,j} = \tilde{a}_n^{-1} \sum_{l=0}^{n-1} U_l U_{l+j}$, $\forall j \in N$. Then, we have

$$
G_1(n) = e^{-\theta h} (n \log n)^{-1/\alpha} h^{-1/\alpha} \gamma_h^2 \sum_{j=1}^{n-1} c_{j,h} \sum_{l=0}^{n-1} U_l U_{l+j}
$$

$$
= e^{-\theta h} C_\alpha^{2/\alpha} \tilde{a}_n^{-1} h^{-1/\alpha} \gamma_h^2 \sum_{j=1}^{\infty} c_{j,h} \sum_{l=0}^{n-1} U_l U_{l+j}
$$

$$
= C_\alpha^{2/\alpha} e^{-\theta h} h^{-2/\alpha} \gamma_h^2 \sum_{j=1}^{\infty} h^1 c_{j,h} H_{n,j}.
$$

(3.31)
Note that the asymptotic distribution of $G_1(n)$ is completely determined by that of $\sum_{j=1}^{\infty} h^{1/\alpha} c_{j,h} H_{n,j}$. For fixed $m \in \mathbb{N}$, we consider the partial sum $\sum_{j=1}^{m} h^{1/\alpha} c_{j,h} H_{n,j}$. By Lemma 3.6, we know that 

$$(H_{n,1}, H_{n,2}, \ldots, H_{n,m}) \Rightarrow (Y_1, Y_2, \ldots, Y_m).$$

By Skorohod’s representation theorem, there exist two sequences of random variables $\{I_{n,j}\}_{j=1}^{m}$ and $\{S_j\}_{j=1}^{m}$ defined on some new probability space $(\Omega, \mathcal{F}, P)$ such that 

$$\mathcal{L}(H_{n,1}, \ldots, H_{n,m}) = \mathcal{L}(I_{n,1}, \ldots, I_{n,m}),$$

$$\mathcal{L}(S_1, \ldots, S_m) = \mathcal{L}(Y_1, \ldots, Y_m),$$

and $I_{n,j} \rightarrow S_j, P$-a.s., $j = 1, \ldots, m$. (Here $m$ can be $\infty$.)

**Claim 1:**

$$\sum_{j=1}^{\infty} h^{1/\alpha} c_{j,h} I_{n,j} - \sum_{j=1}^{\infty} h^{1/\alpha} c_{j,h} S_j \rightarrow \mathcal{P} 0 \quad \text{as } n \rightarrow \infty.$$

**Proof.** Note that $\sup_{j \in \mathbb{N}} (h^{1/\alpha} c_{j,h}) < 1$. Then, we have

$$\mathcal{P} \left( \left| \sum_{j=1}^{\infty} h^{1/\alpha} c_{j,h} (I_{n,j} - S_j) \right| > \varepsilon \right) \leq \mathcal{P} \left( \sum_{j=1}^{\infty} h^{1/\alpha} c_{j,h} |I_{n,j} - S_j| > \varepsilon \right) \leq \sum_{j=1}^{\infty} \mathcal{P} \left( |I_{n,j} - S_j| > 2^{-j} \varepsilon \right). \quad (3.32)$$

Letting $n \rightarrow \infty$, we find

$$\lim_{n \rightarrow \infty} \mathcal{P} \left( \left| \sum_{j=1}^{\infty} h^{1/\alpha} c_{j,h} (I_{n,j} - S_j) \right| > \varepsilon \right) = 0. \quad (3.33)$$

So, Claim 1 is true.

**Claim 2:**

$$\sum_{j=1}^{\infty} h^{1/\alpha} c_{j,h} S_j \Rightarrow Y,$$

where $Y$ has a stable distribution $S_\alpha((\alpha \theta_0)^{-1/\alpha} \sigma_2, 0, 0)$.

**Proof.** Since $\{S_j\}_{j=1}^{\infty}$ is a sequence of independent random variables with the same distribution $S_\alpha(\sigma_2, 0, 0)$, it follows that

$$\sum_{j=1}^{\infty} h^{1/\alpha} c_{j,h} S_j \sim S_\alpha \left( \left( \sum_{j=1}^{\infty} e^{-\alpha \theta_0 j h} \right)^{1/\alpha} \sigma_2, 0, 0 \right) \Rightarrow S_\alpha((\alpha \theta_0)^{-1/\alpha} \sigma_2, 0, 0).$$

By Theorem 25.4 of [24] and using Claims 1 and 2, we find that

$$\sum_{j=1}^{\infty} h^{1/\alpha} c_{j,h} I_{n,j} \Rightarrow Y.$$
and consequently
\[ \sum_{j=1}^{\infty} h^{1/\alpha} c_{j,h} H_{n,j} \Rightarrow Y', \]
where \( Y' \) is a random variable with the same distribution as \( Y \) and \( Y' \) is independent of \( Y_0 \). Hence, 
\[ G_1(n) \Rightarrow C_2^{2/\alpha} Y'. \]
□

Finally, we are in a position to prove Theorem 3.1.

**Proof of Theorem 3.1.** By Lemma 3.4 and Propositions 3.8–3.13, we conclude that
\[
\left( \frac{n}{\log n} \right)^{1/\alpha} h^{1/\alpha} (\hat{\theta}_n - \theta_0) = \Lambda_n - \frac{\Phi_1(n)}{\Phi_2(n)} \Rightarrow -\frac{C_2^{2/\alpha} Y'}{C_2^{2/\alpha} (2\theta_0)^{-1} Y_0} \sim \frac{2\theta_0 (\alpha \theta_0)^{-1/\alpha}}{Y_0}, \tag{3.34}
\]
where \( \tilde{Y} \) has a distribution \( S_\alpha(\sigma_2, 0, 0) \) independent of \( Y_0 \). This completes the proof. □

4. Simulation

We have applied our estimator to the generalized Ornstein–Uhlenbeck process determined by the following stochastic differential equation:
\[
dX_t = -\theta_0 X_t \, dt + dZ_t, \quad X_0 = 1,
\]
where \( \theta_0 = 2 \) and \( Z_t \) is a stable process with index \( \alpha = 1.8 \). We simulate the process on the interval \([0, T]\) with \( T = 200 \).

We plot \( \hat{\theta}_T = \hat{\theta}_n \) (where \( T = nh \)) as a function of \( T \) for \( h = 0.05 \) (Fig. 1a) and \( h = 0.01 \) (Fig. 1b).

For a comparison, the following table describes \( \hat{\theta}(25), \ldots, \hat{\theta}(200) \) for different choices of \( h \). We see from the table that if we fix an \( h \) the estimator may not converge. But if we let both \( h \) converge to 0 and \( T \) converge to \( \infty \), the estimator may converge (Table 1).

We see that we need to let both \( T \) go to infinity and \( h \) go to 0 to have the convergence of \( \hat{\theta}_T \) to \( \theta_0 \).
Table 1

Numerical values of the estimator \( \hat{\theta}_n \).

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Acknowledgments

The authors are grateful to the referee and the associate editor for very helpful comments. The authors also thank Qinghua Zhang (University of Kansas) for the simulations.

The corresponding author is supported by the National Science Foundation under Grant no. DMS0504783. Long is supported by FAU Start-up funding at the C. E. Schmidt College of Science.

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