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Numerical approximations of optimal portfolios in mispriced asymmetric Lévy markets

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We present numerical approximations of optimal portfolios in mispriced Lévy markets under asymmetric information for informed and uninformed investors having logarithmic preference. We apply our numerical scheme to Kou (2002) jump-diffusion markets by deriving analytic formulas for the first two derivatives of the underlying portfolio objective function which depend only on the Lévy measure of the jump-generating process. Optimal portfolios are then simulated using the Box–Muller algorithm, Newton’s method and incomplete Beta functions. Convergence dynamics and trajectories of sample paths of optimal portfolios for both investors are presented at different levels of information asymmetry, mispricing, horizon, asymmetry in the Kou density, jump intensity, volatility, mean-reversion speed, and Sharpe ratios. We also apply the proposed Newton’s algorithm to compute optimal portfolios for investors in Variance Gamma markets via instantaneous centralized moments of returns.

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1. Introduction

Asymmetric information models assume that there are two types of investors in the market—informed and uninformed. The informed investor trades because he or she has non-public informational advantage, while the uninformed is a liquidity trader or hedger without an information advantage. The informed investor partially reveals information to the uninformed investor through trades. Kelly and Ljungqvist (2012) and Easley and O’Hara (2004) provide evidence of the importance of information asymmetry in asset pricing. They find that prices and demand for the risky asset by uninformed investors fall as information asymmetry increases. Their results confirm that information asymmetry is priced, and imply that liquidity is a primary channel that links asymmetry to prices.

Mispricing1 is the difference between the asset price and its fundamental value. It is modeled by a mean-reverting Ornstein–Uhlenbeck process. Because of the exponential decay associated with mean-reverting processes, it has an associated half-life which is inversely proportional to its mean-reversion rate. Half-life is a measure of the slowness of a mean-reverting process, and is measured in trading days. It may be defined as the average time it takes the mean-reverting process to get pulled half-way back to its long-term mean. In this context, half-life gives an estimate of how long we should expect the mispricing to remain far from zero. Thus, a half-life of 10 days means that it takes 20 trading days on average for the mispricing to revert to zero. We define mean-reversion time as the inverse of the mean-reversion rate, and use it as an equivalent measure of mispricing. In other words, mean-reversion speed or mean-reversion time is a proxy for mispricing. In this paper, we combine the concepts of asymmetric information, mispricing, and mean-reversion, and apply them to the Kou (2002) jump-diffusion and Variance Gamma (VG) markets to study the impact on the optimal demand for the risky asset by investors having logarithmic preference.

Portfolio allocation problems2 have been extensively studied in various settings since the seminal work of Markowitz (1952).


Recently, Buckley et al. (2014) show that optimal portfolios satisfy some nonlinear stochastic equations for both informed and uninformed investors in the asymmetric and mispriced Lévy markets. Usually there are no analytic solutions to these nonlinear equations. Numerical and analytic approximations are therefore critical for portfolio allocation problems. We solve our problem by presenting a simple but highly efficient approximation to our asset allocation problem in the asymmetric and mispriced Kou and VG market setting. In particular, we apply the theory presented in the recent studies by Buckley et al. (2012), Buckley et al. (2014) and Buckley et al. (2015) to compute numerical approximations of the optimal portfolios with the aid of Newton’s method, the Box–Muller algorithm, and the incomplete Beta functions.

Kou (2002) jump-diffusion model deals with a finite activity process in the sense that only a small number of jumps occur per unit of time. Empirical tests in Ramenazi and Zeng (2007) demonstrate that Kou’s double exponential jump-diffusion model fits stock data better than Merton’s normal jump-diffusion model. It is also very tractable, and therefore, is a natural candidate to apply our general numerical scheme. For the Kou markets, we derive analytic formulas for the first two derivatives of the underlying portfolio objective function presented in Buckley et al. (2014) which depends only on the Lévy measure of the process generating the jumps. Using the parameters in Kou (2002) and Ramenazi and Zeng (2007), we then implement our model by simulating paths of optimal portfolios for each investor at various levels of mispricing, information asymmetry, investment horizon, volatility, Sharpe ratio, jump probability, and jump size.

The Variance Gamma (VG) model introduced by Madan and Seneta (1990) and Madan, Carr, and Chang (1998) has found extensive applications in modeling stock returns, option pricing and structural credit risk models. It is of infinite activity, that is, there are infinitely many arrivals of small jumps per unit of time. We also employ our theory to the Variance Gamma (VG) Lévy markets to examine how asset mispricing and information asymmetry affect the optimal portfolio and maximum expected utilities of investors in this market. Cvitanić et al. (2008) report that jumps in asset prices lead to higher moments that must be captured when modeling returns and allocating portfolios. Following Cvitanić et al. (2008) and Buckley et al. (2014), we employ instantaneous centralized moments of returns (ICMR) in our analysis of the VG markets and then combine Newton’s method with the ICMR to produce numerical approximations of the optimal portfolios of the investors at different levels of mispricing, asymmetric information, jump intensity, volatility, and horizon.

To the best of our knowledge, this paper is the first to report on the impact of mispricing and asymmetric information in the Kou (2002) and Variance Gamma markets, and therefore, contributes to the literature on mispricing and asymmetric information asset pricing models.

The rest of the paper is organized as follows: the model is briefly reviewed in Section 2. In Section 3, Newton’s method is proposed to give numerical computation of optimal portfolios in the asymmetric Lévy markets. The Kou jump-diffusion market is introduced in Section 4. Newton’s algorithm depends on the first two derivatives of a non-trivial objective function $G(t)$, which is a moment generating function. In this section, we also develop analytic formulas for said derivatives in terms of the cumulative distribution functions of incomplete Beta random variables. The mispriced VG market is introduced in Section 5, where an explicit formula is given for the optimal portfolio. We also report details of the VG market, including ICMR and their effects on mean, variance, skewness and kurtosis of the return process. Concluding remarks are given in Section 6. Numerical results for the Kou and VG markets under various levels of asymmetric information, volatility, mean-reversion speed, Poisson arrival rates and asymmetric distributions are reported in the Appendices.

2. Review of the model and optimal portfolios

In this section, we briefly review the model and the corresponding optimal portfolios in the asymmetric Lévy markets discussed in Buckley et al. (2014). The economy consists of two assets; a risk-free asset $B$, called bank account or money market, with price

$$B_t = \exp \left( \int_0^t r_s \, ds \right),$$

and a risky asset $S$, called stock. The bank account earns a continuously compounded risk-free interest rate $r_t$, while the continuous component of the stock’s total percentage appreciation rate or expected return is $\mu_t$, at time $t \in [0,T]$, where $T$ is the investment horizon. The stock is subject to continuous volatility $\sigma_t > 0$. The market parameters are $\mu_t, r_t, \sigma_t, t \in [0,T]$ and are assumed to be deterministic functions. The stock has log return dynamics:

$$d(\log S_t) = \left( \mu_t - \frac{1}{2} \sigma_t^2 \right) dt + \sigma_t dY_t + dX_t, \quad t \in [0,T].$$

We denote $dY_t = p W_t + q U_t, p^2 + q^2 = 1, \, p \geq 0, \, q \geq 0$. $dU_t = -\lambda(U_t dt + dB_t), \lambda > 0, U_0 = 0$.

