REGULARIZED MODELS OF PHASE TRANSFORMATION IN ONE-DIMENSIONAL NONLINEAR ELASTICITY

A Dissertation
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by
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We investigate the asymptotic behavior of two models of solid-solid phase transitions in one-dimensional nonlinear elasticity. These models illustrate two different methods of regularizing the continuum mechanical equations of elasticity for a multi-phase material. Both systems are studied in the framework of infinite-dimensional gradient dynamical systems theory, and we discuss the issues involved in the application of this method.

First, we consider a viscoelastic model with an added capillarity term which models higher-order interfacial effects and a term which penalizes large displacements to model the fine structure observed in some materials. Standard semigroup arguments establish the existence, uniqueness, and regularity of strong solutions. We show that this system possesses a global attractor which can be characterized by the LaSalle invariance principle as the unstable set of the equilibrium states. We study the local bifurcation of equilibria, and when the fine structure term is sufficiently small, we show that these bifurcating branches comprise all of the equilibria. We also discuss the structure of the attractor and the asymptotic behavior.
of solutions.

The second model introduces an order parameter to determine the phase of the material independently of the strain. This phase-field system is composed of two coupled evolution equations, the mechanical equation of elasticity, which is hyperbolic, and a parabolic equation in the order parameter. We establish the global existence of classical solutions to this system in the special case of a linear elastic constitutive relation. Due to the lack of smoothing in the hyperbolic equation, the asymptotic behavior of solutions is difficult to study using standard methods and invariance principles in the theory of gradient systems. However, we show that, under suitable assumptions, all solutions asymptotically approach the equilibrium set in the weak topology, while the phase-field stabilizes strongly. We also present numerical simulations which illustrate the interaction of acoustic disturbances and phase boundaries. In these experiments, acoustic vibrations are shown to continue to ring after the phase-field has stabilized.
Biographical Sketch

William Kalies was born on July 28, 1967 in Sandusky, Ohio. After graduating from Huron High School in the fall of 1985, he began his undergraduate study at The Ohio State University. After receiving his Bachelor of Science degree in 1989, he went to graduate school as a National Science Foundation Graduate Fellow at Cornell University. Upon completion of his Ph.D. in the summer of 1994, he accepted a postdoctoral position at the Center for Dynamical Systems and Nonlinear Studies at the Georgia Institute of Technology.
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# Table of Contents

1 Introduction: Elastic Phase Transformation .................................................. 1  
1.1 Introduction ................................................................................. 1  
1.2 One-Dimensional Multiphase Elasticity ........................................... 3  
1.3 The Viscoelastic Model with Capillarity ........................................... 5  
1.4 The Phase-Field Model .................................................................. 8  
1.5 The Mathematical Setting: Semigroups and Infinite-Dimensional Dynamical Systems ................................................................. 11  
1.6 The Contents of This Thesis ............................................................. 15  

2 Existence of Solutions: The Viscoelastic Model with Capillarity .......... 18  
2.1 Introduction ................................................................................. 18  
2.2 Global Existence and Regularity ..................................................... 19  

3 Equilibrium States and Dynamics of the Viscoelastic Model ............... 23  
3.1 Introduction ................................................................................. 23  
3.2 Gradient Dynamics ...................................................................... 24  
3.3 Local Bifurcations ...................................................................... 27  
3.4 Equilibria for small $\alpha$ ............................................................ 31  
3.5 Dynamics for small $\alpha$ ............................................................. 36  

4 Existence of Solutions: The Phase-Field Model .................................. 42  
4.1 Introduction ................................................................................. 42  
4.2 Preliminary Transformations ......................................................... 43  
4.3 Local Existence .......................................................................... 46  
4.4 Regularity and Global Existence .................................................... 49  

5 Equilibrium States and Dynamics of the Phase-Field Model ............... 55  
5.1 Introduction ................................................................................. 55  
5.2 Equilibrium States ..................................................................... 59  
5.3 Wave Ringing ............................................................................ 64  
5.4 Gradient Dynamics ..................................................................... 66
List of Figures

3.1 Global Bifurcation Diagram for $\alpha = 0$. The branch indicated by a solid curve is stable. We have not determined the stability of the other non-trivial branches; see §3.5. .......................... 32

3.2 Solution to the viscoelastic model with capillarity at times $t = 200$ and $t = 2000$. Between these times, the first three transition layers move to the right and the last moves to the left. ......................... 41

4.1 Characteristics through the point $(x, t)$............................ 45

6.1 Initial data consisting of an acoustic pulse moving to the right with an amplitude of $u_x = 1$ added to the pure "zero-phase" end of a bar in a two-phase equilibrium state. ............................... 82

6.2 The solution at time $t = .6$ at which the acoustic wave has passed through the phase boundary producing a lower energy reflected wave. Note that the phase boundary has not moved noticeably from its original position. ........................................... 83

6.3 The solution at time $t = 1.9$ at which the acoustic wave has reflected back and passed through the phase boundary a second time. The amplitude (and energy) of the acoustic pulse is smaller, and again the phase boundary has not moved. ............................... 84

6.4 The solution at time $t = 50$ when the system has apparently re-stabilized to the original equilibrium. ................................. 85

6.5 This plot shows the "stair-step" manner in which the energy decays toward $E_1$, the energy of the original equilibrium. Energy losses occur due to interaction of the acoustic wave with the phase boundary. 86

6.6 Initial data with an acoustic pulse moving to the right with an amplitude of $u_x = 10$ added to a two-phase equilibrium. ............... 87

6.7 The solution at time $t = .5$ at which the acoustic wave has interacted with the phase boundary, causing it to sharpen. ................. 88

6.8 The solution at time $t = 1$ shows the acoustic wave pushing the phase boundary to the right and expanding the zero-phase behind it. 89

6.9 The solution at time $t = 2$ at which the acoustic wave has reflected back, pushed the phase boundary to the left, and expanded the one-phase behind it. ................................. 90
6.10 The solution at time $t = 3$ shows that on the third pass the phase boundary stalls and the acoustic wave passes through it. 

6.11 The solution at times $t = 20, 22, 24, 26, 28$, and $30$ are pictured simultaneously to show that the phase boundary appears to no longer move appreciably from its equilibrium position. The fields almost repeat after each round trip with the propagating acoustic waves slowly dissipating each time they pass through the phase boundary.

6.12 The solution at times $t = 74$ and $75$ are pictured simultaneously, which show "standing" acoustic waves continuing to ring.

6.13 This plot shows the decay of energy toward the energy of the single phase boundary equilibrium, $E_1$. 

6.14 This plot shows the solution with initial data which has $\varphi \equiv .2$ and an acoustic wave of amplitude $u_x = 1$ at the times $t = 0$ and $2$. Since $.2 < a$, the equations are decoupled. The phase approaches the constant zero-phase, and the acoustic wave propagates without any loss in energy.

6.15 Initial data with three phase boundaries which is odd about $\frac{1}{2}$. 

6.16 The solution at times $t = 1$ and $25$. On this time interval, the solution remains close to an equilibrium with three phase boundaries and maintains its symmetry about $\frac{1}{2}$. 

6.17 The solution at times $t = 30.6$ and $30.8$. On this time interval, two phase boundaries are destroyed, and the solution retains its symmetry.

6.18 The solution at times $t = 31$ and $31.2$. After the destruction of the phase boundaries, acoustic waves are produced which propagate symmetrically.

6.19 The solution at times $t = 41.4$ and $42.4$ shows that the solution continues to ring.

6.20 This plot shows the energy decaying toward the energy of the equilibrium with three phase boundaries. Near the time $t = 30$, the solution suddenly moves toward the single phase boundary equilibrium which has energy, $E_1$.

6.21 Initial data with seven phase boundaries which is odd about $\frac{1}{2}$. 

6.22 The solution at time $t = 1.8$ has become close to the middle constant phase equilibrium with small acoustic disturbances.

6.23 The solution at time $t = 2.4$ is approaching the two-phase equilibrium.

6.24 The solution at times $t = 6.4$ and $7.4$ shows a large ringing effect.

6.25 This plot shows the energy decaying at first toward the energy of the middle equilibrium and then toward the energy of the single phase boundary equilibrium, $E_1$. However, there is a large energy difference due to the wave ringing.

6.26 The solution at times $t = 998.5$ and $t = 999.5$ at which the ringing waves have been nearly damped out.
6.27 The solution to the system with the Fried-Gurtin scaling at times $t = 1999, 1999.5, \text{ and } 2000$ at which the ringing acoustic waves have not been damped appreciably. 

6.28 Wave ringing in the system with the Fried-Gurtin scaling near the time $t = 10,000$. 
Chapter 1

Introduction: Elastic Phase Transformation

1.1 Introduction

In the theory of nonlinear elasticity, materials which undergo a transformation in phase, such as shape-memory alloys, are often modeled using non-convex stored energy functionals, as proposed by Lifshitz [1948] and Ericksen [1975]. The analysis of such functionals from a variational point of view has contributed to the understanding of observed phenomena such as hysteresis and microstructure, and has stimulated the application of variational methods to partial differential equations.

Implicit in such static analyses is the assumption that the underlying dynamical process is dissipative, and hence that the system attempts to minimize its stored energy. The energy minimization problem is complicated by the fact that non-convex functionals typically possess many local minima or minimizing sequences. Therefore, a selection criterion is necessary to distinguish the physically relevant states. Analysis of dynamic behavior can be used to resolve this nonuniqueness or
to determine whether minimizing sequences are explored. Since static stability and
dynamic stability do not always agree, an examination of the dynamic problem is
also necessary to determine the stability properties of equilibria, Pego [1987] and
James [1980].

The dynamic models are also examples of dissipative evolutionary partial dif-
ferential equations (PDE's) to which a geometric analysis can be applied as in
finite-dimensional dynamical systems theory. The prototypical example of such a
system, closely related to those studied in this thesis, is the bistable scalar reaction-
diffusion equation for which a complete description of the attractor and the ap-
proach to equilibrium can be given, (cf. Henry [1981], Brunovsky and Fiedler
[1988], Carr and Pego [1990], and Bronsard and Kohn [1990]).

In the continuum mechanical equations of elasticity, the lack of convexity in the
stored energy causes a change of type between hyperbolic and elliptic. This mixed
type equation does not admit even local solutions for arbitrary initial data, and the
initial boundary value (Cauchy) problem is ill-posed. To study the dynamics of the
system, additional constitutive assumptions must be made, and there are basically
two approaches. Abeyratne and Knowles [1990] have studied models for which they
provide a nucleation criterion and a kinetic relation to govern the generation and
movement of phase boundaries. In this thesis, however, the alternative approach
of regularization will be considered.

To regularize a system, higher order terms are often added to the differential
equation. These terms model (usually small) physical effects such as viscosity and
surface energy and improve the equations mathematically by smoothing the solu-
tions. The solutions to the modified system are studied in the limit of vanishing
regularizing terms to identify the significant solutions to the original equations,
Slemrod [1989] and Abeyratne and Knowles [1991]. The technique of regularization has been applied to many models from the well-known Burgers equation to the inclusion of density gradient terms to study the liquid-vapor transition in a van der Waals gas, and it has been studied from both a dynamic and variational point of view, (cf. Fonseca and Tartar [1989] and the references therein). The purpose of this thesis is to study the dynamics of regularized models of solid-solid phase transitions in order to better understand multiphase elasticity and to provide examples of infinite-dimensional dynamical systems. In particular, we will study two scalar models which employ different methods of regularizing the continuum mechanical equation of elasticity. These systems can be regarded as simple models which exhibit behavior observed in more realistic higher-dimensional problems in continuum mechanics.

1.2 One-Dimensional Multiphase Elasticity

The properties of an elastic solid are determined by its stored energy. Consider a body occupying a reference region $\Omega \subset \mathbb{R}^n$, and let $u : \Omega \to \mathbb{R}^n$ be the displacement of the body in a deformed state. In the simplest case, the free energy, $W$, is assumed to be a function of the deformation gradient, $\nabla u$, and the total stored energy takes the form

$$I[u] = \int_{\Omega} W(\nabla u) \, dx.$$  

In general, continuum mechanical considerations, such as material frame indifference and material symmetries, impose restrictions on the form of the free energy. Since we will restrict attention to one-dimensional models for which the theory is simpler, a complete description will not be provided here. For more details concerning the continuum mechanics of elastic solids, the reader is referred to Ogden
For a multiphase material, the free energy is a non-convex function of the deformation gradient. In this thesis, we will consider only two-phase elastic bars for which $W(u_x)$ is a double-well potential. The prototypical example is

$$W(u_x) = \frac{1}{4}(u_x^2 - 1)^2. \quad (1.1)$$

In this case, the Piola-Kirchoff stress, $\sigma$, is a cubic function of the strain, $u_x$,

$$\sigma(u_x) \equiv W'(u_x) = u_x^3 - u_x.$$

The two phases of the material are represented by the maximal intervals containing $-1$ and $1$ on which $W$ is convex, $I_{-1} = (-\infty, -1/\sqrt{3})$ and $I_1 = (1/\sqrt{3}, \infty)$, respectively. At a point $x \in \Omega = [0, 1]$, the bar is in the $-1$ phase if the strain $u_x \in I_{-1}$, and likewise for the $+1$ phase.

In the above formulation, the mechanical equation of elasticity expressing the balance of linear momentum becomes

$$u_{tt} = \sigma(u_x)_x. \quad (1.2)$$

For $|u_x| > 1/\sqrt{3}$, equation (1.2) is a hyperbolic PDE, but for $|u_x| < 1/\sqrt{3}$, it is elliptic, and the Cauchy problem is ill-posed.

To study the dynamics of this system, we will regularize equation (1.2) by two different methods. The first method adds higher-order viscosity and capillarity terms directly to equation (1.2), and is justified by appealing to the viscosity-capillarity criterion that the physically relevant motions are limits of solutions to the regularized system in the limit of vanishing viscosity and capillarity, Slemrod [1989]. The second method introduces a smooth order parameter to determine the phase independently of the strain. This phase-field model has also been shown
to agree with the more conventional sharp interface theory as in Abeyratne and Knowles [1990] in the limit of certain vanishing parameters (cf. Fried and Gurtin [1993]). In the next two sections, we introduce the models to be studied.

1.3 The Viscoelastic Model with Capillarity

The first system to be studied in this thesis is expressed by the initial boundary value problem

\[
\begin{align*}
  u_{tt} &= (\sigma(u_x) + \beta u_x) - \gamma u_{xxxx} - \alpha u, \\
  u(0, t) &= u(1, t) = u_{xx}(0, t) = u_{xx}(1, t) = 0, \\
  u(x, 0) &= u_0(x), \quad \text{and} \quad u_t(x, 0) = u_1(x),
\end{align*}
\]

(1.3)

where \( \beta > 0, \alpha \geq 0, \) and \( 0 < \gamma \ll 1. \) Two terms have been added to regularize (1.2), a linear viscosity \( \beta u_x \) and a capillarity term \( -\gamma u_{xxxx} \) which models the effects of interfacial energy. The term \( -\alpha u \) which penalizes large displacements has also been included to model the fine structure observed in some materials. The system can be thought of as describing a viscoelastic bar with capillarity which is bonded to an elastic foundation with bonding stiffness \( \alpha. \) The boundary conditions correspond to a hard loading device in which the ends of the bar are fixed. One can also consider a soft loading which specifies a particular load at one end, and these traction boundary conditions have the form \( \sigma(u_x(1, t)) - \gamma u_{xxx}(1, t) = P(t) \) and \( u_{xx}(1, t) = 0 \) at the right endpoint.