$$X_t = \int_0^t \int \mathcal{N}(ds, dx),$$

where $X_t$ is a pure jump Lévy process having a sigma-finite Lévy measure $\nu$ on $\mathcal{B}(\mathbb{R} - \{0\})$, with triple $(\gamma, \rho, \nu)$, where $\gamma = \int_{|x|>1} \nu(dx)$, $W$ and $B$ are independent standard Brownian motions independent of $X$. $N$ is a Poisson random measure on $\mathcal{B}(\mathbb{R} - \{0\})$ that is linked to the stock. It counts the jumps of $X$ in the time interval $(0,t)$. The return of the stock has three components: a continuous component $\mu_t^* = \mu_t - \frac{1}{2} \sigma_t^2$, a diffusive component $\sigma_t dY_t$ which is random, and a discontinuous component $dX_t$, which is also random. The continuous component of the stock’s return $\mu_t^*$ and the volatility $\sigma_t$, are assumed to be deterministic functions with $\sigma = \lim_{t \to \infty} \sigma_t = \sigma_\infty > 0$. We take $\sigma_t$ to be a constant. The process $Y = (Y_t)_{t \geq 0}$ is the continuous random component of excess return, and the mean-reverting O-U process $U_t = \int_0^t e^{-\lambda(t-s)} dB_s$ represents the mispricing/fads. $U_t = (U_t)$ is a mean-reverting Ornstein–Uhlenbeck process with mean-reversion rate $\lambda$. Because of the exponential decay associated with a mean-reverting process, its half-life is $H = \frac{\ln 2}{\lambda}$. Note that we can use half-life $H$, mean-reversion rate $\lambda$ and the mean-reversion time $t = 1/\lambda$ as equivalent measures of mispricing. Large values of the mean-reversion time $t$ correspond to high levels of mispricing. Namely the mispricing process $U$ takes a long time to revert to zero, which indicates that mispricing is persistent.

Applying Itô’s formula to (2.2) yields the percentage return dynamics:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dY_t + \int(e^\epsilon - 1)N(dt, dx), \quad t \in [0,T].$$

and its solution is given by

$$S_t = S_0 \exp \left( \int_0^t (\mu_s - \frac{1}{2} \sigma_s^2) ds + \int_0^t \sigma_s dY_s + X_t \right).$$

We assume that there are two classes of investors, which consist of informed and uninformed investors. The informed investor, indexed by $i = 1$, observes both the asset prices and its fundamental value (through the mispricing process $U$). The information flow
Theorem 1. Let $G(\pi) = \int_0^1 \log(1 + \pi(e^x - 1)dx), \pi \in [0, 1].$ If $\int_0^1 e^{u^2} - 1)\varphi(dx) < \infty$, then

1. $G''(\pi) < 0$, That is, $G$ is strictly concave on $[0, 1].$

2. Let $i \in [0, 1].$ For the $i$th investor, there is a unique optimal portfolio $\pi^{\star i}$ for the stock dynamics (2.2), given by

$$\pi^{\star i} = \frac{\theta_i}{\sigma_i} + \frac{G'(\pi^{\star i})}{\sigma_i^2}, \quad \text{provided} \pi^{\star i} \in [0, 1] \subset A_i(x), \text{the admissible set.} \quad (2.9)$$

3. The maximum expected logarithmic utility from terminal wealth for the $i$th investor, having $x > 0$ of initial capital, is

$$u'(x) = \log x + \frac{1}{2} E \int_0^T (\theta_i^2)^2 dt + E \int_0^T f_i(\pi^{\star i}) dt. \quad (2.10)$$

where $f_i(\pi) = -\frac{1}{2} (\pi_i^2 \sigma_i^2 - \theta_i^2) + G(\pi_i), \quad \mu_i = \mu_t + v_i \sigma_i,$ and $\theta_i = \frac{\mu_i - r_i}{\sigma_i}$ are Sharpe ratios.

Remark 1. When both investors have equal information, the market is symmetric with $q = 0$ and $Y_r$ reduces to a standard Brownian motion. In this case, $\mu_i = \mu_t$ for $i \in [0, 1]$ and the optimal portfolio $\pi^\star_i$ satisfies

$$\pi^\star_i = \frac{\mu_t - r_i + G'(\pi^\star_i)}{\sigma_i^2}. \quad (2.11)$$

Remark 2. The optimal portfolios (please see Theorem 1) are basically fixed points of the first derivative of the function $G(.)$, which is a moment generating function (the function $G(.)$ generates instantaneous centralized moments of returns). Each optimal portfolio is the Merton optimal (for the symmetric market) plus an excess holding of risky asset which is inversely proportional to the diffusive variance (sigma squared) and directly proportional to $G(.)$, the growth rate of $G(.)$ at the risky asset proportion level. The optimal portfolios are therefore very sensitive to information asymmetry, mean reversion speed, volatility, Sharpe ratio and activity of the Lévy measure. For example, as the diffusive volatility increases, the agents basically hold the deterministic Merton optimal. The excess stock holding is positive or negative in lockstep with the sign and magnitude of $G(.)$; that is, whether $G(.)$ is increasing or decreasing at that stock proportion.

3. Computation of optimal portfolio via Newton's method

3.1. Newton’s algorithm in the symmetric market

By Remark 1 in Section 2, the exact unique optimal portfolio in a symmetric market is

$$\pi^\star = \frac{\theta_t}{\sigma_t} + \frac{G'(\pi^\star)}{\sigma_t^2} = \frac{\mu_t - r_t + G'(\pi^\star)}{\sigma_t^2},$$

where $\theta_t = \frac{\mu_t - r_t}{\sigma_t}$ is the Sharpe ratio. This is a non-linear equation which must be solved numerically. For simplicity, we drop the superscript $*$ in the portfolio $\pi^\star$.

In this section, we employ Newton’s method to achieve this objective. For each $t \in [0, T]$, define $g: [0, 1] \rightarrow \mathbb{R}$ by the prescription, $g(\pi_t) = \pi_t - \frac{\mu_t}{\sigma_t} - \frac{G'(\pi_t)}{\sigma_t^2}$. We generate a sequence $\{\pi_t(n)\}$ which converges to $\pi_t$, via the algorithm:

$$\pi_t(0) = \frac{\theta_t}{\sigma_t}, \quad \pi_t(n + 1) = \frac{\pi_t(n) - g(\pi_t(n))}{g'(\pi_t(n))}, \quad n \geq 0. \quad (3.1)$$

In term of the derivatives $G'(\alpha)$ and $G''(\alpha)$, we have the equivalent algorithm:

Set $\epsilon = 0.5 \times 10^{-d}$, where $d \in [5, 6, 7, 8, 9, 10].$

$$\pi_t(0) = \frac{\theta_t}{\sigma_t}, \quad \pi_t(n + 1) = \frac{-\pi_t(n) G''(\pi_t(n)) + \theta_t \sigma_t + G'(\pi_t(n))}{\sigma_t^2 - G''(\pi_t(n))}, \quad n \geq 0. \quad (3.2)$$

$$\epsilon_t(n) = |\pi_t(n + 1) - \pi_t(n)| \quad (3.4)$$

Stop if $\epsilon_t(n) < \epsilon$, and take $\pi_t = \pi_t(n + 1)$, Else. set $n = n + 1$ and repeat search.
3.2. Newton’s algorithms in the asymmetric market

Let \( i \in [0, 1] \). We use Newton’s method to estimate the optimal portfolios \( \pi_i^t \) at each time \( t \in [0, T] \) (again, the superscript \( * \) is dropped). Observe from Eq. (2.9) that the formula for the optimal portfolios involves Gaussian random variables such as

\[
u_i^1 = -\lambda q B_i^1, \quad \text{and} \quad \nu_i^0 = -\lambda B_i^0,
\]

where

\[
B_i^1 = U_t \int_0^t e^{-\lambda(t-s)} dB_s \sim \mathcal{N}\left(0, \frac{1-e^{-2\lambda t}}{2\lambda}\right).
\]

\[
\tilde{B}_i^0 = \int_0^t e^{-\lambda(t-s)} (1 + \gamma(s)) dB^0_s \sim \mathcal{N}\left(0, \int_0^t e^{-2\lambda(t-s)} (1 + \gamma(s))^2 ds\right),
\]

\[
\gamma(s) = \frac{q^2}{1 + p \tanh(\lambda p s)} - 1.
\]

\( B_t \) and \( B_t^0 \) are independent standard Brownian motions, and \( \mathcal{N}(0, \delta^2) \) is a Gaussian distribution with mean 0 and variance \( \delta^2 \). The optimal portfolio of the \( i \)th investor is

\[
\pi_i^t = \frac{\theta_i}{\sigma_i} + \frac{\nu_i^1}{\sigma_i^2} + G(\sigma_i^t)\sigma_i^t,
\]

\( i = 1 \) for the informed investor, and

\[
\pi_i^0 = \frac{\theta_i}{\sigma_i} + \frac{\nu_i^0}{\sigma_i^2} + G(\sigma_i^0)\sigma_i^0,
\]

for the uninformed investor.