The addition of the viscosity term alone is sufficient to establish the global existence of strong solutions to (1.3) and to provide a strong dissipation. The total
energy (kinetic plus potential),

\[
E[u](t) = \frac{1}{2} \int_0^1 \left[ u_t^2 + \alpha u^2 + \gamma u_{xx}^2 + \frac{1}{2}(u_x^2 - 1)^2 \right] dx,
\]

is a Lyapunov functional, since we have the energy inequality,

\[
\dot{E}[u](t) = -\int_0^1 \beta u_{xx}^2 dx \leq 0.
\]

Hence, \( E(t) \) decreases along solutions as \( t \to \infty \), and this gradient-like behavior prohibits any chaotic or periodic motions. The asymptotic behavior of the system (1.3) with \( \gamma = \alpha = 0 \) has been studied by Andrews and Ball [1982] and Pego [1987].

In this case, the potential energy (1.1) has uncountably many minimizers with discontinuous strain. Pego [1987] has shown that any such equilibrium \( \hat{u} \) satisfying the criterion

\[
\sigma'(\hat{u}_x) = 3\hat{u}_x^2 - 1 \geq \sigma_0 > 0,
\]

is linearly dynamically stable, and that even smooth solutions can approach an equilibrium with discontinuous strain.

The analysis is continued in Ball et.al. [1991] where the energy penalty term, \( -\alpha u \), is added. In this case, the potential energy

\[
I[u] = \frac{1}{2} \int_0^1 \left[ \alpha u^2 + \frac{1}{2}(u_x^2 - 1)^2 \right] dx,
\]

possesses minimizing sequences of sawtooth functions \( \{u_k\} \) such that \( I[u_k] \to 0 = \inf_u I[u] \), but there is no global minimizer \( \hat{u} \) such that \( I[\hat{u}] = 0 \). Any such minimizer would have to satisfy both \( \hat{u} = 0 \) a.e. and \( \hat{u}_x = \pm 1 \) a.e. which cannot happen in reasonable function spaces. The question of interest is whether solutions can dynamically explore minimizing sequences, i.e. \( u(x, t_k) = u_k(x) \) for some \( t_k \to \infty \), which corresponds to the mixture of phases becoming finer and finer. The main
results of Ball et.al. [1991] are that no solution minimizes energy (i.e. explores minimizing sequences), that phase boundaries (jumps in $u_x$), once created, do not move, and that any equilibrium satisfying the stability criterion (1.4) is linearly stable. These results suggest that typical solutions should approach an equilibrium which is a weak relative minimizer of the energy (cf. Ball et.al. [1991]), but this has not been proven except for small energy initial data, Friesecke and MacLeod [1994].

One difficulty in analyzing (1.3) with $\gamma = 0$ is that there are uncountably many equilibria, and it is not known whether orbits are precompact. When $\gamma > 0$, we will show that orbits are precompact, so that if the equilibria are isolated, every solution will approach an equilibrium. Also, the potential energy has global minimizers, $\hat{u}$, with $I[\hat{u}] = \inf I = O(\alpha^{1/3} \gamma^{1/6})$, Müller [1989]. These global minima of $I$ have a periodic structure, and Müller has suggested that the addition of the singular strain gradient term in the energy provides a selection criterion for choosing the physically relevant minimizing sequences of (1.5) and an explanation for the periodicity in the fine structure observed in some materials, (cf. Khachaturyan and Shatalov [1969]). Ball and James [1987] had earlier proposed the use of stored energy functionals which lack lower semicontinuity and possess minimizing sequences, such as (1.5), to address the geometric incompatibility and resulting fine structures, such as twin bands, which appear in some materials, (cf. Basinski and Christian [1954]). In this thesis, we are concerned with the dynamics of the system when the capillarity term is present. In this case, the phase boundaries are no longer jumps in $u_x$ but smooth transition layers with increasingly large strain gradients as $\gamma \to 0$ which can now move. We conjecture that typical solutions do approach a global minimum, although the structure of the attractor and the asymptotic behavior are
not completely understood.

1.4 The Phase-Field Model

In this section, we describe a second system which is derived from a different type of regularized model for phase transformation recently proposed by Fried and Gurtin [1993]. In their model, the field equation does not change type and remains hyperbolic. However, it is coupled to a parabolic equation through a (vector) order parameter which determines the phase. To illustrate some of the features of this model, we sketch the derivation of the one-dimensional system to be studied below, but the reader is referred to Fried and Gurtin [1993] for more details.

We consider the simplest case of a scalar order parameter, \( \varphi(x, t) \), and assume that the free energy has the form

\[
\phi(u_x, \varphi, \varphi_x) = W(u_x, \varphi) + f(\varphi) + \frac{\gamma}{2}|\varphi_x|^2,
\]

where \( u(x, t) \) is the displacement of the point \( x \) in the reference configuration under the deformation \( y(x, t) = x + u(x, t) \). With this form of free energy, the phase-field equations (7.20 and 7.21 of Fried and Gurtin [1993]) are

\[
\rho u_{tt} = W'_{ux}(u_x, \varphi)_x
\]

\[
\beta \varphi_t = \gamma \varphi_{xx} - W'_\varphi(u_x, \varphi) - f'(\varphi).
\]

Note that the regularization in this model is due to the constitutive assumption that the free energy depends on the gradient of the phase variable \( \varphi \).

To model a two-phase material, the exchange energy, \( f \), is assumed to be a double-well potential, and specifically we will use \( f(\varphi) = \mu(\varphi(1 - \varphi))^2/2 \). The two phases, "phase zero" and "phase one," are represented by the maximal intervals
containing 0 and 1 on which \( f \) is convex, i.e. \( I_0 = (-\infty, (3 - \sqrt{3})/6] \) and \( I_1 = [(3 + \sqrt{3})/6, \infty) \) respectively. Then, we take the strain energy, \( W \), to be a convex combination of typical strain energies for single-phase materials with the volume fraction, \( \alpha \), being a function of the order parameter. So \( W \) has the form

\[
W(u_x, \varphi) = (1 - \alpha(\varphi))W_0(u_x) + \alpha(\varphi)W_1(u_x),
\]

where \( W_0 \) and \( W_1 \) are convex with minima at \( u_x = 0 \) and \( u_x = 1 \). In this thesis, we will use the simplest possible quadratic functions,

\[
W_0(u_x) = \frac{1}{2}u_x^2 \quad \text{and} \quad W_1(u_x) = \frac{1}{2}(u_x - 1)^2.
\]

The volume fraction, \( \alpha(\varphi) \), will be an increasing, twice continuously differentiable function with \( 0 \leq \alpha(\varphi) \leq 1 \). In addition we shall often assume \( \alpha(\varphi) = 0 \) on \( I_0 \) and \( \alpha(\varphi) = 1 \) on \( I_1 \). This leads to the phase-field equations

\[
\begin{align*}
\varphi_t &= \frac{\gamma}{\beta}\varphi_{xx} + \frac{\mu}{\beta}\varphi(\varphi - 1)(1 - 2\varphi) + \frac{1}{2\beta}\alpha'(\varphi)(2u_x - 1). \\
\end{align*}
\]

(1.6)

As before, there are two types of boundary conditions which are generally placed on the \( u \) variable,

\[
\begin{align*}
u(0, t) = 0 \quad \text{and} \quad u(1, t) = \Lambda \\
or \quad u(0, t) = 0 \quad \text{and} \quad u_x(1, t) - \alpha(\varphi(1, t)) = P(t),
\end{align*}
\]

which correspond to fixing the total displacement of the bar to be \( \Lambda \) or the total load to be \( P(t) \) respectively. We also impose homogeneous Neumann boundary conditions on the phase variable \( \varphi \),

\[
\varphi_x(0, t) = \varphi_x(1, t) = 0.
\]
This seems to be the simplest system exhibiting the key features of the Fried-Gurtin theory. Nonlinearity (and lack of convexity) is only required in the dependence on the phase variable $\varphi$; the strain energy $W(u_x, \varphi)$, as part of the total energy $\psi$, is quadratic in the displacement gradient $u_x$, consistent with simple linear elasticity theory. In contrast, the strain energy $W(u_x)$ in the viscoelastic model (1.3) is not convex, causing the change of type in the mechanical equation.

The total energy of the phase-field system,

$$E(t) = \int_0^1 \left[ \frac{1}{2} u_t^2 + W(u_x, \varphi) + f(\varphi) + \frac{\gamma}{2} \varphi_x^2 \right] dx,$$

is a Lyapunov function with

$$\frac{d}{dt} E[u, \varphi](t) = -\int_0^1 \beta \varphi_t^2 \, dx \leq 0.$$

Thus, the dissipation of energy is due to the change in the phase-field which occurs primarily near phase boundaries. Since the mechanical energy is not damped directly, there is apparently a subtle interaction between the motion of phase boundaries and acoustic vibrations. This somewhat weak dissipation and the strong coupling of the equations causes difficulties in determining the asymptotic behavior of the system.

Fried and Gurtin [1993] have demonstrated connections between their theory and the more conventional theory, of which the viscoelastic models are examples, in which phase boundaries correspond to discontinuities or sharp interfaces in the strain. They show that the phase-field theory allows for the types of kinetics of transition layers found in the sharp interface theory of Abeyratne and Knowles [1990] and Gurtin and Struthers [1990], and that the phase-field model is in some sense more general than those based on the viscosity-capillarity or maximal dissipation criteria, (cf. Abeyratne and Knowles [1991] and [1992]).
1.5 The Mathematical Setting: Semigroups and Infinite-Dimensional Dynamical Systems

In this thesis, we study the two models systems described above from the point of view of dynamical systems theory. Although the philosophy behind this approach is rather simple since both systems exhibit gradient dynamics, the application of the language and techniques of dynamical systems to evolutionary PDE's requires some functional analysis. In particular, we use the theory of semigroups of linear operators to establish the existence of solutions, and we also use infinite-dimensional versions of certain elements of dynamical systems theory such as center manifold theory, Lyapunov functionals, and gradient dynamics. In this section, we summarize the notation and general results utilized in the sequel. The material is taken primarily from Henry [1981], Pazy [1983], Hale [1988], Carr [1981], and Pego [1987].

Throughout this thesis,

$$\|f\| = \left( \frac{1}{0} f(x) \, dx \right)^{\frac{1}{2}}$$

and

$$\langle f, g \rangle = \frac{1}{0} f(x) \bar{g}(x) \, dx,$$

denote the norm and inner product of $L^2$ functions on $[0, 1]$; the overbar denotes complex conjugation. For $k > 0$, $H^k[0, 1]$ denotes the Sobolev space of functions whose derivatives of all orders up to $k$ belong to $L^2$ with norm

$$\|f\|_k = \left( \sum_{n=0}^{k} \|f^{(n)}\|^2 \right)^{\frac{1}{2}}.$$

For any subspace $Y$ of $L^2[0, 1]$, we use the notation $Y_\alpha$ for the subspace of functions in $Y$ with zero mean, i.e. $L^2$-orthogonal to constant functions, $\frac{1}{0} f(x) \, dx = 0$. 

Both models discussed in this thesis can be transformed into systems in which the primary differential operator involved is the Laplacian with Neumann boundary conditions operating on either $L^2[0,1]$ or $L^2_0[0,1]$ with domain $D(-\Delta) = \{ \varphi \in H^2[0,1] : \varphi_x(0) = \varphi_x(1) = 0 \}$ or $D_a(-\Delta)$. In both cases, $-\Delta$ is a sectorial operator and generates an analytic semigroup, $e^{\Delta t}$. Also, the resolvent of $-\Delta$ is compact, and hence $e^{\Delta t}$ is a semigroup of compact operators.

The application of the theory of analytic semigroups to solve nonlinear parabolic PDE's involves the use of the fractional powers of sectorial operators to determine the underlying topology in which the existence theory is valid. In general, if $A$ is a sectorial operator on a Banach space $X$ with $\text{Re}\, \sigma(A) \geq \delta > 0$, one can define the fractional powers of $A$ as in §1.5 of Henry [1981]. In this case, for each $0 \leq \nu \leq 1$, $A^\nu$ is a closed, invertible, unbounded operator with $D(A) \subset D(A^\nu)$, and we define $X^\nu = D(A^\nu)$ with the graph norm $\|z\|_\nu = \|A^\nu z\| + \|z\|$ which is equivalent to $\|A^\nu z\|$ since $A^\nu$ is invertible.

Perhaps the most important feature of analytic semigroups is that for $t > 0$, $e^{-At}z \in D(A)$ for all $z \in X$, which is a smoothing property for the solutions to the abstract parabolic equation $z_t + Az = 0$. This regularity is also present in the solutions to the semilinear equation $z_t + Az = f(z)$ obtained via the variation of constants formula,

$$z = e^{-At}z_0 + \int_0^t e^{-A(t-s)}f(z(s)) \, ds.$$ 

If $f : X^\alpha \to X$ is Hölder continuous, then $z \in C((0,T), D(A)) \cap C^1((0,T), X^\gamma)$ for any $\gamma < 1$, by Theorems 3.3.3 and 3.5.2 of Henry [1981]. These results are the basis for the existence and regularity theory for both systems studied in this thesis.

To study the dynamical behavior of these systems, we will employ some of the standard techniques of dynamical systems theory. For example, since both
are gradient systems, we will rely on arguments involving Lyapunov functions and invariance principles which are similar to those in the finite-dimensional theory, (cf. Hale [1988]). However, we will also use some techniques which are not as easily applied in a general infinite-dimensional setting, such as those which are often based on linearization, eg. stability of equilibria, hyperbolicity, local bifurcation, and center manifold theory.

As an example of the problems that can arise, consider the linear differential equation

\[
\frac{d}{dt} z = Az, \quad z(0) = z_0,
\]

and let \( T(t)z_0 \) denote its solution. In the finite-dimensional case, the asymptotic stability of the trivial solution is equivalent to uniform exponential stability, i.e. the following are equivalent

1) \( T(t)z_0 \to 0 \) for all \( z_0 \),

2) \( \|T(t)\| \leq Me^{-\omega t} \) for some \( \omega > 0 \), and

3) \( s(A) = \sup \{ \Re \lambda : \lambda \in \sigma(A) \} < 0 \).

In particular, exponential asymptotic stability can be deduced from the spectrum of \( A \).

When \( A \) is an unbounded operator on a Banach space, this equivalence does not hold in general. In fact, there are examples of operators for which \( s(A) < 0 \) but the trivial solution is not asymptotically stable (cf. Arendt et. al. [1980]). This has implications in the application of the techniques and terminology of dynamical systems to infinite-dimensional equations. For example, the exponential splitting of solutions to (1.7) in finite dimensions is called hyperbolicity, and it is equivalent to the condition that the spectrum of \( A \) does not intersect the imaginary axis.
In infinite dimensions, we refer to these two notions as hyperbolicity and spectral hyperbolicity respectively, as they are not generally equivalent. Also, Carr and Malhardeen [1980] give an example of a well-behaved operator \( A \) on a Hilbert space whose spectrum consists only of isolated imaginary eigenvalues for which the trivial solution to (1.7) is unstable. This example shows that the application of center manifold and local bifurcation theory must be done with care (cf. Carr [1981]).