Remark 3. Since \( B_i^1 \) and \( B_i^0 \) are integrated Brownian motions (Gaussian random variables) for each \( t \in [0, T] \), we randomly generate these values for use in Newton’s algorithm at each time \( t \).

Consider the normal random variable \( \kappa \sim \mathcal{N}(0, \delta^2) \). Then

\[
\xi_t^i = \frac{\kappa_t}{\delta_t} \sim \mathcal{N}(0, 1).
\]

Based on Eq. (2.9) in Theorem 1, we propose the following:

Algorithm. Let \( i \in [0, 1] \), \( t \in [0, T] \), and \( \theta, q \in [0, 1] \), \( p^2 + q^2 = 1 \).

**Step 1:** Randomly generate \( u_t \), independent uniform variables on \( [0, 1] \).

**Step 2:** Generate independent standard normal random variables \( \xi_0^i \) and \( \xi_1^i \) by the Box-Muller method as follows:

\[
\xi_0^i = \sqrt{-2 \ln u_1} \cos(2\pi u_0),
\]

\[
\xi_1^i = \sqrt{-2 \ln u_1} \sin(2\pi u_0).
\]

**Step 3:** The optimal portfolios for the investors are generated numerically from

\[
\pi_i^1 = \frac{\theta_i}{\sigma_i} - \frac{\lambda q B_i^{1t}}{\sigma_i} + G(\sigma_i^{1t})\sigma_i^{1t},
\]

and for the uninformed investor, it is

\[
\pi_i^0 = \frac{\theta_i}{\sigma_i} - \frac{\lambda B_i^{0t}}{\sigma_i} + G(\sigma_i^{0t})\sigma_i^{0t}.
\]

**Step 4:** Generate \( v_i^0 \) and \( v_i^1 \) recursively.

3.2.1. Generating \( v_i^1 \)

\[
v_i^1 = -\lambda q B_i^{1t},
\]

where

\[
B_i^{1t} = U_t \int_0^t e^{-\lambda(t-s)} dB_s
\]

is distributed \( \sim \mathcal{N}\left(0, \frac{1-e^{-2\lambda t}}{2\lambda}\right) \). Let \( h = \frac{1}{252} \) be the time step. Set

\[
t = \kappa h, \ k = 0, 1, 2, \ldots, N = \frac{T}{h}, \ T = 1.
\]

\[
\tilde{B}_i^{1t}(0) = 0.
\]

\[
\tilde{B}_i^{1t}(t + h) = e^{-\lambda h} \tilde{B}_i^{1t}(t) + \left(\frac{1-e^{-2\lambda h}}{2\lambda}\right) \xi_1^i.
\]

Thus, in terms of \( k \), where \( k = 0, 1, 2, \ldots, N = \frac{T}{h}, \ T = 1 \).

\[
\tilde{B}_i^{1t}(k + h) = e^{-\lambda h} \tilde{B}_i^{1t}(k) + \left(\frac{1-e^{-2\lambda h}}{2\lambda}\right) \xi_1^i.
\]

where \( \xi_1^i \) are drawn from the standard normal variable \( \mathcal{N}(0, 1) \).

3.2.2. Generating \( v_i^0 \)

\[
v_i^0 = -\lambda B_i^{0t},
\]

where

\[
\tilde{B}_i^{0t} = U_t \int_0^t e^{-\lambda(t-s)} (1 + \gamma(s)) dB^0_s
\]

\[
\gamma(s) = \frac{q^2}{1 + p \tanh(\lambda p s)} - 1.
\]

\( \tilde{B}_i^{0t} \) is distributed \( \sim \mathcal{N}\left(0, \int_0^t e^{-2\lambda(t-s)} (1 + \gamma(s))^2 ds\right) \). Set

\[
t = \kappa h, \ k = 0, 1, 2, \ldots, N = \frac{T}{h}, \ T = 1.
\]

\[
\tilde{B}_i^{0t}(0) = 0.
\]

\[
\tilde{B}_i^{0t}(t + h) = e^{-\lambda h} \tilde{B}_i^{0t}(t) + \int_t^{t+h} e^{-\lambda(t+s-h)} (1 + \gamma(s)) dB^0_s
\]

\[
e^{-\lambda h} \tilde{B}_i^{0t}(t) + e^{-\lambda h} (1 + \gamma(t)) \sqrt{h} \xi_0^i
\]

\[
e^{-\lambda h} \tilde{B}_i^{0t}(t) + q^2 \left(\frac{e^{-\lambda h}}{1 + p \tanh(\lambda p t)}\right) \sqrt{h} \xi_0^i.
\]

Thus, in terms of \( k \), where \( k = 0, 1, 2, \ldots, N = \frac{T}{h}, \)

\[
\tilde{B}_i^{0t}(k + h) \approx e^{-\lambda h} \tilde{B}_i^{0t}(k) + e^{-\lambda h} (1 + \gamma(k)) \sqrt{h} \xi_0^i
\]

\[
e^{-\lambda h} \tilde{B}_i^{0t}(k) + q^2 \left(\frac{e^{-\lambda h}}{1 + p \tanh(\lambda p k)}\right) \sqrt{h} \xi_0^i,
\]

where \( \xi_0^i \) are drawn from the standard normal variable \( \mathcal{N}(0, 1) \).

**Step 5:** The optimal portfolios \( \pi_i^1 \) are estimated by \( \pi_i^1(n) \) using Newton’s method as follows:

\[
\pi_i^1(0) = \frac{\theta_i}{\sigma_i} \quad \text{(3.17)}
\]

\[
\pi_i^1(n + 1) = -\pi_i^1(n) G'(\pi_i^1(n)) + \frac{\theta_i}{\sigma_i} + G(\pi_i^1(n)) \quad \text{for } n \geq 0.
\]

\[
\text{(3.18)}
\]
\[ \epsilon_i(n) = |\pi_i(n+1) - \pi_i(n)| \]  
with Sharpe ratios

\[ \theta_i = \frac{\mu_i - r}{\sigma_i}, \]  

\[ \theta_i^0 = \theta_i - \lambda \tilde{B}_i^0, \]  

\[ \theta_i^1 = \theta_i - \lambda \theta_i \tilde{B}_i^1. \]

Stop if \( \epsilon_i(n) < \epsilon \), and take \( \pi_i = \pi_i(n+1) \). Else, set \( n = n + 1 \) and repeat search.

**Remark 4.** In Newton’s algorithm, the computation of the approximate optimal portfolios involves the derivatives \( G(\pi) \) and \( G'(\pi) \). In general, it is very difficult or impossible to derive closed-form analytic formulas for these derivatives in the general Lévy markets. However, in the next section, we show that analytic formulas exist for \( G(\pi) \) and \( G'(\pi) \) in the Kou jump-diffusion market although it requires some non-trivial mathematical derivations. We also note that optimal portfolios bigger than 1 are truncated at 1, while portfolios less than 0 are truncated at 0.

### 4. The Kou jump-diffusion market

It is well-known that asset return distributions have heavier left and right tails than the normal distributions. Jump-diffusion models since the pioneering work of Merton (1976) are among the most popular alternative models proposed to address this issue, and they are especially useful to price options with short maturities. Brownian motion and normal distribution have been widely used in the Black–Scholes–Merton option-pricing framework to model the return of assets. However, two puzzles emerge from many empirical investigations: the leptokurtic feature that the return distribution of assets may have a higher peak and two (asymmetric) heavier tails than those of the normal distribution, and an empirical phenomenon called “volatility smile” in option markets. **Kou (2002)** argues that the existing models either do not explain the observed behavior of stock prices, or are too complex to yield analytical solutions for many American options.