For general results about the asymptotic behavior of semigroups generated by unbounded operators on a Banach space, the reader is referred to Arendt et al. [1980] and the references therein. However, we will discuss briefly a specific case in which many of the above problems do not occur. Suppose \( A \) is an unbounded operator on a separable Hilbert space, \( X \), which generates a strongly continuous semigroup. Further suppose that \( \sigma(A) \) consists only of isolated simple eigenvalues \( \lambda_n \) with corresponding eigenfunctions \( \varphi_n \). If the eigenfunctions \( \{\varphi_n\} \) are an orthonormal basis of \( X \), then it is clear that the notions of stability and hyperbolicity can be determined from the spectrum \( \{\lambda_n\} \) just as in the finite-dimensional case. However, the orthogonality of \( \{\varphi_n\} \) is too stringent a requirement to be useful in many applications. Instead, we will consider the situation where the \( \{\varphi_n\} \) are equivalent to an orthonormal basis. Such a set is called a Riesz basis (cf. Gohberg and Krein [1969]) and is defined by the conditions

1) \( \forall f \in X, \exists c_n \) such that \( f = \sum_n c_n \varphi_n \) in \( X \), and

2) \( \exists a, b > 0 \) such that \( a \sum_n |c_n|^2 \leq \|f\|^2 \leq b \sum_n |c_n|^2 \).

Stability and hyperbolicity are determined by the spectrum just as in the orthonormal case. Also, local bifurcation and center manifold theory can be applied to non-
linear equations of the form $z_t = Az + N(z)$ as in Chapter 6 of Carr [1981]. The linearization of the viscoelastic model satisfies the above hypothesis as described in §3.3, but the phase-field model is not as well-behaved (see §5.2).

In this thesis, we will distinguish between three types of solutions to an evolution equation of the form $z_t + Az = F(z, t)$ where $A$ is a linear differential operator on a Banach space $X$ with domain $D(A)$, which generates a strongly continuous semigroup. If the initial data, $z_0 \in X$, then we will refer to a solution in $X$ of the integral equation

$$z(t) = e^{-At}z_0 + \int_{0}^{t} e^{-A(t-s)}F(z(s), s) \, ds,$$

obtained from the variation of constants formula, as a mild solution. If the initial data $z_0 \in D(A)$ and the differential equation is satisfied in $X$, then we will call $z(t)$ a strong solution. Additionally, if the solution has continuous derivatives in space and time up to the appropriate order, we will refer to it as a classical solution.

Throughout this thesis, we adopt the convention that arbitrary constants denoted by $C$ or $K$ may change at each occurrence, and any important dependence of these constants on other quantities will be appropriately noted. Also, other labels denoting functions, operators, or spaces such as $f$, $A$, and $X$ may change their meaning from section to section.

1.6 The Contents of This Thesis

Chapter 2 contains a proof of the global existence and regularity of strong solutions to the viscoelastic model with capillarity and homogeneous displacement boundary conditions (Theorem 2.2.1). The dynamic behavior of this model is the subject of Chapter 3. The results show that the system can be analyzed using
the techniques for gradient dynamical systems. In particular, the solutions are precompact, which is not known for viscoelastic models without capillarity, and there exists a global attractor which is the unstable set of the equilibria (Lemma 3.2.1 f.f.). The equilibrium states satisfy a nonlinear fourth-order boundary value problem which is difficult to solve. However, the linearization near the trivial equilibrium is sufficiently well-behaved that a local bifurcation analysis yields some information (Lemmas 3.3.1 and 3.3.2). More results are obtained via perturbation methods in the case that the fine structure parameter $\alpha$ is small.

The remainder of this thesis is devoted to the study of the phase-field model as described in §1.4. Chapter 4 establishes the global existence of classical solutions to the phase-field system with either displacement or traction boundary conditions (Theorem 4.2.1). Even though the system possesses a Lyapunov functional, the strong coupling between the mechanical and phase-field equations presents some difficulty in determining the asymptotic behavior. In Chapter 5, we discuss the problems involved in applying the techniques used on the viscoelastic model. For example, we do not know whether the orbits of the phase-field system are precompact. However, we describe a method for studying the asymptotic behavior in the weak topology. In particular, we prove, under suitable assumptions concerning the volume fraction, that the weak $\omega$-limit set of any strong solution is nonempty and is contained in the equilibrium set (Corollary 5.4.1 to Theorem 5.4.1), and the phase-field stabilizes strongly. We also discuss the equilibrium states, their stability properties, and a wave ringing phenomenon which results from the mechanical and phase-field equations becoming decoupled or nearly decoupled on certain subintervals. Chapter 6 describes the results of numerical experimentation with the phase-field model. In particular, we explore acoustic wave and phase boundary in-
teractions, the stability of equilibria, and wave ringing. We summarize and discuss open problems in Chapter 7.
Chapter 2

Existence of Solutions: The Viscoelastic Model with Capillarity

2.1 Introduction

In this chapter, we study the existence and regularity of solutions to the viscoelastic model with capillarity (1.3). The proof relies on a change of variables which transforms the equation into a parabolic system to which semigroup theory can be applied (Henry [1981] and Pazy [1983]). This transformation was used on the viscoelastic models without capillarity originally by Andrews [1980] and subsequently by Pego [1987] and Ball et.al. [1991].

Let $X = L^2_0[0, 1]$ be the Hilbert space of all $L^2$ functions with zero mean, i.e. orthogonal to constant functions. From the introduction, recall the following properties of the Laplacian with Neumann boundary conditions operating on $X$ with domain $D(-\Delta) = \{p \in H^2[0, 1] : p_x(0) = p_x(1) = 0\} \cap L^2_0$. The Laplacian is a
sectorial operator on $X$ with a compact inverse. Therefore, its spectrum, $\sigma(-\Delta)$, consists only of eigenvalues, all of which lie on the positive real axis. Since zero is in the resolvent set, $\rho(-\Delta)$, the fractional powers of $-\Delta$ are defined, and we take $X^\nu = D(-\Delta^\nu)$ with the graph norm. From §1.3 of Henry [1981], $X^0 = X$, $X^1 = D(A)$, and $X^{1/2} = H^1_0[0,1]$. From the Nirenberg-Gagliardo inequalities and the Sobolev imbedding theorems, $X^\nu$ is continuously imbedded in $C^1_0[0,1]$ for $\nu > 3/4$, (cf. Theorem 1.6.1 of Henry [1981]).

2.2 Global Existence and Regularity

We consider the initial boundary value problem from (1.3)

\[
\begin{align*}
  u_{tt} &= (u_x^3 - u_x + \beta u_{xt} - \gamma u_{xxx})_x - \alpha u, \\
  u(0, t) &= u(1, t) = u_{xx}(0, t) = u_{xx}(1, t) = 0, \\
  u(x, 0) &= u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x).
\end{align*}
\]

(2.1)

For $\beta, \gamma > 0$ and $\alpha \geq 0$, we show that there exist strong global solutions to (2.1).

**Theorem 2.2.1** Suppose $u_0 \in H^2 \cap H^1_0$ and $u_1 \in L^2$. Then for all $T > 0$, there exists a unique strong solution of (2.1) on $[0, T)$ in the following sense:

\[
\begin{align*}
  u &\in C([0, T), H^2 \cap H^1_0) \cap C^1((0, T), C^2) \cap C((0, T), H^3) \\
  u_t &\in C([0, T), L^2) \cap C^1((0, T), C) \cap C((0, T), H^1_0) \\
  u_{tt} &\in C((0, T), C) \quad \text{and} \quad u_x^3 - u_x + \beta u_{xt} - \gamma u_{xxx} \in C((0, T), C^1),
\end{align*}
\]

and (2.1) holds pointwise for $t > 0$ and $0 < x < 1$. Also, the energy inequality

\[
\frac{d}{dt} E = -\beta \|u_{xt}\|^2
\]

(2.2)
holds for $t > 0$ where

$$E[u](t) = \frac{1}{2} \| u_t \|^2 + \frac{\alpha}{2} \| u \|^2 + \frac{\gamma}{2} \| u_{xx} \|^2 + \frac{1}{4} \int_0^1 (u_x^2 - 1)^2 \, dx.$$  (2.3)

**Proof:** As in Ball et.al. [1991], we transform (2.1) into a semilinear parabolic system and apply the results of Henry [1981]. The transformed variables are defined by

$$p(x, t) = \int_0^x u_t(s, t) \, ds - \int_0^x u_t(s, t) \, ds \, dx$$  (2.4)

$$q(x, t) = \beta u_x(x, t) - p(x, t).$$

This yields the system

$$\begin{pmatrix}
p_t \\
q_t
\end{pmatrix} + \begin{pmatrix}
\beta - \frac{\gamma}{\beta} & -\frac{\gamma}{\beta} \\
\frac{\gamma}{\beta} & \frac{\gamma}{\beta}
\end{pmatrix} \begin{pmatrix}
-\Delta & 0 \\
0 & -\Delta
\end{pmatrix} \begin{pmatrix}
p \\
q
\end{pmatrix} = f(p, q) \begin{pmatrix}
1 \\
-1
\end{pmatrix},$$  (2.5)

$$p_x(0, t) = p_x(1, t) = q_x(0, t) = q_x(1, t) = \int_0^1 p(x, t) \, dx = \int_0^1 q(x, t) \, dx = 0,$$

where

$$f(p, q) = \sigma \left( \frac{p + q}{\beta} \right) - \int_0^1 \sigma \left( \frac{p + q}{\beta} \right) \, dx - \alpha B \left( \frac{p + q}{\beta} \right)$$

with $\sigma(w) = w^3 - w$ and

$$B(w) = \int_0^x \int_0^y w(z) \, dz \, dy - \int_0^x \int_0^y w(z) \, dz \, dy \, dx.$$

It suffices to show the global existence of solutions to (2.5). Let $A$ be the operator on $Y = L^2_a \times L^2_a$ with $D(A) = D(-\Delta) \times D(-\Delta)$ given by

$$A = \begin{pmatrix}
\beta - \frac{\gamma}{\beta} & -\frac{\gamma}{\beta} \\
\frac{\gamma}{\beta} & \frac{\gamma}{\beta}
\end{pmatrix} \begin{pmatrix}
-\Delta & 0 \\
0 & -\Delta
\end{pmatrix} = C(-\Delta I).$$

The first step is to show that $A$ generates an analytic semigroup on $Y$. This is the content of Lemma 2.2.1. The remainder of the proof is similar to that in Ball et.al. [1991].
Lemma 2.2.1 A is a sectorial operator on $L^2_a \times L^2_a$.

Proof: Since the Laplacian has a compact inverse, it follows that $A$ also has a compact inverse, and $\sigma(A)$ consists only of eigenvalues. The eigenvalues of $C$ are $\lambda_{\pm} = \frac{1}{2}(\beta \pm \sqrt{\beta^2 - 4\gamma})$. We will assume $\beta^2 \neq 4\gamma$ in which case we can diagonalize $C$. So, there exists some bounded operator $P$ so that $P^{-1}AP = \Lambda$ with

$$
\Lambda = \begin{pmatrix}
-\lambda_+ \Delta & 0 \\
0 & -\lambda_- \Delta
\end{pmatrix}.
$$

Therefore, $\sigma(A) = \sigma(\Lambda) = \{n^2\pi^2 \lambda_{\pm} : n > 0\}$. For $\beta^2 > 4\gamma$, $\lambda_{\pm}$ are real, distinct and positive so that $\sigma(A) \subset [n^2\lambda_-, \infty)$. For $\beta^2 < 4\gamma$, $\lambda_{\pm}$ are complex with positive real part, and $\sigma(A)$ is contained in the sector $\{\lambda \in \mathbb{C} : |\arg \lambda| \leq \varphi = \arg \lambda_+\}$. The Laplacian with Neumann boundary conditions in $L^2_a$ has the property that for any sector $S_\theta = \{\lambda \in \mathbb{C} : \theta \leq |\arg \lambda| \leq \pi\}$ we have the resolvent estimate $\| (\lambda - \Delta)^{-1} \| \leq M_\theta/|\lambda|$ for $\lambda \in S_\theta$, (cf. §1.3 of Henry [1981]). Therefore, if we choose $\theta$ so that $\theta > \varphi$, then $S_\theta \in \rho(A)$ and $\lambda \in S_\theta$ implies $\lambda/\lambda_{\pm} \in S_{\theta-\varphi}$. This gives the following resolvent estimate for $A$

$$
\| (\lambda - A)^{-1} \| = \| P(\lambda - \Lambda)^{-1} P^{-1} \| \leq \| P \| \| P^{-1} \| \left( \| (\lambda - \lambda_+ \Delta)^{-1} \| + \| (\lambda - \lambda_- \Delta)^{-1} \| \right) \leq \| P \| \| P^{-1} \| \frac{2M_{\theta-\varphi}}{|\lambda|},
$$

for $\lambda \in S_\theta$. Therefore, $A$ is sectorial. \(\square\)

Using (2.4), we find that the initial data $(p_0, q_0) \in H^1_a \times H^1_a = D((-\Delta)^{\frac{1}{2}}) \times D((-\Delta)^{\frac{1}{2}}) = Y^{\frac{1}{2}}$. Also the nonlinear term $f(p, q) : Y^{\frac{1}{2}} \to L^2_a$ is locally Lipschitz since $\sigma$ is locally Lipschitz. Therefore, since $A$ is sectorial, we obtain a local solution to (2.5) using Theorem 3.3.3 of Henry [1981] with $(p, q) \in C((0, T), D(A))$. 
Theorem 3.5.2 of Henry [1981] implies that the solution is regular with \((p, q) \in C^1((0, T), Y^\nu)\) for any \(\nu < 1\). So by choosing \(\nu > 3/4\), we see that \((p, q) \in C^1((0, T), C^1_\delta)\).

By adding the two equations in (2.5), we obtain \(p_t + q_t = -\beta p_{xx}\), and from (2.4b), we have

\[
u = \int_0^x \left( \frac{p + q}{\beta} \right) dx \quad \text{and} \quad u_{xt} = -p_{xx}.\]

From these relationships, \(u, u_t\), and \(u_{tt}\) have the regularity stated in Theorem 2.2.1. Since \((u^2_x - u_x + \beta u_{xt} - \gamma u_{xxx})_x = u_{tt} + \alpha u\), each term in (2.1a) is continuous, and the equation holds pointwise.

To obtain global solutions we use the energy, \(E\). A simple calculation involving integration by parts shows that the energy inequality (2.2) holds, and \(E\) is decreasing along solutions. In the \((p, q)\) variables,

\[
E[p, q](t) = \frac{1}{2} \|p_x\|^2 + \frac{\alpha}{2\beta} \|B(p + q)x\|^2 + \frac{\gamma}{2\beta} \|p_x + q_x\|^2 + \frac{1}{4\beta^4} \int_0^1 ((p + q)^2 - \beta^2)^2 dx.
\]

This implies that \(p\) and \(q\) are bounded in \(H^1_\delta\). By the Poincaré inequality, \(p\) and \(q\) are also bounded in \(L^\infty\), and hence so is \(f(p, q)\). Global existence in time follows from Corollary 3.3.5 of Henry [1981]. Uniqueness is proved in the usual way by applying Gronwall's inequality to the difference between two solutions. \(\Box\)
Chapter 3

Equilibrium States and Dynamics of the Viscoelastic Model

3.1 Introduction

In the previous chapter, the energy, $E$, of the viscoelastic model with capillarity was shown to be decreasing along orbits. Hence, the solutions are bounded and exist globally in time. Now, we investigate the asymptotic behavior of these solutions as $t \to \infty$. Using the machinery of Hale [1988], we show that the system is a gradient dynamical system with $E$ as a Lyapunov function. In particular, the orbits are precompact, and hence if the equilibria are isolated, then all solutions approach an equilibrium. In contrast, for the viscoelastic model without capillarity, precompactness and the approach to equilibrium of orbits are still open questions, Ball et.al. [1991].