Kou incorporates both of these properties by proposing a double exponential\(^3\) jump-diffusion model which is both tractable and applicable for option pricing. Moreover, the model is simple enough to produce analytical solutions for a variety of option-pricing problems, including European call and put options, interest rate derivatives such as swaptions, caps, floors, and bond options, and path-dependent options such as perpetual American options, barrier, and lookback options.

**Kou (2002)** assumes that the price of the underlying asset is modeled by two parts; a continuous part driven by Brownian motion, and a jump part where the logarithm of the jump sizes have a double exponential distribution. Because of its simplicity, the parameters in the model can be easily interpreted, and the analytical solutions for option pricing can be obtained. **Kou (2002)** is a finite activity process in the sense that only a small number of jumps occur per unit of time. Empirical tests in Ramenanzini and Zeng (2007) demonstrate that Kou’s double exponential jump-diffusion model fits stock data better than Merton’s normal jump-diffusion model. It is also very tractable, and therefore is a natural candidate market for application of the theory in Buckley et al. (2014).

The mispriced Kou jump-diffusion model has stock price dynamics

\[ dS_t = \mu_t dt + \sigma_t dY_t + d(\sum_{i=1}^{N_t} (V_t - 1)). \]

where \( Y \) is given in (2.3) and (2.4), \( N \) is the driving Poisson process with intensity \( \lambda \), and the log jump amplitude \( Z = log(V) \) has double exponential distribution with density \( f = f_{\text{Kou}} \) dependent on 3 parameters \( \tilde{p}, \eta_1 \) and \( \eta_2 \), and given by

\[ f_{\text{Kou}}(x) = \tilde{p} \eta_1 \exp(-\eta_1 x) I_{[x<0]} + \tilde{q} \eta_2 \exp(-\eta_2 |x|) I_{[x>0]}. \]

\( \eta_1 > 0, \eta_2 > 0, \tilde{p} + \tilde{q} = 1, \tilde{p} \geq 0, \tilde{q} \geq 0. \)

When \( \tilde{q} = 0 \) (without mispricing) or \( \lambda = 0 \) (without mean-reversion) in \( Y \), this model reduces to the original model introduced by **Kou (2002)**. \( Z \) can be expressed as:

\[ Z = \begin{cases} Z^e, & \text{with probability } \tilde{p}; \\ Z^d, & \text{with probability } \tilde{q}. \end{cases} \]

where

\[ Z^e \sim \text{Exp}(\eta_1), Z^d \sim \text{Exp}(\eta_2), \]

are exponential random variables with means \( \frac{1}{\eta_1} \) and \( \frac{1}{\eta_2} \), respectively. \( Z^e \), the upward jump log-amplitude, occurs with probability \( \tilde{p} \), which is not expected to exceed 100 percent. This leads to the constraint

\[ E(Z^e) = \frac{1}{\eta_1} < 1. \]

\( Z^d \) is the log-amplitude of the downward movements in returns, which occurs with probability \( \tilde{q} = 1 - \tilde{p} \). For this model, we have:

\[ E(Z) = \frac{\tilde{p}}{\eta_1} - \frac{\tilde{q}}{\eta_2}, \]

\[ \text{Var}(Z) = \tilde{p} \tilde{q} \left( \frac{1}{\eta_1} + \frac{1}{\eta_2} \right)^2 + \left( \frac{\tilde{p}}{\eta_1^2} + \frac{\tilde{q}}{\eta_2^2} \right), \]

\[ E(V) = \frac{\tilde{q} \eta_2}{(\eta_2 + 1)} + \frac{\tilde{p} \eta_1}{(\eta_1 - 1)} \]

is the expected jump amplitude. The Kou Lévy density \( \nu_{\text{Kou}} \) is given by:

\[ \nu_{\text{Kou}}(x) = \tilde{\lambda} f_{\text{Kou}}(x) = \tilde{\lambda} \tilde{p} \eta_1 \exp(-\eta_1 x) I_{[x<0]} + \tilde{\lambda} \tilde{q} \eta_2 \exp(-\eta_2 |x|) I_{[x>0]}. \]

4.1. **Analytic formulas for \( G(\pi) \) and \( G'(\pi) \)**

Let \( G(\pi) = \int_0^\infty \log(1 + \pi (e^{x-1}) v(dx)), \pi \in [0, 1] \), where \( v(dx) \) is the Kou Lévy measure. Then

\[ G(\pi) = \int_{-\infty}^\infty \frac{e^x - 1}{1 + \pi (e^{x-1})} v(dx); \]

\[ G'(\pi) = -\int_{-\infty}^\infty \frac{(e^x - 1)^2}{(1 + \pi (e^{x-1}))^2} v(dx). \]

In general, if there are no analytic formulas available for \( G(\pi) \) and \( G'(\pi) \), we can use standard Riemann sums to approximate these integrals. In this section, we develop analytic formulas for the derivatives of \( G(\pi) \) in terms of the incomplete Beta functions (or equivalently, the cumulative distribution functions of Beta random variables). We show that \( G(\pi) \) and \( G'(\pi) \) can be expressed analytically in terms of incomplete Beta functions. Let \( \alpha \in [0, 1] \) and \( \eta > 0 \). Define

\[ D(\alpha, \eta) \triangleq \int_0^\infty \frac{e^x - 1}{1 + \alpha (e^{x-1})} e^{-\eta x} dx, \]

and \( f(\alpha, \eta) \), by the prescription:

\[ f(\alpha, \eta) \triangleq \int_0^\infty \frac{dy}{(1 + \alpha y)(1+y)^\eta}. \]
We introduce the standard incomplete Beta function as follows:
\[ B(\alpha, b; x) = \int_0^x t^{\alpha-1} (1 - t)^{b-1} dt, \quad a > 0, \quad b > 0, \quad x \in [0, 1]. \]

We use \([\eta]\) to denote the integer part of a positive number \(\eta\).

We have the following preliminary result which will be used to establish analytic formulas for \(G(\pi)\) and \(G'(\pi)\) in the Kou jump-diffusion market.

**Proposition 1.** In the Kou jump-diffusion market, we have

1. \(J(\alpha, \eta + 1) = \frac{1}{\eta(1 - \alpha)} - \frac{\alpha}{1 - \alpha} J(\alpha, \eta)\) for \(\alpha \in [0, 1)\) and \(\eta > 0\).
2. \(J(\alpha, \eta) = \frac{\alpha^{\eta+1}}{(1 - \alpha)\eta} B(\eta, 1 - \eta; 1 - \alpha)\) for \(\eta < 1\) and \(\alpha \in [0, 1]\).
3. For any \(\eta > 0\) and \(\alpha \in [0, 1]\),
   \[ J(\alpha, \eta) = \frac{1}{1 - \alpha} \sum_{i=1}^{\eta-1} (-1)^{i-1} \left( \frac{\alpha}{1 - \alpha} \right)^{i-1} \]
   \[ + (-1)^{\lceil \eta \rceil} \frac{\alpha^{\eta-1}}{(1 - \alpha)\eta} B(\eta - [\eta], [\eta] + 1 - \eta; 1 - \alpha). \]
4. \(D(\alpha, \eta) = \frac{1}{\eta(1 - \alpha)} - \frac{1}{1 - \alpha} J(\alpha, \eta)\) for \(\alpha \in [0, 1)\) and \(\eta > 0\).