As a consequence of the gradient dynamics, the system possesses a global attractor which can be characterized in terms of the equilibria and their unstable manifolds. However, the equilibria satisfy the fourth-order boundary value prob-
\[ \gamma u_{xxx} - (u_x^3 - u_x)_x + \alpha u = 0, \tag{3.1} \]
\[ u(0) = u(1) = u_{xx}(0) = u_{xx}(1) = 0. \]

This nonlinear equation is difficult to solve, and we do not have a complete description of the equilibrium set except when \( \alpha = 0 \), in which case (3.2) can be integrated twice to obtain a more manageable second-order equation. However, center manifold theory can be used to study the local bifurcations near the trivial solution in the general case.

There are three essential ingredients to completely describing the dynamics of gradient systems: determining the equilibria and their stability types, finding the connecting orbits between equilibria which characterize the attractor, and describing the approach to equilibrium along the attractor. For the viscoelastic model with capillarity, this can be done fairly completely in the case of traction boundary conditions and no fine structure term, i.e. \( \alpha = 0 \), (cf. Hattori and Mischaikow [1991] and [1992]). A description of their results in relation to the fixed boundary condition case and the substantially more difficult general case of \( \alpha > 0 \) will be given at the end of the chapter, and the results of this chapter are a first step toward understanding the dynamics of the general system.

### 3.2 Gradient Dynamics

The solutions to the viscoelastic system with capillarity in the transformed variables (2.5) form a nonlinear \( C^1 \)-semigroup, \( S(t) \), on \( Y^{\frac{1}{2}} = H_0^1 \times H_0^1 \) defined by \( S(t)(p_0, q_0) = (p(t), q(t)) \). We wish to show that \( S \) is a gradient dynamical system on \( Y^{\frac{1}{2}} \) as defined in Hale [1988]. The first step is to show that positive orbits are
precompact.

**Lemma 3.2.1** For any solution, \((p(t), q(t))\), we have that

1. \(p_t\) and \(q_t\) are bounded in \(H^1_a\) and hence precompact in \(L^2_a\),
2. \(p\) and \(q\) are bounded in \(H^3_a\) and hence precompact in \(H^2_a\), and
3. \(p \to 0\) as \(t \to \infty\) in \(H^2_a\).

**Proof:** Let \(z = (p, q)\) and \(F(z) = f(p, q)(1, -1)\), and \(z(t)\) be the solution to the parabolic equation \(z_t + Az = F(z)\). Then Lemma A.3 of Pego [1987] states that for \(T > 0\), there exists a constant \(C(T)\) such that for any \(t, t_0\) with \(0 < t - t_0 < T\), the following estimate holds,

\[
\|z_t(t)\|_1 \leq C(T) \left[ (t - t_0)^{-1} \|z(t_0)\|_1 + (t - t_0)^{-\frac{1}{2}} \sup_{t_0 < s < t_0 + T} \|F(z(s))\| \right]. \tag{3.2}
\]

In Chapter 2, we showed that \(z(t)\) is bounded in the \(H^1\) norm and \(F(z(t))\) is bounded in the \(L^2\) norm. Therefore, taking \(t - t_0 = 1\) and \(T = 2\), we obtain that \(z_t(t)\) is bounded in the \(H^1\) norm. Precompactness in \(L^2_a\) follows from Rellich's lemma. This proves (1).

From the relations

\[
p_{xx} = -\frac{1}{\beta} (p_t + q_t) \quad \text{and} \quad \frac{\gamma}{\beta} q_{xx} = -q_t - \frac{\gamma}{\beta} p_{xx} - f(p, q),
\]

which were derived in the previous chapter, we see that \(p\) and \(q\) are bounded in the \(H^3\) norm which proves (2).

Finally, from the Poincaré inequality and the energy inequality (2.2), we obtain

\[
\int_0^\infty \|p_x(t)\|^2 dt \leq \int_0^\infty \|p_{xx}(t)\|^2 dt \leq 2E(0).
\]

Since, \(\|p_x(t)\|\) is uniformly \(C^1\) by part (1), \(p \to 0\) in \(H^1_a\) by the Poincaré inequality. But, \(p\) is precompact in \(H^2_a\), and hence \(p \to 0\) as \(t \to \infty\) in \(H^2_a\). \(\Box\)
Remark: The lemma shows that for any solution, \( p \to 0 \) in \( H^2_\alpha \) as \( t \to 0 \). So, asymptotically, \( q = \beta u_x \) approximately satisfies

\[
q_t = \frac{\gamma}{\beta} q_{xx} - \sigma \left( \frac{q}{\beta} \right) + \int_0^1 \sigma \left( \frac{q}{\beta} \right) dx + \frac{\alpha}{\beta} Bq.
\]

This equation is the bistable reaction-diffusion equation with additional nonlocal terms. It seems to be the next most complicated (constrained) gradient system which should in some sense describe the dynamics on the attractor of (2.5), but we will not study it here.

The energy, \( E \), is a Lyapunov function for \( S \). It satisfies: (a) \( E \geq 0 \), (b) \( E[p, q] \to \infty \) as \( \| (p, q) \|_1 \to \infty \), (c) \( \frac{d}{dt} E \leq 0 \), and (d) \( \frac{d}{dt} E[p, q] = 0 \) iff \( (p, q) \) is an equilibrium of (2.5). The last property follows from the fact that if \( \frac{d}{dt} E[p, q] = -\beta \| p_{xx} \|^2 = 0 \), then \( p_{xx} \equiv 0 \), and the boundary conditions imply \( p \equiv 0 \). So, \( (p, q) = (0, q) \) is an equilibrium, and \( q \) satisfies the equation

\[
\frac{\gamma}{\beta} q_{xx} - \sigma \left( \frac{q}{\beta} \right) + \int_0^1 \sigma \left( \frac{q}{\beta} \right) dx + \frac{\alpha}{\beta} Bq = 0. \tag{3.3}
\]

Since \( E \) is a Lyapunov function and positive orbits are precompact, \( S(t) \) is a gradient dynamical system on \( Y^{\frac{1}{2}} \). Let \( \mathcal{E} \) be the set of equilibria of (2.5). By the LaSalle invariance principle (cf. Lemma 3.8.2 of Hale [1988]), the \( \omega \)-limit set of any orbit is a compact, connected set in \( \mathcal{E} \). If \( \mathcal{E} \) is a finite set, then all solutions tend to an equilibrium. In fact, by Lemma 3.2.1, \( p_t \) and \( q_t \) are precompact in \( L^2_\alpha \), and hence \( p_t, q_t \to 0 \) as \( t \to \infty \).

Arguing as in Hale [1988], we will show that \( \mathcal{E} \) is a bounded set in \( Y^{\frac{1}{2}} \). For simplicity, we use the original variable \( u \) and show that \( \mathcal{E} \) is bounded in \( H^2 \cap H^1_0 \).

Let \( u \) be any stationary solution satisfying (3.2). Then, \( u \) is a critical point of the
potential energy

\[ V[u] = \int_0^1 \left[ \frac{\alpha}{2} u^2 + \frac{\gamma}{2} u_{xx}^2 + \frac{1}{4} (u_x^2 - 1)^2 \right] \, dx. \]

Therefore,

\[ \int_0^1 \left[ \alpha uv + \gamma u_{xx} v_{xx} + (u_x^3 - u_x) v_x \right] \, dx = 0 \quad \text{for all } v \in H^2 \cap H_0^1. \tag{3.4} \]

In particular, (3.4) holds for \( v = u \). Let \( J_1 = \{ x \in [0,1] : |u_x(x)| \leq 1 \} \) and \( J_2 = [0,1]/J_1 \). Then, we have

\[ \int_0^1 \gamma u_{xx}^2 \, dx = -\int_0^1 \alpha u^2 \, dx + \int_{J_1} (1 - u_x^2) u_x^2 \, dx - \int_{J_2} (u_x^2 - 1) u_x^2 \, dx \]

\[ = I_1 + I_2 + I_3. \]

Since \( I_1 \) and \( I_3 \) are negative and \( I_2 \leq 1 \), we obtain \( \|u_{xx}\|^2 \leq 1/\gamma \), and hence \( \|u\|_2 \leq C/\sqrt{\gamma} \).

The above results show that the system satisfies all of the hypotheses of Theorems 4.2.4 and 3.8.5 of Hale [1988] for sectorial evolutionary equations and gradient dynamical systems, and we conclude that there is a connected global attractor \( \mathcal{A} \) for \( S \), and

\[ \mathcal{A} = W^u (\mathcal{E}) = \{(p_0, q_0) \in Y^\frac{1}{2} : S(-t)(p_0, q_0) \text{ is defined for } t \geq 0 \}

\[ \text{and } S(-t)(p_0, q_0) \to \mathcal{E} \text{ as } t \to \infty \}. \]

Since (2.5) has an attractor which is the unstable set of \( \mathcal{E} \), the same is true of (2.1), and now we turn to the problem of finding the equilibria.

### 3.3 Local Bifurcations

The equilibria of (2.1) are solutions to the boundary value problem

\[ \gamma u_{xxxx} - (u_x^2 - u_x)_x + \alpha u = 0 \tag{3.5} \]
We begin by studying the bifurcations which occur from the trivial solution \( u \equiv 0 \). For \( \alpha = 0 \), we will be able to produce a global bifurcation diagram, and using a perturbation argument, this will also provide global information about the equilibria for \( \alpha \) sufficiently small. For \( \alpha \) large, the picture is possibly more complicated, and we do not have complete information. For the local results of this section, we do not need to assume that \( \alpha \) is small.

We want to study the local bifurcations of equilibria which occur near \( u \equiv 0 \) as \( \gamma \to 0 \). Again we will use the \((p, q)\) formulation which will allow us to apply the center manifold theorem more easily. The stationary solutions of (2.5) satisfy \( p = 0 \) and equation (3.3). To investigate the local behavior near \((p, q) = 0\), we form the linear operator \( L = A - DF \) on \( Y \) with domain \( D(A) \) where \( DF \) is the Frechét derivative of \( F \),

\[
L(p, q) = \begin{pmatrix}
-\beta p_{xx} + \frac{3}{2}(p_{xx} + q_{xx}) + \frac{1}{2}(p + q) + \frac{\gamma}{3} B(p + q) \\
-\frac{3}{2}(p_{xx} + q_{xx}) - \frac{1}{2}(p + q) + \frac{\gamma}{3} B(p + q)
\end{pmatrix}.
\]

First, we compute the eigenvalues of \( L \). Suppose \( \lambda \) is an eigenvalue, so \( L(p, q) = \lambda(p, q) \). Then \(-\beta p_{xx} = \lambda(p + q)\). Also note that \( B_{p_{xx}} = p \) and \( B_{q_{xx}} = q \). Thus, for \( \lambda \neq 0 \), we can eliminate \( q \) and obtain an equation for \( p \)

\[
\lambda^2 p + \lambda \beta p_{xx} + \gamma p_{xxx} + p_{xx} + \alpha p = 0.
\]

In this case, \( p \in \{ \cos \pi k x : k \geq 1 \} \), and the eigenvalues of \( L \) are

\[
\lambda_k = \frac{1}{2} \left( \beta \pi^2 k^2 \pm \sqrt{\beta^2 \pi^4 k^4 - 4(\gamma \pi^4 k^4 - \pi^2 k^2 + \alpha)} \right) \tag{3.6}
\]

with corresponding eigenvectors

\[
(p, q) = (\lambda_k \cos \pi k x, (\beta k^2 \pi^2 - \lambda_k) \cos \pi k x). \tag{3.7}
\]
If \( \gamma \pi^4 k^4 - \pi^2 k^2 + \alpha = 0 \), then \( \varphi_k = (0, \sqrt{2} \cos \pi k x) \) is the eigenvector of \( \lambda_k = 0 \).

For \( \beta^2 \neq 4 \gamma \), \( L \) has no essential spectrum, i.e. \( \sigma(L) \) consists only of eigenvalues. To see this, note that for \( \lambda \) in the resolvent set of \( A \), \( (\lambda - A)^{-1} = P(\lambda - \Lambda)^{-1} P^{-1} \) for some bounded operator \( P \), and \( (\lambda - A)^{-1} \) is therefore compact. Since \( DF : Y \to Y \) is bounded, \( DF(\lambda - A)^{-1} \) is compact. Thus \( L = A - DF \) is a relatively compact perturbation of \( A \) and has the same essential spectrum as \( A \), which is empty (cf. appendix to Chapter 5 of Henry [1981]). So \( \sigma(L) = \{ \lambda_k : k \geq 1 \} \), since \( \{ \cos k \pi x : k \geq 1 \} \) is an orthonormal basis of \( L^2_a[0,1] \).

In this case, \( L \) is a well-behaved operator on \( Y \). It is sectorial (Theorem 1.3.2 of Henry [1981]) with spectrum consisting only of isolated simple eigenvalues (see Lemma 3.3.1 below) with corresponding eigenvectors (3.7) which form a Riesz basis of \( Y = L^2_a \times L^2_a \). Therefore, no problems arise in the linearized stability and local bifurcation analysis as described in §1.5.

From the above computations, we see that \( (p,q) = 0 \) is hyperbolic (i.e. all eigenvalues of \( L \) have non-zero real part) when \( \gamma \neq (\pi^2 k^2 - \alpha)/\pi^4 k^4 \) for any \( k > 0 \). For fixed \( \alpha \) and \( \beta \), as \( \gamma \) decreases, the eigenvalues successively cross the imaginary axis at the origin. The nature of the bifurcations which occur as \( \gamma \to 0 \) is characterized in the following two lemmas.

**Lemma 3.3.1** The eigenvalues of the zero solution are simple and pass through the origin at \( \gamma_k = (\pi^2 k^2 - \alpha)/\pi^4 k^4 \), for \( \pi^2 k^2 > \alpha \), with nonzero speed \( \lambda'(\gamma_k) = \pi^2 k^2 / \beta \).

**Proof:** The adjoint of \( L(\gamma_k) \) is given by

\[
L^*(\gamma_k)(p,q) = \left( \begin{array}{c} -\beta p_{xx} + \frac{\gamma_k}{\beta} (p_{xx} - q_{xx}) + \frac{1}{\beta} (p - q) + \frac{\alpha}{\beta} B(p - q) \\ \frac{\gamma_k}{\beta} (p_{xx} - q_{xx}) + \frac{1}{\beta} (p - q) + \frac{\alpha}{\beta} B(p - q) \end{array} \right).
\]

A simple calculation shows that the zero-eigenvector of \( L^* \) is \( \varphi_k^* = (0, \sqrt{2} \cos \pi k x) \).

Thus, the geometric multiplicity of \( \lambda = 0 \) is one for both \( L \) and \( L^* \), and \( \ker L(\gamma_k) = \)
ker $L^*(\gamma_k)$. This implies that Range $L(\gamma_k) \cap \ker L(\gamma_k) = 0$, since in this case, the Fredholm alternative gives Range $L(\gamma_k) = \ker L^*(\gamma_k) \perp$. So zero is a simple eigenvalue. Differentiating (3.6) and evaluating at the bifurcation points yields $\lambda'(\gamma_k) = \pi^2 k^2 / \beta$. □

**Lemma 3.3.2** A supercritical pitchfork bifurcation occurs at $\gamma_k$ with $\pi^2 k^2 > \alpha$.

**Proof:** We use a center manifold reduction following the notation of Carr [1981, §6.4], and we will drop the subscript $k$, since the analysis is essentially the same at every bifurcation point. We showed above that $Y = \ker L \oplus \text{Range } L$ and $\ker L = \text{span}\{\varphi\}$. So $z = s\varphi + y$ for some $y \in \text{Range } L$. The system (2.5) can be written $z_t = -Lz + N(z)$ where

$$N(z) = N(p, q) = \frac{1}{\beta^3} \left( (p + q)^3 - \int_0^1 (p + q)^3 \, dx \right) \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$ 

Let $P$ be the projection onto $\ker L$, $P(z) = \langle z, \varphi \rangle \varphi$. Note that

$$P \left( \int_0^1 \frac{1}{\beta^3} (p + q)^3 \, dx \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) = 0.$$ 

In these coordinates with $\lambda(\gamma)$ denoting the eigenvalue that passes through zero, so $\lambda(\gamma_k) = 0$, (2.5) becomes

$$\dot{s}\varphi = \lambda(\gamma)s\varphi + PN(s\varphi + y)$$

$$\dot{y} = -(I - P)Ly + (I - P)N(s\varphi + y)$$

$$\dot{\gamma} = 0.$$ 

There exists a center manifold, $y = h(s, \gamma) = o(s^2 + |\gamma s|)$. We compute
\[ PN(s\varphi + h(s, \gamma)) = \int_0^1 \frac{-\sqrt{2}}{\beta^3} \left( \sqrt{2s \cos \pi k x} + (h_1 + h_2)(s, \gamma) \right)^3 \cos \pi k x \, dx \]

\[ = -\frac{4s^3}{\beta^3} \int_0^1 \cos^4 \pi k x \, dx + o(s^4 + |\gamma s^3|) \]

to obtain the reduced system

\[ \dot{s} = \lambda(\gamma)s - \frac{3}{2\beta^3} s^3 + o(s^4 + |\gamma s^3|). \]

Therefore, the bifurcations are supercritical pitchforks. \( \Box \)

The above results locally characterize the equilibria near the trivial solution for the general case \( \alpha \geq 0 \). We now describe the global bifurcation diagram when \( \alpha = 0 \), and for \( \alpha > 0 \) small.

### 3.4 Equilibria for small \( \alpha \)

In the case \( \alpha = 0 \), the equilibria satisfy \( p = 0 \), and the non-local equation

\[ \gamma v_{xx} + v - v^3 + \int_0^1 v^3 \, dx = 0, \quad v_x(0) = v_x(1) = 0, \tag{3.8} \]

where we have replaced \( q \) with \( v = q/\beta = u_x \) in (3.3). We are interested in solutions for which \( \int_0^1 v \, dx = 0 \), consistent with \( u(0) = u(1) = 0 \). The only solutions to (3.8) with zero mean also appear to have \( \int_0^1 v^3 \, dx = 0 \), and hence satisfy \( \gamma v_{xx} + v - v^3 = 0 \). This seems to be the case from numerical experiments and also appears to be implicit in the properties of the exact elliptic function solutions of \( \gamma v_{xx} + v - v^3 + C = 0 \). However, it and a similar estimate needed in the proof of Lemma 3.4.1, below, are difficult to obtain explicitly and are therefore included in the statements as assumptions. The global bifurcation diagram for the second-order equation (3.8)
without the integral term is well-known and is shown in Figure 3.1 (cf. Smoller and Wasserman [1981]). In particular, for \( n\pi \leq 1/\sqrt{\gamma} < (n + 1)\pi \), there exist exactly \( 2n \) nontrivial equilibria which bifurcate from the trivial solution as in the previous section. We show that the same is true for the fourth-order boundary value problem (3.5) when \( \alpha \) is small enough.

Figure 3.1: Global Bifurcation Diagram for \( \alpha = 0 \). The branch indicated by a solid curve is stable. We have not determined the stability of the other non-trivial branches; see §3.5.
Theorem 3.4.1 Assume that any solution to (3.8) with zero mean also has $\frac{1}{0} v^3 \, dx = 0$. For fixed $\gamma$ with $n\pi \leq 1/\sqrt{\gamma} < (n + 1)\pi$ and $\alpha$ sufficiently small depending on $\gamma$, there are exactly $2n + 1$ solutions to (3.5).

Proof: As explained above, the result is assumed to be true for $\alpha = 0$. We will fix $\gamma > 0$ and consider $\alpha$ to be a small parameter and perturb from this case. Solutions to the boundary value problem (3.5) are orbits of the four-dimensional system

$$u_x = v, \quad v_x = w, \quad w_x = z,$$

$$\gamma z_x = (3v^2 - 1)w - \alpha u,$$

which satisfy $u(0) = w(0) = u(1) = w(1) = 0$. Let $\mu = 1/\sqrt{\gamma}$. We rescale by

$$\bar{u} = \mu u, \quad \bar{v} = v, \quad \bar{w} = \sqrt{\gamma}w, \quad \bar{z} = \gamma z, \quad \bar{z} = \mu x,$$

which gives (dropping the bars)

$$u_x = v, \quad v_x = w, \quad w_x = z,$$

$$z_x = (3v^2 - 1)w - \alpha \gamma u. \quad (3.9)$$

Let $\varphi T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be the flow generated by (3.9), and define $F_\alpha(v_0, z_0) \equiv \varphi_{\mu}^0(0, v_0, 0, z_0)$, the time-$\mu$ map of the flow restricted to $\mathbb{R}^2 = v_0, z_0$-plane. The solutions to (3.5) correspond to points in the intersection $F_\alpha(\mathbb{R}^2) \cap \mathbb{R}^2$. For $\alpha = 0$, these intersection points all lie on the curve $C = \{(v_0, z_0) : v_0 \in (-1, 1) \text{ and } z_0 = v_0^3 - v_0\}$ and are fixed points of $F_0$. By continuity of the flow $\varphi T$ with respect to the vector field and an appeal to the implicit function theorem, we finish the proof of the theorem with the following lemma. Note that we have already determined the behavior near the origin via the local bifurcation results of §3.3.
Lemma 3.4.1 Let \((\bar{v}_0, \bar{z}_0)\) be a point in \(F_0(\mathbb{R}^2) \cap \mathbb{R}^2\) with \(\bar{v}_0 \neq 0\). Assume that 
\[
\frac{\partial}{\partial z} A_1(v_0, 0) > 0
\]
where \(A_1\) is defined by (3.12) below. Then \(F_0(\mathbb{R}^2)\) intersects \(\mathbb{R}^2\) transversely at \((\bar{v}_0, \bar{z}_0)\).

Proof: By definition, the intersection is transverse if \(DF_0(\mathbb{R}^2) \oplus \mathbb{R}^2 = \mathbb{R}^4\). This is equivalent to \(D(P \circ F_0)(v_0, z_0\text{-plane}) = (u_0, w_0\text{-plane})\) where \(P\) is the projection onto the \(u_0, w_0\text{-plane}\). So, we have a transverse intersection iff \(\det D(P \circ F_0) \neq 0\) at \((\bar{v}_0, \bar{z}_0)\). Since the point \((\bar{v}_0, \bar{z}_0)\) lies on \(C\), we restrict \(F_\alpha\) to a small strip \(S\) around \(C\). Let \(\bar{S} = (-1, 1) \times (-\epsilon, \epsilon)\) and define a smooth change of variables \(\Pi : \bar{S} \to S\) by \(\Pi(v_0, C) = (v_0, C + v_0^3 - v_0)\). Let \(G = P \circ F_\alpha \circ \Pi : \bar{S} \to u_0, w_0\text{-plane}\).

Then, \(DG = D(P \circ F_\alpha)D\Pi\), and \(\det DG = \det D(P \circ F_\alpha)\). This transformation is convenient because for \(\alpha = 0\), the system (3.9) has a first integral and can be reduced to

\[
u_x = v, \quad v_x = w, \quad w_x = v^3 - v + C, \quad \text{with} \quad C = z_0 - v_0^3 + v_0. \tag{3.10}\]

Here, \(v_0 = v(0), z_0 = z(0) = w_x(0)\), and we recall that \(u(0) = w(0) = 0\). All of the orbits of this system which intersect the \(v\)-axis more than once are periodic, and we denote the half-period by \(T(v_0, C)\). In this case, we compute

\[
G_1(v_0, C) = u(\mu; v_0, C) = \int_0^\mu v(x; v_0, C) \, dx,
\]

\[
G_2(v_0, C) = w(\mu; v_0, C) = \int_0^\mu (v^3 - v)(x; v_0, C) \, dx + C\mu.
\]

Therefore, to prove transversality, it suffices to show that at \((v_0, C) = (\bar{v}_0, 0)\) the following determinant is non-zero,

\[
\det DG = \det \begin{pmatrix}
\frac{\partial}{\partial v_0} \int_0^\mu v \, dx & \frac{\partial}{\partial C} \int_0^\mu v \, dx \\
\frac{\partial}{\partial v_0} \int_0^\mu v^3 \, dx & \frac{\partial}{\partial C} \int_0^\mu v^3 \, dx + \mu
\end{pmatrix}. \tag{3.11}
\]
Since \((\tilde{v}_0, \tilde{z}_0)\) is a fixed point of \(F_0, \mu = kT(\tilde{v}_0, 0)\) for some integer \(k\). We also define
\[
A_i(\nu_0, C) \equiv \int_0^{T(\nu_0, C)} v^i(x; \nu_0, C) \, dx \quad \text{for } i = 1, 3. \quad (3.12)
\]
By the symmetry of the orbits of (3.10) with \(C = 0\), \(A_i(\nu_0, 0) = 0\) for all \(|\nu_0| < 1\) and \(i = 1, 3\). Using Leibnitz’ Rule, we compute the entries in (3.11) at \((\nu_0, 0)\) as follows
\[
\frac{\partial}{\partial \nu_0} \int_0^\mu v^i(\nu_0, 0) \, dx = \frac{\partial}{\partial \nu_0} \left[ \int_0^{\mu - kT(\nu_0, C)} v^i \, dx \right]_{(\nu_0, 0)} + \mu \frac{\partial}{\partial \nu_0} A_i(\nu_0, C)_{(\nu_0, 0)}
\]
\[
= -k \frac{\partial}{\partial \nu_0} T(\nu_0, C) v^i_{(\nu_0, 0)} + \int_0^{\mu - kT(\nu_0, C)} \frac{\partial}{\partial \nu_0} v^i \, dx_{(\nu_0, 0)}
\]
\[
= -k \frac{\partial}{\partial \nu_0} T(\nu_0, 0) v^i_0.
\]
Similarly, evaluating at \((\nu_0, 0)\), we have
\[
\frac{\partial}{\partial C} \int_0^\mu v^i \, dx = -k \frac{\partial}{\partial C} T(\nu_0, 0) v^i_0 + k \frac{\partial}{\partial C} A_i(\nu_0, 0).
\]
Using the fact that \(\int_0^T w_x \, dx = 0 = A_3 - A_1 + CT\), we compute the determinant in (3.11)
\[
det DG(\nu_0, 0) = k^2(\bar{v}_0^3 - \tilde{v}_0) \frac{\partial}{\partial \nu_0} T(\nu_0, 0) \frac{\partial}{\partial C} A_1(\nu_0, 0).
\]
Since \(v_0 \neq 0, 1\) for the orbits of interest, we conclude that the intersections are transverse as long as \(\frac{\partial}{\partial \nu_0} T(\nu_0, 0) \neq 0\) and \(\frac{\partial}{\partial C} A_1(\nu_0, 0) \neq 0\). It is shown in Chafee and Infante [1974] that \(\frac{\partial}{\partial \nu_0} T(\nu_0, 0) > 0\). By assumption, \(\frac{\partial}{\partial C} A_1(\nu_0, 0) \neq 0\). □

Remark: Numerical experiments with the elliptic function solutions of (3.8) show that \(\frac{\partial}{\partial C} A_1(\nu_0, 0)\) is also strictly positive, but we have not found an analytic proof.
3.5 Dynamics for small $\alpha$

The previous analysis indicates that for $\alpha$ sufficiently small, the global bifurcation diagram for (3.5) is essentially Figure 3.1. Since there are finitely many equilibria, all solutions tend to an equilibrium. To characterize the dynamics completely for this system, we must determine whether the equilibria are hyperbolic and describe the connecting orbits among them. We are also interested in the asymptotics of the approach to equilibrium. For the parabolic equation $v_t = \gamma v_{xx} + v - v^3$, much is known about these issues. Hyperbolicity is established in Smoller et.al. [1981], the structure of the attractor is explored in Brunovsky and Fiedler [1988], and the approach to equilibrium for small $\gamma$ is detailed in Carr and Pego [1990]. In this section, we consider some of these questions for equation (2.5) with $\alpha$ small.

Let $(0, \psi)$ be a non-trivial equilibrium of (2.5) with $\alpha = 0$. Recall that $v = \psi/\beta$ satisfies

$$\gamma v_{xx} + v - v^3 = 0,$$

(as above, we assume that $\int_0^1 v \, dx = 0$ implies $\int_0^1 v^3 \, dx = 0$, cf. equation (3.8)). Linearizing about $(0, \psi)$, we define the linear operator

$$L(p, q) = \begin{pmatrix} -\beta p_{xx} + \frac{1}{\beta}(p_{xx} + q_{xx}) + \frac{1}{\beta} \int_0^1 (1 - 3v^2)(p + q) + \frac{1}{\beta} \int_0^1 3v^2(p + q) \, dx \\ -\gamma (p_{xx} + q_{xx}) - \frac{1}{\beta} (1 - 3v^2)(p + q) - \frac{1}{\beta} \int_0^1 3v^2(p + q) \, dx \end{pmatrix}.$$ 

Note that the same argument given in §3.3 shows that the spectrum of $L$ consists only of real eigenvalues. By definition, $(0, \psi)$ is hyperbolic if all eigenvalues of $L$ are non-zero. Suppose zero is an eigenvalue of $L$, then $p \equiv 0$ and $q$ satisfies

$$\gamma q_{xx} + (1 - 3v^2)q + \int_0^1 3v^2 q \, dx = 0,$$

(3.14)
with $q_x = 0$ at $x = 0, 1$ and $\int_0^1 q \, dx = 0$. To establish the hyperbolicity of $(0, \psi)$, we must show that there are no solutions to (3.14) with these constraints.

This problem can be cast as an inhomogeneous Sturm-Liouville boundary value problem

$$\hat{L} q = \gamma q_{xx} + (1 - 3v^2)q = C,$$  

(3.15)

with boundary conditions $q_x = 0$ at $x = 0, 1$ and the additional constraint that $q$ have zero mean. The operator $\hat{L}$ is the linearization of (3.13). It is shown in Smoller et al. [1980] that the solutions to $\gamma v_{xx} + g(v) = 0$ are hyperbolic if $g(v)/v > g'(v)$ for $v \neq 0$, $g(-A) = g(0) = g(B) = 0$ for some $A, B > 0$, and $g'(0) > 0$. In our case, $g(v) = v - v^3$ satisfies these conditions, and so there is no nontrivial solution to (3.15) for $C = 0$. However, since the homogeneous problem $\hat{L} q = 0$ has no nontrivial solution, by the Fredholm alternative, one does exist for the inhomogeneous problem (3.15) for all $C \neq 0$. To establish hyperbolicity, it is sufficient to prove that this solution has non-zero mean.