**Proof.**

\[ J(\alpha, \eta + 1) = \int_0^\infty \frac{dy}{(1 + \alpha y)(1 + y)^{\eta+1}} \]
\[ = \frac{\alpha}{1 - \alpha} \int_0^\infty \left[ \frac{1}{\alpha(1 + y)} - \frac{1}{(1 + \alpha y)} \right] (1 + y) dy \]
\[ = \frac{\alpha}{1 - \alpha} \int_0^\infty \frac{dy}{1 + \alpha y} \]
\[ = \frac{\alpha}{1 - \alpha} \int_0^\infty \frac{dy}{(1 + y)^{\eta+1}} \]
\[ = \frac{\alpha^{\eta+1}}{(1 - \alpha)\eta} B(\eta, 1 - \eta; 1 - \alpha). \]

(2) Using the change of variable \(1 + \alpha y = \frac{1}{1 - \alpha}\), we find

\[ J(\alpha, \eta) = \int_0^\infty \frac{dy}{(1 + \alpha y)(1 + y)^{\eta}} \]
\[ = \frac{\alpha^{\eta-1}}{(1 - \alpha)^{\eta}} \int_0^{1 - \alpha} t^{\eta-1}(1 - t)^{(1 - \eta) - 1} dt \]
\[ = \frac{\alpha^{\eta-1}}{(1 - \alpha)^{\eta}} B(\eta, 1 - \eta; 1 - \alpha). \]

(3) It is straightforward to use (1) and (2) recursively to get

\[ f(\alpha, \eta) = \frac{1}{(1 - \alpha)(\eta - 1)} - \frac{\alpha}{1 - \alpha} f(\alpha, \eta - 1) \]
\[ = \ldots \]
\[ = \frac{1}{1 - \alpha} \sum_{i=1}^{\lceil \eta \rceil} (-1)^{i-1} \left( \frac{\alpha}{1 - \alpha} \right)^{i-1} \left( \frac{\alpha}{1 - \alpha} \right)^{\lceil \eta \rceil} J(\alpha, \eta - \lceil \eta \rceil) \]
\[ = \frac{1}{1 - \alpha} \sum_{i=1}^{\lceil \eta \rceil} (-1)^{i-1} \left( \frac{\alpha}{1 - \alpha} \right)^{i-1} \left( \frac{\alpha}{1 - \alpha} \right)^{\lceil \eta \rceil} \]
\[ + (-1)^{\lceil \eta \rceil} \frac{\alpha^{\eta-1}}{(1 - \alpha)^{\eta}} B(\eta - \lceil \eta \rceil, \lceil \eta \rceil + 1 - \eta; 1 - \alpha). \]

(4) By change of variable \(y = e^t - 1\) and the recursive formula in part (1), we find

\[ D(\alpha, \eta) = \int_0^\infty \frac{e^t - 1}{1 + \alpha(e^t - 1)} e^{-\eta t} dx \]
\[ = \int_0^\infty \frac{dy}{(1 + \alpha y)(1 + y)^{\eta+1}} \]
\[ = \frac{\lambda}{\pi} \hat{\beta} \eta_1 D(\pi, \eta_1) - \frac{\lambda}{\pi} \hat{\eta} \eta_2 D(1 - \pi, \eta_2). \]

**Remark 5.** The incomplete Beta function (or equivalently the Beta cumulative distribution function) is readily available in many statistical packages. It can also be directly implemented in MATLAB. Consequently, \(J(\alpha, \eta)\) and \(D(\alpha, \eta)\) can be computed directly from the incomplete Beta function.

The \(D(\cdot, \cdot)\) function above are inputs for building the derivative of \(G\), as the following result shows.

**Theorem 2.** For the Kou model, with parameters \(\eta_1 \geq 1, \eta_2 > 0, \lambda > 0, \beta + \bar{q} = 1, \beta, \bar{q} \geq 0\),

\[ G'(\pi) = \hat{\lambda} \hat{\beta} \eta_1 D(\pi, \eta_1) - \hat{\lambda} \hat{\eta} \eta_2 D(1 - \pi, \eta_2). \]

**Proof.** By change of variable, we find

\[ G'(\pi) = \frac{\lambda}{\pi} \hat{\beta} \eta_1 \int_0^\infty \frac{e^t - 1}{1 + \pi(e^t - 1)} e^{-\eta t} dy \]
\[ + \frac{\lambda}{\pi} \hat{\eta} \eta_2 \int_0^\infty \frac{e^t - 1}{1 + \pi(e^t - 1)} e^{-\eta t} dy \]
\[ = \frac{\lambda}{\pi} \hat{\beta} \eta_1 D(\pi, \eta_1) - \frac{\lambda}{\pi} \hat{\eta} \eta_2 D(1 - \pi, \eta_2). \]

**Remark 6.** The behavior of \(G'(\cdot)\), the growth rate at the risky portfolio holding, depends only on the Lévy measure. **Theorem 2** posits that \(G'(\cdot)\) is strictly increasing in the rate at which jumps occur, the probability of upward jumps, and the upward log-jump amplitude. It is strictly decreasing in the probability of downward jumps and downward log-jump amplitude.

To implement Newton's algorithm for the Kou market, we also need the second derivative of \(G'(\cdot)\). An analytic formula for \(G''(\pi)\) now follows.

**Theorem 3.** Let \(\pi \in (0, 1)\). For the Kou model with parameters \(\eta_1 \geq 1, \eta_2 > 0, \lambda > 0, \beta + \bar{q} = 1, \beta, \bar{q} \geq 0\),

\[ G''(\pi) = \frac{\lambda}{\pi} \hat{\beta} \eta_1 D(\pi, \eta_1) + \frac{\lambda}{\pi} \hat{\eta} \eta_2 D(1 - \pi, \eta_2). \]

**Proof.** We compute \(G''(\pi)\) directly by taking derivative of \(G'(\pi)\) given in (4.9) with respect to \(\pi\)

\[ G''(\pi) = \frac{\lambda}{\pi} \hat{\beta} \eta_1 D'(\pi, \eta_1) + \frac{\lambda}{\pi} \hat{\eta} \eta_2 D'(1 - \pi, \eta_2). \]

By part (4) of **Proposition 1**, it follows that for \(\alpha \in (0, 1)\) and \(\eta > 0\)

\[ D' (\alpha, \eta) = \frac{1}{\eta(1 - \alpha)} - \frac{1}{1 - \alpha} J(\alpha, \eta) - \frac{1}{\eta(1 - \alpha)^2}. \]
By using integration by parts as well as the recursive formula in part (1) of Proposition 1, we find
\[
f(\alpha, \eta) = - \int_0^\infty \frac{ydy}{(1 + \alpha y)^{(1 + \eta)^2}} - \frac{1}{\alpha} \int_0^\infty \frac{dy}{(1 + \alpha y)(1 + \eta)^2} + \frac{1}{\alpha} \int_0^\infty \frac{dy}{(1 + \alpha y)^2(1 + \eta)^2} - \frac{1}{\alpha} f(\alpha, \eta) + \frac{1}{\alpha \eta} \left( \frac{1}{1 - \alpha} - f(\alpha, \eta + 1) \right) - \frac{1}{\alpha(1 - \alpha)} \int f(\alpha, \eta).
\]

Therefore, we have
\[
D(\alpha, \eta) = \frac{1}{(1 - \alpha)^2} \left( \frac{1}{1 - \alpha} + \frac{1}{\alpha} \frac{\eta + \alpha - 1}{\alpha(1 - \alpha)^2} f(\alpha, \eta) - \frac{1}{\eta(1 - \alpha)^2} \right).
\]

which immediately yields (4.10). □

**Remark 7.** By employing these analytic formulas for \( G(\tau) \) and \( G'(\tau) \) presented in Theorems 2 and 3, we can use Newton’s algorithms proposed in Sections 3.1 and 3.2 to numerically compute the optimal portfolios for both uninformed and informed investors in the Kou jump-diffusion market. The corresponding numerical results are reported in Appendix A.