Let $C_k$ be the branch of non-trivial solutions to (3.13) which bifurcates from the trivial solution at $\gamma = 1/k^2\pi^2$. On each of these branches, the linearization $\hat{L}$ has infinitely many non-zero eigenvalues $\lambda_n$ with corresponding eigenfunctions $\varphi_n$. The solution, $q$, to (3.15) can be expressed as an eigenfunction expansion in the usual way. A simple calculation gives

$$q(x) = \sum_{n \geq 1} C(1, \varphi_n) \lambda_n^{-1} \varphi_n(x),$$

and so

$$\int_0^1 q(x) \, dx = \langle 1, q \rangle = \sum_{n \geq 1} C(1, \varphi_n)^2 \lambda_n^{-1}.$$ 

On $C_1$, all the eigenvalues $\lambda_n$ are positive, and thus the mean of $q$ is non-zero. This implies that the solution to the boundary value problem (3.15) does not satisfy the
zero mean constraint, and hence that the $C_1$ branch of equilibria are hyperbolic with respect to $L$. In fact, appealing to the bifurcation analysis, we can conclude that the equilibria on this branch are all stable (see Figure 3.1). Unfortunately, this argument does not extend to the other branches $C_k$, for $k > 1$, because $\dot{L}$ has negative as well as positive eigenvalues along these branches. We conjecture that all of the non-trivial equilibria are hyperbolic, but are as yet unable to prove this. Note that if an equilibrium is hyperbolic for $\alpha = 0$, it remains so for $\alpha$ small.

In the case that the equilibria are hyperbolic, then the analysis in Chapter 24 of Smoller [1983], which applies the Conley index theory to semilinear parabolic systems, will also prove for our system that the $C_k$ branch of equilibria have $(k-1)$-dimensional unstable manifolds and that there are orbits connecting the zero solution to each of the other equilibria. So, for example, when $\pi < 1/\sqrt{\gamma} < 2\pi$ the global attractor consists of three equilibria, the zero solution with a one-dimensional unstable manifold and two attractors $u_+$ and $u_-$, and two orbits connecting zero to $u_+$ and $u_-.$

Hattori and Mischaikow have studied the viscoelastic model with capillarity (1.3), but with traction boundary conditions at the right endpoint,

$$\sigma(ux(1,t)) + \beta u_{xt}(1,t) - \gamma u_{xxx}(1,t) = P, \quad \text{and} \quad u_{xx}(1,t) = 0,$$

and the fine structure parameter $\alpha = 0$. In this case, the estimates needed for the existence theory are more complicated than those of Chapter 2, due to the unusual boundary conditions, but the zero mean constraint does not occur as in the fixed boundary case. Hence, in the traction boundary case, the hyperbolicity of the equilibria follows directly from Smoller et.al. [1980], and the characterization of the equilibria and the structure of the attractor can be completely determined using topological methods such as connection matrices and the Conley index (cf.
Hattori and Mischakow [1991]).

For both types of boundary conditions (and for $\alpha \geq 0$), all solutions approach an equilibrium by the LaSalle invariance principle. In addition, one expects that most solutions approach a global minimizer of the energy. For two-phase materials, after an initial transient period, the strain-field becomes twinned, i.e. the strain is "piecewise" in one of the two pure phases with small transition layers between them. One dynamical question of interest is how these phase boundaries move and interact so that the solution approaches a global minimizer. For bistable scalar reaction-diffusion equations, (e.g. $u_t = \varepsilon^2 u_{xx} + u - u^3$) this question has been answered fairly completely. Bronsard and Kohn [1990], Carr and Pego [1990], Fusco and Hale [1989], and Grant [1991] have all studied this problem, and it is known that the transition layers move exponentially slowly (i.e. $O(e^{-1/\varepsilon})$) for an exponentially long time. When two transition layers collide or one hits a boundary, the number of layers is reduced, and this process continues until the solution reaches a homogeneous state (for Neumann boundary conditions). Hattori and Mischakow [1992] have shown that the phase boundaries for the viscoelastic model with capillarity (either boundary condition and $\alpha = 0$) also move exponentially slowly (in the capillarity parameter) on exponentially long time intervals.

Methods such as maximum principles and Sturm-Liouville theory used on the scalar reaction-diffusion equation to characterize the structure of the attractor (in terms of the equilibria and connecting orbits) and the dynamics near the attractor (in terms of the slow motion of transition layers) are well-suited only for second-order or scalar problems. However, the technique used by Hattori and Mischakow [1992] to study phase boundary motion in the viscoelastic model with capillarity (and $\alpha = 0$), which is the same as that used by Bronsard and Kohn [1990] on
the reaction-diffusion equation, consists primarily of energy estimates which can perhaps be extended to $\alpha > 0$.

The main difficulty in applying these methods to the case when $\alpha > 0$ is that the energy penalty involves the displacement $u$ which is a non-local quantity with respect to the strain $u_x$. Preliminary analysis and numerical simulations suggest that the energy penalty causes the length of the intervals on which $u_x$ is homogeneous to equalize and that the motion of phase boundaries in a twinned state need not be exponentially slow if there are disproportionately large intervals on which the strain is constant. Figure 3.2 shows a solution to the system with $\alpha = \beta = 1$ and $\gamma = .0001$ at times $t = 200$ and $t = 2000$ indicating a fairly rapid approach to a periodic state having equally-spaced phase boundaries.
Figure 3.2: Solution to the viscoelastic model with capillarity at times $t = 200$ and $t = 2000$. Between these times, the first three transition layers move to the right and the last moves to the left.
Chapter 4

Existence of Solutions: The Phase-Field Model

4.1 Introduction

In this chapter, we establish the global existence of solutions to the phase-field system (1.6). These solutions are not only strong but also classical if the initial data are smooth. However, it is important to note that the strict conditions imposed on the data in Theorem 4.2.1 are not necessary to obtain strong solutions in various Sobolev spaces, and later we will view the phase-field model as a dynamical system on these weaker spaces.

Recall from Chapter 1 the following results about the Laplacian with Neumann boundary conditions. Let $X = L^2[0, 1]$ and $A = -(\gamma / \beta)\Delta$ operating on $X$ with $D(A) = \{\varphi \in H^2[0, 1] : \varphi(0) = \varphi(1) = 0\}$. Then, $A$ is sectorial and generates an analytic semigroup of compact operators $e^{-At}$, and for any $\alpha > 0$ and any $\nu \in [0, 1]$ we have the estimates (Henry [1981])

$$
\|e^{-At}\varphi\| \leq C(\gamma, \beta)e^{\alpha t}\|\varphi\| \quad \text{and} \quad \|\hat{A}^\nu e^{-At}\varphi\| \leq C(\gamma, \beta, \nu)t^{-\nu}e^{\alpha t}\|\varphi\|.
$$
The fractional powers are defined for the operator $\hat{A} = A + aI$ so that zero is in
the resolvent set of $\hat{A}$. Note that $D(\hat{A}) = D(A)$. The above estimate holds for any
choice of $a > 0$ (cf. Henry [1981] and Pazy [1983]). Let $X^\nu = D(\hat{A}^\nu)$ with the
graph norm. From Henry, $X^0 = X$, $X^1 = D(A)$, and $X^{1/2} = H^1[0,1]$. Also, $X^\nu$
is continuously imbedded into $C[0,1]$ for $\nu > 1/4$ and into $C^1[0,1]$ for $\nu > 3/4$. In
particular, choosing $\nu = 1/2$ in the above estimate, we obtain

$$\|e^{-A t} \varphi\|_1 \leq C(\gamma, \beta) t^{-1/2} e^{a t} \|\varphi\|. \quad (4.1)$$

4.2 Preliminary Transformations

The phase-field system is described by the following initial boundary value problem

$$u_{tt} = (u_x - \alpha(\varphi))_x,$$
$$\varphi_t = \gamma \varphi_{xx} - \frac{1}{\beta} f'(\varphi) + \frac{1}{2\beta} \alpha'(\varphi)(2u_x - 1), \quad (4.2)$$
$$u(0,t) = 0, \quad u(1,t) = \Lambda, \quad \varphi_x(0,t) = \varphi_x(1,t) = 0,$$
$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \text{and} \quad \varphi(x,0) = \varphi_0(x).$$

The hyperbolic equation (4.2 a) requires compatibility conditions between the boundary
conditions and initial data, namely

$$u_0(0) = 0, \quad u_0(1) = \Lambda, \quad u_1(0) = u_1(1) = 0,$$
$$\text{and} \quad u''_0 - \alpha'(\varphi_0)\varphi'_0 = u''_0 = 0 \quad \text{at} \quad x = 0, 1.$$

A simple calculation shows that (4.2 a, b) are equivalent via $u = (p + q)/2$ to the
first-order system

$$p_t = p_x - \alpha(\varphi),$$
$$q_t = -q_x + \alpha(\varphi), \quad (4.3)$$
$$\varphi_t = \frac{\gamma}{\beta} \varphi_{xx} + g(\varphi, p_x, q_x),$$
where $g(\varphi, p_x, q_x) = -(1/\beta)[f'(\varphi) - \alpha'(\varphi)(p_x + q_x - 1)/2]$. Under this transformation, the initial conditions for $p$ and $q$ become

$$p_0(x) = u_0(x) + \int_0^x u_1(y) \, dy \quad \text{and} \quad q_0(x) = u_0(x) - \int_0^x u_1(y) \, dy,$$

and the boundary and compatibility conditions are

$$p(0) = -q(0) \quad \text{and} \quad p(1) = 2\Lambda - q(1),$$

and $p_0^{(n)} = (-1)^{n+1}q_0^{(n)}$ at $x = 0, 1$ for $n = 1, 2$. \hspace{1cm} (4.4)

We formulate (4.3) as a system of integral equations as follows. Using the semigroup $e^{-At}$ and the variation of constants formula, the integral equation for $\varphi$ is

$$\varphi = e^{-At}\varphi_0 + \int_0^t e^{-A(t-s)}g(\varphi(s), p_x(s), q_x(s)) \, ds. \quad \hspace{1cm} (4.5)$$

When integrating the $p$ and $q$ equations, we need to consider reflections from the boundaries. We will restrict attention to the unit square in $(x, t)$-space, in which case only one reflection can occur, and consider local existence in time. The integral equations are expressed in terms of the backward characteristics through a point $(x, t)$ as shown in Figure 4.1. The latter are given by

$$c_1(s) = x + t - s \quad \text{and} \quad c_2(s) = 2 - x - t + s,$$

$$c_3(s) = x - t + s \quad \text{and} \quad c_4(s) = t - x - s,$$

for the $p$-equation and the $q$-equation respectively.
Figure 4.1: Characteristics through the point \((x, t)\).

So the integral equations for the system (4.3) are

\[
p = \begin{cases} 
  p_0(c_1(0)) - \int_0^t \alpha(\varphi(c_1(s), s)) \, ds & \text{if } c_1(0) \leq 1 \\
  2\Lambda - q_0(c_2(0)) - \int_0^{c_1(1)} \alpha(\varphi(c_2(s), s)) \, ds - \int_{c_1(1)}^t \alpha(\varphi(c_1(s), s)) \, ds & \text{if } c_1(0) \geq 1 
\end{cases}
\]

\[
q = \begin{cases} 
  q_0(c_3(0)) + \int_0^t \alpha(\varphi(c_3(s), s)) \, ds & \text{if } c_3(0) \geq 0 \\
  -p_0(c_4(0)) + \int_0^{c_4(0)} \alpha(\varphi(c_4(s), s)) \, ds + \int_{c_4(0)}^t \alpha(\varphi(c_3(s), s)) \, ds & \text{if } c_3(0) \leq 0 
\end{cases}
\]

and \( \varphi = e^{-\Lambda t} \varphi_0 + \int_0^t e^{-\Lambda (t-s)} g(\varphi(s), p_x(s), q_x(s)) \, ds \). \tag{4.6} \]

In the next two sections, we prove the following existence result.
Theorem 4.2.1 (Classical Existence) Suppose \( u_0, \varphi_0 \in C^2[0, 1] \) and \( u_1 \in C^1[0, 1] \) satisfy the conditions

\[
    u_0(0) = 0, \quad u_0(1) = \Lambda, \quad u_1(0) = u_1(1) = 0, \quad u_0''(0) = u_0''(1) = 0, \quad \text{and} \quad \varphi_0'(0) = \varphi_0'(1) = 0.
\]

Assume that \( \alpha(\varphi) \) is a bounded function whose first and second derivatives are uniformly bounded, and \( \alpha(0) = 0 \). Then, on any time interval \([0, T]\), there exists a classical solution \((u, \varphi)\) to the initial boundary value problem

\[
\begin{align*}
    u_{tt} &= (u_x - \alpha(\varphi))_x, \\
    \varphi_t &= \frac{\gamma}{\beta} \varphi_{xx} - \frac{1}{\beta} f'(\varphi) + \frac{1}{2\beta} \alpha'(\varphi)(2u_x - 1), \\
    u(0, t) &= 0, \quad u(1, t) = \Lambda, \quad \varphi_x(0, t) = \varphi_x(1, t) = 0, \\
    u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), \quad \text{and} \quad \varphi(x, 0) = \varphi_0(x),
\end{align*}
\]

such that

\[
\begin{align*}
    u &\in C^2([0, 1] \times [0, T]) \\
    \varphi &\in C([0, T], C^2[0, 1]) \cap C^1([0, T], C^1[0, 1]),
\end{align*}
\]

and the system of equations (4.7) holds pointwise.

4.3 Local Existence

We first prove the local existence of weak solutions to the integral equations (4.6) in \( Y = H^1 \times H^1 \times H^1 \). We will use the notation \( y = (p, q, \varphi) \in Y \) and \( y_0 = (p_0, q_0, \varphi_0) \). For \( T < 1 \), the norm of \( C([0, T], Y) \) will be denoted by \( ||y(t)|| = \sup\{||y(t)||_1 : 0 \leq t \leq T\} \). Let \( z(t) \) denote the terms in (4.6) explicitly involving the initial data \( p_0, q_0, \) and \( \varphi_0 \), i.e.
\[
   z(t) = \begin{cases} 
   p_0(x + t) & \text{if } x + t \leq 1 \\
   2\Lambda - q_0(2 - x - t) & \text{if } x + t \geq 1 \\
   q_0(x - t) & \text{if } x - t \geq 0 \\
   -p_0(t - x) & \text{if } x - t \leq 0 \\
   e^{-A t}\phi_0 & 
   \end{cases}.
\]

(4.8)

Note that by our assumptions on \( p_0 \) and \( q_0 \), \( z(t) \in C([0, T], Y) \), and define \( \Gamma = \{ y(t) \in C([0, T], Y) : ||y(t) - z(t)|| \leq ||y_0||_1 \} \). Then \( \Gamma \) is a non-empty, complete space, and using (4.1), we have the estimate

\[
   ||y|| \leq 2\Lambda + C(\gamma, \beta)||y_0||_1 \quad \text{for all } y(t) \in \Gamma.
\]

(4.9)

To simplify the estimates below, we will suppress the dependence of constants on the parameters \( \gamma, \beta, \) and \( \Lambda \). Throughout, the constant \( C \) will change from line to line, and its dependence on important quantities will be duly noted.

Let \( M \) be a bound and global Lipschitz constant for \( \alpha \) and \( \alpha' \). We now compute the a priori estimates necessary to obtain local existence.