5. The Variance Gamma market

We closely follow Madan (2010) and Madan and Seneta (1990) for a description of the Variance Gamma (VG) process. Application and further developments are reported in Angelos (2013), Carr and Wu (2004), Hirs and Madan (2004), Madan et al. (1998) and Seneta (2004). Madan (2010) describes the Variance Gamma process \( X(t) \) as a Brownian motion with drift \( \theta \) and volatility \( \sigma \) time changed by a gamma process with a mean rate of one unit, and variance rate \( \nu \). Thus, for a standard Brownian motion \( B(t) \), and a gamma process \( \Gamma(t; \nu) \),

\[
X(t; \sigma, \nu, \theta) = B(t; \nu) + \sigma \Gamma(t; \nu)
\]

is a Variance Gamma process, with parameters \( (\sigma, \nu, \theta) \). It is a positive increasing process with independent and identically distributed increments over disjoint intervals of length \( h \), that have a gamma density

\[
\frac{1}{\nu + \Gamma(\frac{h}{\nu})} x^{\nu-1} \exp\left(-\frac{x}{\nu}\right).
\]

It is a Lévy process with characteristic function

\[
E[\exp(iuX(t))] = \left( 1 - \frac{i u \theta}{\nu} + \frac{1}{\sigma^2} u^2 \Gamma(\frac{h}{\nu}) \right)^{\frac{1}{2}}.
\]

The Variance Gamma process is a pure jump Lévy process with parameters \( C, G, \) and \( M \) (cf. Carr, Geman, Madan, and Yor, 2002), and density

\[
\nu(x) = C \left( \frac{x}{|x|} \right) I_{[x>0]} + C \left( \frac{-x}{|x|} \right) I_{[x<0]}.
\]

where

\[
C = \frac{1}{\nu}
\]

\[
G = \frac{1}{\sigma} \left( \frac{2}{\nu} + \frac{\theta^2}{\sigma^2} \right) + \frac{\theta}{\sigma^2}
\]

\[
M = \frac{1}{\sigma} \sqrt{\left( \frac{2}{\nu} + \frac{\theta^2}{\sigma^2} \right) - \frac{\theta}{\sigma^2}}
\]

The Variance Gamma process is of finite variation and can be written as the difference of two increasing processes. The total positive and negative variations are \( \frac{\pi}{\sigma} \) and \( \frac{\pi}{\sigma} \), respectively.

5.1. The mispriced asymmetric VG market

The mispriced asymmetric VG market has stock price dynamic:

\[
d(\log S_t) = \left( \mu - \frac{1}{2} \sigma^2 \right) dt + \sigma dY_t + dX_t, \quad t \in [0, T].
\]

(5.2)

where \( Y_t = p W_t + q U_t, p^2 + q^2 = 1, \ p \geq 0, \ q > 0, \) is defined by Eq. (2.3), \( q^2 \) is the proportion of asymmetric information in the market. \( W_t \) is standard Brownian motion, while \( U = (U_t) \) is a mean-reverting Ornstein–Uhlenbeck process with mean-reversion rate \( \lambda > 0, \) defined by Eq. (2.4). \( \lambda \) is a measure of mispricing.

Our model (5.2), has percentage return:

\[
dS_t = \mu_t dt + \sigma_t dY_t + \int \left( e^{x} - 1 \right) N(dt, dx), \quad t \in [0, T].
\]

(5.3)

Let \( X^{VG} \equiv \{X_t \}_{t \geq 0} \) be a Variance Gamma process with Lévy triple \( (\gamma, 0, \nu_{VG}) \), where \( \gamma = \int_0^1 \alpha \nu_{VG}(\alpha) d\alpha \). We assume that \( X^{VG} \) has parameters \( C, G, M, \) and Lévy measure \( \nu(x) \equiv \nu_{VG}(x) = C \exp\left(-G|x|\right) \frac{|x|}{x} I_{[x>0]} + C \exp\left(-Mx\right) \frac{|x|}{x} I_{[x<0]} \), (5.4)

where \( C > 0, \ G > 0, \ M > 0, \ G \geq M \).

5.2. The symmetric VG market

This market occurs when \( q = 0 \) (that is, there is no asymmetry). It consists of a single risk–free bond \( B \) that earns risk–free interest \( r_t \), and has price dynamic given by (2.1).

\[
dS_t = \mu_t dt + \sigma_t dB_t + \int \left( e^{x} - 1 \right) N(dt, dx), \quad t \in [0, T].
\]

(5.5)

\[
X^{VG}_t = \int_{0}^{t} \mu(x) dx, \quad t \in [0, T].
\]

where \( N(t, A) \) is the Poisson Random measure on \( \mathbb{R}^+ \times (\mathbb{R} - \{0\}), \ A \in B(\mathbb{R} - \{0\}), \) that counts the jumps of \( X^{VG} \) in \( (0, t), \ t \in [0, T], \) \( T > 0 \) (cf. Applebaum, 2004). The total return on the stock is:

\[
b_t = \mu_t + M_t = \mu_t + \int_{0}^{t} (e^{x} - 1) \nu(dx). \quad \text{VG process is a pure jump Lévy process with an infinite arrival rate of small jumps since} \int_0^1|x|\nu(x)dx = \infty \quad \text{and having paths of finite variation with} \int_0^1|x|\nu(x)dx = \infty.
\]

5.3. Instantaneous centralized moments of returns

In this section we present a general result for instantaneous centralized moments of returns. Let \( \nu(\cdot) \) be the Lévy measure of an arbitrary pure jump Lévy process \( X \).

**Definition 1** (Instantaneous centralized moments of returns). Define the objects \( M_j \) and \( K_j \) by the prescriptions:

\[
M_j = \int_{R} (\exp(x) - 1)^j \nu(dx),
\]

(5.7)

\[
K_j = \int_{R} (\exp(x) - 1)^j \nu(dx).
\]

(5.8)

\( M_j \) is called the jth instantaneous centralized moments of returns of the Lévy process \( X \), with measure \( \nu(\cdot) \). \( K_j \) is a kernel used to calculate \( M_j \).
The following lemma is easily derived from the binomial theorem. It will be quite useful in the sequel.

**Lemma 1.** If there exists $k \in \mathbb{N}$ such that $\int_{\mathbb{R}}(e^{x^j} - 1)\nu(dx) < \infty$ for each $0 \leq j \leq k$, then $M_j$ and $K_j$ exist, and

$$M_j = \sum_{i=1}^{j} (-1)^{j-i} \binom{j}{i} K_i.$$  

(5.9)

### 5.4. Polynomial approximation of $G(\alpha)$

Newton's algorithm requires explicit values for the first and second derivatives of $G(\alpha) = \int_{\mathbb{R}} \log(1 + \alpha \nu(dx))$, which must be estimated if an analytic formula is not available. In the sequel, we estimate $G(\alpha)$ by its $k$th degree Taylor series $G_k(\alpha)$.

We use the instantaneous central moments of returns $M_j$, to approximate the function $G$ and hence $G'$ and $G''$, by a truncated Taylor series. The larger the value of $k$, the better the approximation of $G$. We have the following result which is general for all models where $M_k$ exists.

**Proposition 2.** If there exists $k \in \mathbb{N}$ such that $M_j = \int_{\mathbb{R}}(e^{x^j} - 1)\nu(dx) < \infty$ for each $1 \leq j \leq k$, then $G_k(\alpha)$ has a $k$th degree polynomial approximation $G_k(\alpha)$ given by:

$$G_k(\alpha) = \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j} M_j \alpha^j, \quad \alpha \in [0, 1].$$  

(5.10)

with

$$G_k(\alpha) = \sum_{j=1}^{k} (-1)^{j-1} M_j \alpha^j + \text{and} \quad G_k''(\alpha)$$

$$= \sum_{j=2}^{k} (-1)^{j-1} (j - 1) M_j \alpha^{j-2}. \quad (5.11)$$

**Proof.** This results follows directly by expanding $G(\alpha)$ as a Taylor series about $\alpha = 0$. □

**Remark 8.** For the VG market with parameters $C, G, M$, the approximating polynomial $G_k$ is controlled by $M$, and we chose $k = [M]$, the greatest integer less than $M$. The derivatives $G_k'$ and $G_k''$ will be required in Newton's algorithm to compute the optimal portfolios.