**A priori Estimates:** \((p, q, \varphi) \) and \((\dot{p}, \dot{q}, \dot{\varphi}) \in \Gamma\).

\[
   \|\alpha(\varphi) - \alpha(\dot{\varphi})\| \leq M\|\varphi - \dot{\varphi}\|
\]

\[
   \|\alpha(\varphi)_x - \alpha(\dot{\varphi})_x\| \leq \|\alpha'(\varphi)\varphi_x - \alpha'(\dot{\varphi})\dot{\varphi}_x\| + \|\alpha'(\dot{\varphi})\varphi_x - \alpha'(\dot{\varphi})\dot{\varphi}_x\|
\]

\[
   \leq ||\varphi||_1 \|\alpha'(\varphi) - \alpha'(\dot{\varphi})\| + M\|\varphi_x - \dot{\varphi}_x\|
\]

\[
   \leq M(||\varphi||_1 + 1)||\varphi - \dot{\varphi}||_1,
\]

consequently, using (4.9), we have

\[
   ||\alpha(\varphi) - \alpha(\dot{\varphi})||_1 \leq C(||y_0||_1)||\varphi - \dot{\varphi}||_1.
\]

(4.10)
Let

\[ F(\varphi) = -\frac{1}{\beta} f'(\varphi) - \frac{1}{2\beta} \alpha'(\varphi), \]

\[ G(\varphi, p_x, q_x) = \frac{1}{2\beta} \alpha'(\varphi)(p_x + q_x). \]

Then, \( g = F + G. \) Since \( F \) is locally Lipschitz, let \( N \) be a bound and also a Lipschitz constant for \( F \) in \( \{ \varphi : \|\varphi\|_{\infty} \leq 2\Lambda + C(\gamma, \beta)\|y_0\|_1 \} \), where \( C(\gamma, \beta) \) is the constant in (4.9). Note that \( \|\varphi\|_{\infty} \leq \|y\|_1 \leq 2\Lambda + C(\gamma, \beta)\|y_0\|_1 \) in \( \Gamma \) by (4.9), so that

\[ \|F(\varphi) - F(\varphi)\| \leq N\|\varphi - \varphi\|. \quad (4.11) \]

Also, estimating as above using the inequality (4.9), we obtain

\[ \|G(\varphi, p_x, q_x) - G(\varphi, p_x, q_x)\| \]

\[ \leq \frac{M}{2\beta} (\|p - \hat{p}\|_1 + \|q - \hat{q}\|_1) + \frac{M}{2\beta} (\|\hat{p}\|_1 + \|\hat{q}\|_1) \|\varphi - \varphi\| \]

\[ \leq C(\|y_0\|_1)\|y - \hat{y}\|_1. \quad (4.12) \]

Let \( \Phi[y] \) be the right-hand side of the integral equations (4.6). We shall prove that for \( T \) small enough depending on \( \|y_0\|_1, \Phi : \Gamma \rightarrow \Gamma \) and \( \Phi \) is a contraction. Using the estimates (4.1), (4.9), (4.10), (4.11), (4.12), and Fubini's theorem, we have

\[ \|\Phi[y](t) - z(t)\|_1 \leq \int_0^t 4\|\alpha(\varphi(\cdot, s))\| ds + \int_0^t e^{-A(t-s)} g(\varphi, p_x, q_x)(s) ds \]

\[ \leq C(\|y_0\|_1) [8Mt + t^{1/2} (N + M\dot{C}(\|y_0\|_1))] \|y\|_1. \]

Therefore, again using (4.9), we have, for all \( t \leq T, \)

\[ \|\Phi[y](t) - z(t)\| \leq C(\|y_0\|_1) [T + \sqrt{T}]. \]

Similarly,

\[ \|\Phi[y] - \Phi[\hat{y}]\| \leq C(\|y_0\|_1) [T + \sqrt{T}] \|y - \hat{y}\|. \]
Thus, by the Banach fixed point theorem, there exists a solution to the integral equations in \( \Gamma \subset C([0, T], Y) \) for \( T \) small enough depending only on \( \|y_0\|_1 \). We deduce that this is a unique solution by combining a standard Gronwall estimate with those above.

4.4 Regularity and Global Existence

In this section, we show that the weak solutions found in the previous section are in fact classical solutions to the differential equations which exist globally in time. To this end, we will assume that \( p_0, q_0, \) and \( \varphi_0 \in C^2[0, 1] \) satisfy the compatibility conditions in (4.4). In this case, \( z(t) \), as defined by equation (4.8), is in \( C^2[0, 1] \), and \( z_1(t) \) and \( z_2(t) \), the \( p \) and \( q \) components of \( z(t) \), are also \( C^2 \) in time on \( [0, T] \). We will establish the desired regularity of solutions in several steps in which information about \( \varphi \) is used to gain information about \( p \) and \( q \) or vice-versa.

Since \( \varphi \) is a continuous solution of the integral equation (4.5) into \( H^1 = X^1/2 \), by Lemma 3.3.2 of Henry [1981], we conclude that \( \varphi \) is locally Hölder continuous on \( (0, T] \) of any order \( \delta < \frac{1}{2} \) into \( H^1 \), and the following inequality holds (see also Pazy [1983]),

\[
\|\varphi(s + h) - \varphi(s)\|_1 \leq K(s)h^\delta \quad \text{with} \quad K(s) = O(s^{-\frac{1}{2} - \delta}e^{as}). \tag{4.13}
\]

In particular, \( K(s) \) is integrable on \( [0, T] \). Combining the inequalities (4.10) and (4.13), we obtain,

\[
\|\alpha(\varphi(s + h)) - \alpha(\varphi(s))\|_1 \leq C(\|y_0\|_1)K(s)h^\delta. \tag{4.14}
\]

Next, we use this estimate to show that \( p \) and \( q \) are also locally Hölder continuous of order \( \delta \), but for this we need to introduce some notation.
We will define

\[ H(x, s; t) = \left( \chi_{[0,1-t] \times [0,t]} + \chi_{[1-t,1] \times [x+t-1,t]} \right) \alpha(\varphi(x+t-s,s)) + \left( \chi_{[1-t,1] \times [0,x+t-1]} \right) \alpha(\varphi(2-x-t+s,s)), \]

where \( \chi_B \) is the characteristic function of the set \( B \subset [0,1] \times [0,T] \), so that

\[ p(t) = z_1(t) - \int_0^t H(s; t) \, ds. \]

Let \( 0 < t < t + h \leq T \), then

\[ p(t + h) = z_1(t + h) - \int_0^h H(s; t + h) \, ds - \int_h^{t+h} H(s; t + h) \, ds. \]

We change variables in the last integral by \( s \to s - h \), and it becomes

\[ \int_0^t \left[ (\chi_{[0,1-t-h] \times [0,t]} + \chi_{[1-t-h,1] \times [x+t-1,t]} \right) \alpha(\varphi(x+t-s,s+h)) + \chi_{[1-t-h,1] \times [0,x+t-1]} \alpha(\varphi(2-x-t+s,s+h)) \right] ds. \]

Let \( H_h(x, s; t) \) denote the above integrand.

With this notation, we can now estimate \( \|p(t + h) - p(t)\|_1 \) as follows,

\[ p(t + h) - p(t) = z_1(t + h) - z_1(t) - \int_0^h H(s; t + h) \, ds \]

\[ - \int_0^t \left[ H_h(s; t) - H(s; t) \right] \, ds. \]

Comparing the kernels \( H_h(x, s; t) \) and \( H(x, s; t) \), we find that

\[ \int_0^t \|H_h(s) - H(s)\|_1 \, ds \leq 2 \int_0^t \|\alpha(\varphi(s + h)) - \alpha(\varphi(s))\|_1 \, ds. \tag{4.15} \]

Using (4.14), (4.15), Fubini's theorem, and the fact that \( z_1(t) \) is twice continuously differentiable, we obtain the estimate

\[ \|p(t + h) - p(t)\|_1 \leq Ch + 2h\|\alpha(\varphi)\| + \hat{C}(T)h^6. \tag{4.16} \]
Therefore, $p$ is locally Hölder continuous of order $\delta$ on $(0, T]$ into $H^1$. A completely analogous estimate shows that the same is true for $q$.

From this, we find that $g(\varphi(t), t) = g(\varphi(t), p_x(t), q_x(t))$ is locally Hölder continuous into $L^2$. In fact, using (4.11), (4.12), (4.13), and (4.16), we obtain

$$\|g(\varphi(s + h), s + h) - g(\varphi(s), s)\| \leq C\|y - y\|_1$$
$$\leq C(\|y_0\|_1) K(s) h^\delta, \quad (4.17)$$

where $K(s)$ is integrable on $[0, T]$. Define $G(t)$ to be the integral term of equation (4.5),

$$G(t) = \int_0^t e^{-A(t-s)} g(\varphi(s), s) \, ds.$$  

Applying Lemma 3.5.1 of Henry [1981] to $G(t)$, we conclude that $G(t)$ is continuously differentiable on $(0, T)$ into $X^\nu$ for any $\nu < \delta$. In particular, we can choose $\delta$ and $\nu$ so that $\frac{1}{4} < \nu < \frac{1}{2}$, in which case $\varphi_t \in X^\nu \subset C[0, 1]$. We also use Lemma 3.3.2 of Henry to find that $\varphi \in D(A)$ for $t > 0$, and the differential equation

$$\varphi_t = \frac{\gamma}{\beta} \varphi_{xx} + g(\varphi, p_x, q_x) \quad (4.18)$$

holds in the sense of $L^2$ functions. Since $\varphi \in D(A) \subset C^1[0, 1]$, the integral equations for $p$ and $q$ can be differentiated and $p, q \in C^1[0, 1]$, which implies that $g(\varphi, p_x, q_x)$ is continuous. We also know that $\varphi_t$ is continuous, and so (4.18) implies that $\varphi \in C^2[0, 1]$. By again differentiating the integral equations for $p$ and $q$, we find that $p, q \in C^2[0, 1]$. Specifically, differentiating $p$ with respect to $x$ we obtain,

$$p_x = \begin{cases} \left. p_0 - \int_0^t \alpha'(\varphi(c_1)) \varphi_x(c_1) \, ds \right|_{c_1(1)} & \text{if } c_1(0) \leq 1, \\ q_0 + \int_0^{c_1(1)} \alpha'(\varphi(c_2)) \varphi_x(c_2) \, ds - \int_{c_1(1)}^t \alpha'(\varphi(c_1)) \varphi_x(c_1) \, ds & \text{if } c_1(0) \geq 1. \end{cases}$$
Since $x$ and $t$ occur together in the integral equation for $p$ (except in the limits of integration), we see that $p \in C^2([0,1] \times [0,T])$. For example,

$$
\begin{align*}
pt &= \begin{cases} 
p_0'(x + t) - \alpha(\varphi(x,t)) - \int_0^t \alpha'(\varphi)\varphi_z(c_1) \, ds, 
q_0'(2 - x - t) - \alpha(\varphi(x,t)) + \int_0^{c_1(1)} \alpha'(\varphi)\varphi_z(c_2) \, ds - \int_{c_1(1)}^t \alpha'(\varphi)\varphi_z(c_1) \, ds.
\end{cases}
\end{align*}
$$

So that $p_t = p_x - \alpha(\varphi)$. A similar argument shows that $q \in C^2([0,1] \times [0,T])$, and the differential equation for $u$ is satisfied by $u = (p + q)/2$ classically. Therefore, $p_x(t)$ and $q_x(t)$ are locally Lipschitz continuous in $t$, and hence $g(\varphi, t)$ is locally Lipschitz in both variables. From Theorem 3.5.2 of Henry [1981], we find that

$$
\varphi \in C([0,T), H^1) \cap C^1((0,T), C^1) \cap C((0,T), D(A)),
$$

and the differential equation (4.18) also holds classically. Note that we could not apply this theorem until we had proven that $g(\varphi, t)$ is locally Lipschitz in $t$, at which point all that was left to conclude was that $\varphi \in C^1((0,T), C^1)$.

To finish the proof of Theorem 4.2.1, we prove the following lemma.

**Lemma 4.4.1** Suppose $y(t) = (p(t), q(t), \varphi(t))$ is a solution to (4.7) on some interval $[0, T)$ with $T < \infty$. Then, there is some $y_1 \in Y$ such that $\lim_{t \to T} y(t) = y_1$ in $Y$.

Since a local solution exists for any initial data $y_1 \in Y$ by the results of §4.3, the lemma implies that any solution can be extended past any finite time $T$. Hence, we have global existence.

**Proof:** Without loss of generality, we can assume that $T \leq 1$. If $T > 1$, then consider the solution with initial data $y(T - 1)$ defined on $[0, 1)$.
First, we show that $\|y(t)\|_1$ is bounded on $[0, T)$, using the total energy,

$$E(t) = \int_0^1 \left[ \frac{1}{2} u_t^2 + W(u_x, \varphi) + f(\varphi) + \frac{\gamma^2}{2} \varphi_x^2 \right] dx$$

(4.19)

$$= \frac{1}{2} \int_0^1 \left[ u_t^2 + (1 - \alpha(\varphi)) u_x^2 + \alpha(\varphi)(u_x - 1)^2 + 2f(\varphi) + \gamma \varphi_x^2 \right] dx$$

$$= \frac{1}{2} \int_0^1 \left[ u_t^2 + (u_x - \alpha(\varphi))^2 + \alpha(\varphi)(1 - \alpha(\varphi)) + 2f(\varphi) + \gamma \varphi_x^2 \right] dx.$$

A simple calculation, using the classical solutions that we have obtained, shows that $E$ is continuously differentiable, and using integration by parts, we find that

$$\frac{d}{dt} E[u, \varphi](t) = -\int_0^1 \beta \varphi_t^2 \, dx \leq 0.$$  (4.20)

Therefore, $E$ is a Lyapunov functional, and $E(t) \leq E(0)$ for any $t \in [0, T)$. This implies that $\varphi$ is bounded in the $H^1$ norm on $[0, T)$. From the integral equations for $p$ and $q$, we see that their $H^1$ norms are both bounded by $\|p_0\|_1 + \|q_0\|_1 + CT_1 \|\varphi\|_1$. Therefore, $\|y(t)\|_1$ is bounded on $[0, T)$.

In the previous section, we showed that $y(t)$ is Hölder continuous on $(0, T)$ into $Y$, and from equations (4.13) and (4.16), for $0 < \tau < t < T$ we have

$$\|y(t) - y(\tau)\|_1 \leq C(T)(t - \tau)^\delta.$$

Therefore, $\lim_{t \to T} y(t)$ exists in $Y$. This completes the proof of Lemma 4.4.1 and Theorem 4.2.1. □

Remark: The above proof is also valid in the case where the fixed boundary conditions (4.2c) are replaced by the traction boundary conditions

$$u(0, t) = 0, \quad u_x(1, t) - \alpha(\varphi(1, t)) = P(t), \quad \varphi_x(0, t) = \varphi_x(1, t) = 0.$$
Using (4.3), the right boundary condition in the \( p, q \) variables becomes 
\[ p(1,t) - q(1,t) = 2P(t), \]
or by integrating in time
\[ p(1,t) = q(1,t) + p_0(1) - q_0(1) + \int_0^t 2P(s) \, ds. \]

Thus, the only change in the integral equations is in the \( p \)-equation (4.6a) in which
\[ 2\Lambda - q_0(2 - x - t) \]
is replaced by
\[ \int_0^{x+t-1} 2P(s) \, ds + p_0(1) - q_0(1) + q_0(2 - x - t), \]
with the corresponding change in the compatibility conditions. However, the essential estimates are identical to the fixed boundary case.