### 5.5. The ICMR for the VG market

**Definition 2.** The $k$th central moment of instantaneous returns for the VG market is $M_k$, with kernel $K_k$, define by the prescriptions:

$$M_k = \int_{\mathbb{R}}(e^{x^k} - 1)^k \nu_G(x)dx \quad \text{and} \quad K_k = \int_{\mathbb{R}}(e^{x^k} - 1)\nu_G(x)dx. \quad (5.12)$$

The following result simplifies future computations of the ICMR for the VG market.

**Lemma 2.** Let $u, v > 0$. Then

$$\int_{0^+}^{\infty} \frac{(e^{-ux} - e^{-vx})}{x} dx = \ln\left(\frac{u}{v}\right). \quad (5.13)$$

**Proof.** Fix $v > u > 0$ and $t > 0$.

$$\int_{0^+}^{\infty} \frac{(e^{-ux} - e^{-vx})}{x} dx = \int_{0^+}^{\infty} t(e^{-tx}) dx = \int_{0^+}^{\infty} \left[ \int_{0^+}^{t} e^{-tx} dx \right] dx$$

$$= -\int_{0^+}^{\infty} \left[ \int_{0}^{v} e^{-tx} dx \right] dt = -\int_{0^+}^{\infty} \frac{e^{-tx}}{-t} \left|_{t=0^+}^{\infty} \right| dt = \int_{0^+}^{\infty} 1/t dt$$

$$= \ln\left(\frac{v}{u}\right).$$

□

We are now able to compute the VG kernel $K_k$.

**Lemma 3.** Let $X^{VG}$ be a Variance Gamma process with parameters $C, G$, and $M$. Then for $0 \leq s < M$,

$$K_s = C \log\left[ \frac{GM}{(G + s)(M - s)} \right]. \quad (5.14)$$

and $M_k$ is given by **Lemma 1**, provided $k \leq [M]$.

**Proof.** Let $0 \leq s < M$. From **Lemma 2**, we get

$$K_s = \int_{\mathbb{R}} (e^{x^k} - 1)\nu_G(x)dx$$

$$= C \int_{0}^{\infty} \left( e^{x^k} - 1 \right) \frac{e^{sx}}{x} dx + C \int_{0}^{\infty} \left( e^{x^k} - 1 \right) \frac{e^{-mx}}{x} dx$$

$$= C \int_{0}^{\infty} \left( e^{-(G+s)x} - G - (G+s)x - e^{-mx} \right) \frac{dx}{x}$$

Thus

$$K_s = C \left[ \log\left( \frac{G}{G+s} \right) + \log\left( \frac{M}{M-s} \right) \right] = C \log\left[ \frac{GM}{(G + s)(M - s)} \right].$$

□

With these explicit values for $K_1, K_2, \ldots, K_5$, in hand, we can now compute the first five (5) instantaneous central moments of returns, $M_1, M_2, \ldots, M_5$, which are required for the estimation of the optimal portfolio, $\pi$. Let $X$ be the VG process with parameters $C, G$, and $M$. It follows easily from **Lemma 1** that

$$M_1 = K_1 = \log\left[ \frac{GM}{(G + 1)(M - 1)} \right] \text{ if } M > 1.$$  

$$M_2 = C \log\left[ \frac{(G+1)^2(M-1)^2}{GM(G+2)(M-2)} \right] \text{ if } M > 2.$$  

$$M_3 = C \log\left[ \frac{GM(G+2)^2(M-3)^2}{(G+3)(M-3)(G+4)(M-4)^2} \right] \text{ if } M > 3.$$  

$$M_4 = C \log\left[ \frac{GM(G+4)^2(M-4)^2}{(G+5)(M-4)(G+6)(M-6)^2} \right] \text{ if } M > 4.$$  

$$M_5 = C \log\left[ \frac{GM(G+6)^2(M-6)^2}{(G+7)(M-6)(G+8)(M-8)^2} \right] \text{ if } M > 5.$$  

We now give a general formula for $k$th instantaneous centralize moment of return, $M_k$.

**Lemma 4.** Let $X$ be a VG process with parameters $C, G$, and $M$. Let $k \in \mathbb{N}$, with $k \leq [M]$. Then the $k$th ICMR is

$$M_k = C \log A_k \equiv C \sum_{j=0}^{k} (-1)^{k-j+1} \binom{k}{j} \log((G + j)(M - j)) \quad (5.15)$$

where

$$A_k \triangleq \prod_{j=0}^{k} ((G + j)(M - j))^{(-1)^{j-k+1} \binom{k}{j}). \quad (5.16)$$
Proof. If $M > k$ then by Lemma 1, $K_t$ exist for $s = 0, 1 \ldots, k$, and

\[
M_k = C \sum_{j=1}^{k} (-1)^{j-k} \binom{k}{j} \log \left[ \frac{GM}{(G+j)(M-j)} \right],
\]

\[
= C \sum_{j=1}^{k} \log \left[ \frac{GM}{(G+j)(M-j)} \right] (-1)^{j-k} \binom{k}{j}
\]

\[
= C \sum_{j=1}^{k} \log [GM] (-1)^{j-k} \binom{k}{j} + C \sum_{j=1}^{k} \log [(G+j)(M-j)] (-1)^{j-k} \binom{k}{j}
\]

Therefore

\[
M_k = C \log [GM] \sum_{j=1}^{k} (-1)^{j-k} \binom{k}{j} + \log \prod_{j=1}^{k} [(G+j)(M-j)] (-1)^{j-k} \binom{k}{j}
\]

\[
= C \log [GM] (-1)^{k-1} \binom{k}{k} + \log \prod_{j=1}^{k} [(G+j)(M-j)] (-1)^{k-1} \binom{k}{k}
\]

\[
= C \log \left( \prod_{j=0}^{k} [(G+j)(M-j)](-1)^{j-k} \binom{j}{k} \right) = C \log A_k.
\]

\[\square\]

5.6. Optimal demand in the VG market

Let $\pi$ be the optimal portfolio that maximizes the expected logarithmic utility from terminal wealth at time $T > 0$, with initial wealth $x > 0$. We now give explicit formulas for the approximation of $\pi$, based on the Taylor series approximation of $G_{VG}(\pi)$.

Theorem 4. Let $\pi^t$ be the optimal portfolio that maximizes the expected logarithmic utility from terminal wealth for the $t$th investor in the VG market with parameters $r, \mu, \sigma, C, G, M$. Assume that $k \leq [M]$, and that $\pi^t_k$ is the approximation of $\pi^t$ based on the $k$th degree polynomial approximation $G_k$, of the instantaneous central moment generating function $G(\cdot)$, given by Proposition 2. The optimal portfolio of the $t$th investor is

\[\pi^t_k = \frac{\mu^t - r}{\sigma^2} + \frac{\gamma^t_j \sigma^2}{\sigma^2} + \frac{G_k(\pi^t_k)}{\sigma^2} .\]

(5.17)

where $\gamma^t_j$ is given by Eq. (3.5).

Proof. This result follows directly from Theorem 1. \[\square\]

Observe from Theorem 4 that the optimal portfolios are basically fixed points of the first derivative of the function $G_k(\cdot)$, which is a moment generating function (the function $G_k$ generates instantaneous central moments of returns). Each optimal portfolio is the Merton optimal (for the symmetric market without jumps) plus an excess holding of risky asset. This excess consists of two parts. The first is purely random, and depends on information asymmetry, mean-reversion speed and volatility, while the second additive component is inversely proportional to the diffusive variance (sigma squared) and directly proportional to $G(\cdot)$, the growth rate of $G(\cdot)$ at the risky asset proportion level. Consequently, the optimal portfolios are very sensitive to information asymmetry, mean reversion speed, volatility, Sharpe ratio and activity of the Levy measure. For example, as the diffusive volatility increases, the agents basically hold the deterministic Merton optimal. The excess stock holding is basically positive or negative in lockstep with the sign and magnitude of $G_k(\cdot)$, that is, whether $G_k$ is increasing or decreasing at that stock proportion.

The behavior of $G(\cdot)$ depends only on the Levy measure, which is driven by the parameters. To implement Newton’s algorithm for the VG market, we also need the second derivative of $G(\cdot)$. There are many independent parameters (in addition to functions thereof) involved in the respective derivatives, and only extensive simulation can provide a proper sensitivity analysis of the impact of these parameters on the optimal portfolios.

Remark 9. For the VG market with parameters $C, G, M$, the approximating polynomial $G_k$ is controlled by $M$, and we chose $k = [M]$, the greatest integer less than $M$. An observation of the data in Table 1 of Carr et al. (2002) shows that the values of $M$ range between 25 and 138. Thus polynomials of degree $k \geq 25$ may be used to approximate $\pi$ using Newton’s method. However, it is best to use a smaller $k$ value if $M_k$ is very small, for example if it is less than say, $0.5 \times 10^{-4}$.

Remark 10. By using the formulas for $G(\pi)$ and $G(\pi)$ presented in Proposition 2, we employ Newton’s algorithms proposed in Sections 3.1 and 3.1 to numerically compute the optimal portfolios for both uninformed and informed investors in the VG jump–diffusion market. The corresponding numerical results for Bank of America stock, based on the parameters $C = 65.65, G = 47.38, M = 46.98$ in Carr et al. (2002), are reported in Appendix B.

6. Conclusion

We apply the theory of mispricing models under asymmetric information developed in Buckley et al. (2012), Buckley et al. (2014) and Buckley et al. (2015) to estimate optimal portfolios in the mispriced Kou jump–diffusion market. We propose a simple numerical scheme using Newton’s method to compute approximate optimal demand at various levels of mispricing, information asymmetry, jump intensity, jump frequency, and investment horizon. In particular, we apply the proposed algorithm to the Kou jump–diffusion markets by first developing analytic formulas for the first and second derivatives of the underlying objective function $G(\cdot) = G_{Kou}$ which depends only on the Levy measure of the driving jump process. We also present numerical simulations of paths of optimal portfolios using the same input parameters in Kou (2002).

We also apply our theory to Variance Gamma markets in Madan and Seneta (1990). For these special markets, we obtain optimal portfolios and their approximations based on an in-depth study of the instantaneous central moments (ICMR), $M_k$, for the return process, and its moment generating function, $G(\cdot) = G_{VG}$. We show that the optimal portfolios are fixed points of functions and Taylor polynomials created from $M_k$, the ICMR, which are dependent only on the Levy measure of the jump process driving the market. Using the parameters in Carr et al. (2002), we simulate paths of optimal portfolios for each investor holding Bank of America stock.

Acknowledgment

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Appendix A. Numerical results for Kou market

We present sample paths of optimal portfolios for the asymmetric Kou jump–diffusion market using the proposed algorithms in Sections 3.1 and 3.2. Information asymmetry is set at 0 percent,
25 percent, 50 percent, 75 percent and 100 percent. In each simulation, we assume that the market coefficients \( r_t, \mu_t, \sigma^2_t \) are constants. \( \tilde{p} \) is the probability of an upward jump; \( \tilde{q} \) is the probability of a downward jump; \( \lambda \) is the rate at which jumps occur per unit time; \( \eta_1 \) and \( \eta_2 \) are inverse of the log of upward and downward jumps, respectively; \( q^2 \) is the level of information asymmetry, \( \lambda \) is the mean-reversion speed, while \( \delta = \frac{1}{2} \) is the mean-reversion time for the mispricing process \( U \). Both \( \lambda \) and \( \delta \) are measures of mispricing; \( \mu \) is the expected return on the stock; \( r \) is the risk-free rate, and \( \sigma^2 \) is the continuous component of volatility.

Kou Lévy density: \( \nu_{\text{Kou}}(x) = \lambda (\tilde{p} \eta_1 \exp(-\eta_1 x) + \tilde{q} \eta_2 \exp(-\eta_2 x)) \).

### A.1. Convergence at 75 percent asymmetric information

**Input parameters:**
- \( \tilde{p} = 0.4 \)
- \( \tilde{q} = 0.6 \)
- \( \lambda = 3 \)
- \( \eta_1 = 25 \)
- \( \eta_2 = 50 \)
- \( q^2 = 0.75 \)
- \( \lambda = 0.2 \)
- \( \mu = 0.10 \)
- \( r = 0.05 \)
- \( \sigma^2 = 0.32^2 \)

### Table A.3. Optimal portfolios \((\pi_t)\) as a function of asymmetric information \((q^2)\) and investment horizon \((t)\)

<table>
<thead>
<tr>
<th>( t )</th>
<th>( q^2 = 0 )</th>
<th>( q^2 = 0.25 )</th>
<th>( q^2 = 0.5 )</th>
<th>( q^2 = 0.75 )</th>
<th>( q^2 = 1.00 )</th>
</tr>
</thead>
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<td>0.752949</td>
<td>0.752949</td>
<td>0.752949</td>
</tr>
<tr>
<td>0</td>
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<td>0.752949</td>
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<td>0.752949</td>
<td>0.752949</td>
</tr>
<tr>
<td>0</td>
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<td>0.752949</td>
<td>0.752949</td>
<td>0.752949</td>
</tr>
<tr>
<td>0</td>
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<td>0.752949</td>
<td>0.752949</td>
<td>0.752949</td>
</tr>
</tbody>
</table>

The above table reports the optimal portfolios of investors at different levels of asymmetric information and investment horizons. Asymmetric information ranges from 0 percent to 100 percent, in steps of 25 percent, and investment horizon ranges between 0 and 1 year. Mean-reversion time is 5 trading days, volatility is 30 percent, and the jump-diffusion rate is 3 jumps per year.

### Appendix B. Numerical results for Variance Gamma market using Bank of America stock

**B.1. Optimal portfolios \((\pi_t)\) as a function of asymmetric information \((q^2)\) and investment horizon \((t)\)**

**Input parameters:**
- \( q^2 \in [0, 1] \)
- \( t \in [0, 1] \)
- \( \lambda = 0.2 \)
- \( \mu = 0.06 \)
- \( r = 0.05 \)
- \( \sigma^2 = 0.2428^2 \)
- \( C = 65.65 \)
- \( M = 49.98 \)

### Table B.2. Trajectories of optimal portfolios at 75 percent asymmetric information in Kou model

The figure below shows the trajectories of optimal portfolios for informed and uninformed investors over 252 trading days. The input parameters are \( \tilde{p} = 0.4 \), \( \tilde{q} = 0.6 \), \( \lambda = 3 \), \( \eta_1 = 25 \), \( \eta_2 = 50 \), \( q^2 = 0.75 \), \( \lambda = 0.2 \), \( \mu = 0.10 \), \( r = 0.05 \), \( \sigma^2 = 0.32^2 \).

### A.2. Trajectories of optimal portfolios at 75 percent asymmetric information in Kou model

The above table reports the optimal portfolios of investors at different levels of asymmetric information and investment horizons for Bank of America stock. Asymmetric information ranges from 0 percent to 100 percent, in steps of 25 percent, and
investment horizon ranges between 0 and 1 year. Mean-reversion time is 5 trading years, volatility is 24.28 percent.

B.2. Trajectories of optimal portfolios at 75 percent asymmetric information in VG model

The figure above shows the trajectories of optimal portfolios for informed and uninformed investors over 252 trading days. The input parameters are: \(q^2 = 0.75, t \in [0, 1], \lambda = 0.2, \mu = 0.10, r = 0.05, \sigma = 0.2428, C = 65.65, G = 47.38, M = 46.98.

References