In subsequent chapters, we will also consider the dynamics of the system when
the volume fraction \( \alpha(\varphi) \) is strictly increasing with \( \alpha(\varphi) \to 0 \) as \( \varphi \to -\infty \), \( \alpha(\varphi) \to 1 \) as \( \varphi \to \infty \), \( \alpha(0) \) small, and \( \alpha(1) \) near 1. The existence proof for that case is essentially the same as the one above with a few minor modifications of the estimates since \( g(0, p_x, q_x) \neq 0 \).
Chapter 5

Equilibrium States and Dynamics of the Phase-Field Model

5.1 Introduction

Due to the dependence of the energy (4.19) on the gradient of the order parameter, the phase-field system (4.2) is a regularized model of phase transformation. Smooth solutions exist, as the existence theory of Chapter 4 shows, and the transitions between phases are characterized not by sharp interfaces but by thin layers in which the order parameter and the strain have large gradients. However, this regularization is fundamentally different from the viscoelastic models discussed in previous chapters. Energy dissipation occurs throughout a viscoelastic bar, and in the absence of capillarity, phase boundaries do not move, and energy is not minimized globally (Theorems 4.1 and 4.10 in Ball et.al. [1991]). In the presence of both viscosity and (small) capillarity effects, energy is dissipated due to both work against viscous forces and the movement of phase boundaries. It appears that solutions can minimize energy globally via the slow motion of phase boundaries,
In contrast, the energy inequality (4.20) suggests that energy in the phase-field system is dissipated primarily through the movement of phase boundaries which need not be slow. The field equation (4.2a) is a forced linear wave equation, 
\[ u_{tt} - u_{xx} = \alpha'(\varphi) \varphi_x, \]
which permits the propagation of undamped acoustic waves. The interaction between the phase boundaries and acoustic disturbances can be complex, and one dynamical question of interest is whether this dissipation mechanism is sufficient to stabilize the system. For example, suppose that there exists an equilibrium with a single phase boundary, and that an acoustic wave pulse is sent in from one of the pure phases to interact with it. Besides the movement of the transition layer, one expects to find both transmitted and reflected waves which continue to propagate in the bar. We would like to know whether there is enough dissipation to damp out these disturbances and re-equilibrate the system. In the next chapter, we explore this question in detail by numerical simulation. Similar issues are studied for systems featuring a maximum dissipation criterion in Lin and Pence [1992].

In this chapter, we study analytically the dynamics of the phase-field model with a double-well exchange energy, 
\[ f(\varphi) = \mu \varphi^2 (1 - \varphi)^2 / 2, \]

\[ u_{tt} = (u_x - \alpha(\varphi))_x, \]

\[ \beta \varphi_t = \gamma \varphi_{xx} + \mu \varphi(\varphi - 1)(1 - 2\varphi) + \alpha'(\varphi)(u_x - \frac{1}{2}), \quad (5.1) \]

\[ u(0, t) = 0, \quad u(1, t) = \Lambda, \quad \varphi_x(0, t) = \varphi_x(1, t) = 0, \]

\[ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) \quad \text{and} \quad \varphi(x, 0) = \varphi_0(x). \]

However, we first clarify our assumptions concerning the volume fraction \( \alpha(\varphi) \) which plays an important role in the nature of the dynamics. We will assume that
\( \alpha(\varphi) \in C^3(\mathbb{R}) \) and has one of the following two forms:

(H1): For some \( 0 < a < b < 1 \), \( \alpha(\varphi) \equiv 0 \) for \( \varphi \in (-\infty, a] \), \( \alpha(\varphi) \equiv 1 \) for \( \varphi \in [b, \infty) \), and \( \alpha(\varphi) \) is strictly increasing on \([a, b] \).

(H2): \( \alpha(\varphi) \) is strictly increasing on \( \mathbb{R} \), with \( \alpha(\varphi) \to 0 \) as \( \varphi \to -\infty \), \( \alpha(\varphi) \to 1 \) as \( \varphi \to \infty \), with \( \alpha(0), \alpha'(0), \alpha'(1) \) small, and \( \alpha(1) \) near 1.

For convenience, we also require \( \alpha(\varphi) \) to be odd about the point \( (\varphi, \alpha(\varphi)) = (1/2, 1/2) \). We will think of (H1) as a limiting case of (H2) as \( \alpha(0) \to 0 \) and \( \alpha(1) \to 1 \).

It will also be convenient to make a simple change of variables, \( w(x, t) = u(x, t) - \Lambda x \), so that the boundary conditions are all homogeneous. In these variables, the phase-field system (5.1) becomes

\[
\begin{align*}
w_t &= (w_x - \alpha(\varphi))_x, \\
\beta \varphi_t &= \gamma \varphi_{xx} + \mu \varphi(\varphi - 1)(1 - 2\varphi) + \alpha'(\varphi)(w_x + \Lambda - \frac{1}{2}), \\
w(0, t) &= w(1, t) = \varphi_x(0, t) = \varphi_x(1, t) = 0, \\
w(x, 0) &= u_0(x) - \Lambda x, \quad w_t(x, 0) = u_1(x) \quad \text{and} \quad \varphi(x, 0) = \varphi_0(x).
\end{align*}
\]

In the analysis below, we will switch between the \((u, \varphi)\) and \((w, \varphi)\) variables whenever necessary.

The phase-field system also has several parameters, \( \beta, \gamma, \) and \( \mu \), which reflect the constitutive properties of the model. To focus our investigation, we will sometimes restrict our attention to what is perhaps the most physically relevant situation in which the above parameters are scaled to formally agree with the sharp interface theory of Gurtin and Struthers [1990] and Abeyratne and Knowles [1990].
In this scaling introduced by Fried and Gurtin [1993], equation (5.1) becomes

\[ \varepsilon \tilde{\beta} \varphi_t = \varepsilon \tilde{\gamma} \varphi_{xx} + \frac{\tilde{\mu}}{\varepsilon} \varphi(\varphi - 1)(1 - 2\varphi) + \alpha'(\varphi)(u_x - \frac{1}{2}), \]  

(5.3)

where \( \tilde{\beta} \), \( \tilde{\gamma} \), and \( \tilde{\mu} \) are \( O(1) \) as \( \varepsilon \to 0 \). This scaling determines the relative importance of the terms involving the phase variable.

The phase-field system can be formulated as an abstract semilinear parabolic equation on the Hilbert space \( X = H^1_0 \times L^2 \times H^1 \) by writing (5.2) as a first-order system of the form \( z_t + Az = F(z) \),

\[
\begin{pmatrix}
  w_t \\
  v_t \\
  \varphi_t
\end{pmatrix} +
\begin{pmatrix}
  0 & -I & 0 \\
  -\Delta_D & 0 & 0 \\
  0 & 0 & -\frac{\tilde{\gamma}}{3} \Delta_N
\end{pmatrix}
\begin{pmatrix}
  w \\
  v \\
  \varphi
\end{pmatrix} =
\begin{pmatrix}
  0 \\
  -\alpha(\varphi)_x \\
  -\frac{1}{3} f''(\varphi) + \frac{1}{3\beta} \alpha'(\varphi)(w_x + \Lambda - \frac{1}{2})
\end{pmatrix},
\]

and we will need to consider mild solutions, i.e. functions, \( z(t) \in C([0, T], X) \) which satisfy the integral equation obtained from the variation of constants formula,

\[ z(t) = e^{-At}z_0 + \int_0^t e^{-A(t-s)} F'(z(s)) \, ds. \]  

(5.4)

The local existence theory of Chapter 4 shows that a mild solution exists locally for every \( z_0 \in X \). Since strong solutions are bounded (using the Lyapunov function) and dense in \( X \) (cf. Theorem 4.2.7 of Pazy [1983]), the mild solutions are also bounded and hence globally defined. Note that the operator \( A \) with \( D(A) = D(-\Delta_D) \times H^1_0 \times D(-\Delta_N) \subset X \) generates a strongly continuous semigroup on \( X \) which is the direct sum of the group generated by the wave operator on \( H^1_0 \times L^2 \) and the analytic semigroup generated by the Laplacian on \( L^2 \). In the above notation, \( D(\Delta_D) = H^2 \cap H^1_0 \) is the domain of the Dirichlet Laplacian, and \( D(\Delta_N) = H^2 \cap \{ \varphi \in H^2 : \varphi_x(0) = \varphi_x(1) = 0 \} \) is the domain of the Laplacian with Neumann boundary conditions. The nonlinear function \( F' : X \to Y = H^1_0 \times L^2 \times L^2 \) is locally Lipschitz continuous.
5.2 Equilibrium States

Any stationary solution \((u, \varphi)\) of the phase-field system (5.1) must satisfy \(u_t = (u_x - \alpha(\varphi))_x = 0\) which implies that \(u_x = \alpha(\varphi) + P\) where \(P\) is the constant load applied to the ends of the bar to maintain its deformed length, \(L = 1 + \Lambda\).

Substituting into (5.1), we obtain a second-order equation in \(\varphi\) depending on \(P\)

\[
\gamma \varphi_{xx} + \mu \varphi (\varphi - 1)(1 - 2\varphi) + \alpha'(\varphi)(\alpha(\varphi) + P - \frac{1}{2}) = 0.
\]  

(5.5)

This equation has a first integral

\[
H_P(\varphi, \varphi_x) = \frac{1}{2} (\gamma \varphi_x^2 - \mu \varphi^2 (1 - \varphi^2) + \alpha(\varphi)(\alpha(\varphi) + 2P - 1)).
\]  

(5.6)

Simple phase plane methods show that only periodic orbits of (5.5) can satisfy both boundary conditions. Thus, the equilibria of the phase-field system are periodic orbits of (5.5) for which some integer multiple of the half-period equals one and which satisfy the conditions

\[
\varphi_x(0) = \varphi_x(1) = 0 \quad \text{and} \quad \Lambda = \int_0^1 \alpha(\varphi) \, dx + P.
\]  

(5.7)

For a fixed value of the displacement \(\Lambda\), the number of equilibria, with the stress \(P\) varying, depends on \(\gamma, \mu\), and the form of the volume fraction \(\alpha(\varphi)\), but for fixed \(\Lambda\) and \(P\) there can be only finitely many.

We begin with the equilibria for which the phase variable \(\varphi\) is constant. For all such stationary solutions, the strain is also constant, and hence \(u(x) = \Lambda x\).

In the (H1) case, \(\varphi \equiv 0\) and \(\varphi \equiv 1\) are solutions since \(\alpha'(0) = \alpha'(1) = 0\). Due to the form of \(\alpha(\varphi)\), there is one other equilibrium for which \(\varphi \equiv \hat{\varphi}\) is constant with \(a < \hat{\varphi} < b\) which satisfies \(f'(\hat{\varphi}) = \mu \hat{\varphi}(1 - \hat{\varphi})(1 - 2\hat{\varphi}) = \alpha'(\hat{\varphi})(\Lambda - 1/2)\). In particular, if \(\Lambda = 1/2\), then \(\hat{\varphi} = 1/2\). In the (H2) case, there are also three constant
phase equilibria which are close to those above. We will call the equilibria near \( \varphi \equiv 0 \) and \( \varphi \equiv 1 \) the outer equilibria, and the other one will be referred to as the middle equilibria.

**Lemma 5.2.1** If \( \alpha(\varphi) \) satisfies the hypothesis \((H1)\), then the outer constant phase equilibria \((w, v, \varphi) \equiv (0, 0, 0) \) and \((0, 0, 1) \) are stable in \( X \) but not asymptotically stable.

**Proof:** Let \( \epsilon > 0 \), and consider the equilibrium \((0, 0, 0)\); the other case is similar. Let \((w, v, \varphi)(t)\) be the solution through \((w_0, v_0, \varphi_0) \subseteq D(A)\). If \( \|\varphi_0\|_\infty < a \), then \( \alpha'(\varphi(t)) \equiv 0 \) on some time interval \( 0 \leq t < T \), and the equations decouple so that

\[
\begin{align*}
w_{tt} &= w_{xx}, \\
\beta \varphi_t &= \gamma \varphi_{xx} + \mu \varphi(\varphi - 1)(1 - 2\varphi).
\end{align*}
\] (5.8)

It is well-known that the trivial solution is asymptotically stable for the semilinear parabolic equation \((5.8\, b)\), (cf. Smoller [1983]). Therefore, we can choose a \( \delta_1 > 0 \) such that \( \|\varphi_0\|_1 < \delta_1 \) implies that \( \|\varphi(t)\|_\infty \leq \|\varphi(t)\|_1 < a \) for all \( t \geq 0 \). Since \((w, v)\) solves the linear wave equation \((5.8\, a)\), the “mechanical energy” is preserved, i.e.

\[
\|w_x(t)\|^2 + \|v(t)\|^2 = \|w'_0\|^2 + \|v_0\|^2,
\]

for all \( t \geq 0 \). Finally, let \( \delta = \min\{\delta_1, \epsilon/\sqrt{3}\} \). Then, for \( \|(w_0, v_0, \varphi_0)\|_X < \delta \), we have

\[
\|(w, v, \varphi)(t)\|_X^2 = \|w_x(t)\|^2 + \|v(t)\|^2 + \|\varphi(t)\|^2_1
\]

\[
= \|w'_0\|^2 + \|v_0\|^2 + \|\varphi(t)\|^2_1 < \epsilon^2.
\]

This proves stability in \( X \).

For every \( \epsilon > 0 \), we can find a nontrivial solution \((\hat{w}, \hat{v})(t)\) to the linear wave equation with \( \|\hat{w}_x(t)\|^2 + \|\hat{v}(t)\|^2 = \epsilon^2 \) for all \( t \geq 0 \). Therefore, \((\hat{w}, \hat{v}, 0)\) is a periodic
solution to (5.2) with \(\|\hat{w}(t), \hat{v}(t), 0\|_X = \epsilon\) for all \(t \geq 0\), and hence \((0, 0, 0)\) is not asymptotically stable. \(\square\)

In the (H2) case, the stability of the outer constant phase equilibria is not as easily established, but we can prove linear stability by considering the system (5.2) linearized about such an equilibrium, \(\varphi(x) \equiv \psi\),

\[
\begin{align*}
  w_{tt} &= w_{xx} - (\alpha'(\psi)\varphi)_x, \\
  \beta \varphi_t &= \gamma \varphi_{xx} - f''(\psi)\varphi + \alpha''(\psi)(\alpha(\psi) + P + \Lambda - \frac{1}{2})\varphi + \alpha'(\psi)w_x,
\end{align*}
\]

with homogeneous boundary conditions. The existence theory of Chapter 4 with some modification of the estimates provides the local existence of solutions to the linearized system for any equilibrium \(\psi(x)\).

**Lemma 5.2.2** If \(\alpha(\varphi)\) satisfies the condition (H2), then the outer constant phase equilibria are linearly stable.

**Proof:** A simple computation shows that the functional

\[
V[w, \varphi] = \frac{1}{2} \int_0^1 \left( w_t^2 + (w_x - \alpha'(\psi)\varphi)^2 + h(\psi)\varphi^2 + \gamma \varphi_x^2 \right) dx,
\]

is a Lyapunov functional for (5.9) where \(h(\psi) = f''(\psi) - \alpha''(\psi)(\alpha(\psi) + P + \Lambda - (1/2)) + (\alpha'(\psi))^2\) and

\[
\frac{d}{dt} V[w, \varphi] = -\int_0^1 \beta \varphi_x^2 dx.
\]

Therefore, solutions to (5.9) exist globally. Since \(\alpha'(\psi)\) and \(\alpha''(\psi)\) are small and \(f''(\psi) > 0\), we see that \(h(\psi) > 0\) and \(V\) is positive definite. Therefore, the zero solution is stable in \(X = H^1_0 \times L^2 \times H^1\), and hence the outer constant phase equilibria are linearly stable. \(\square\)

To determine the bifurcation and stability properties of arbitrary equilibria, we now study the linearization of (5.2) more closely. Let \((\omega(x), 0, \psi(x))\) be any
equilibrium of (5.2). Then \( \omega(x) = \alpha(\psi(x)) + P \) for some constant \( P \), and the linearization at \((\omega(x), 0, \psi(x))\), \( L = A - DF : D(A) \to Y = H_0^1 \times L^2 \times L^2 \), can be expressed completely in terms of \( \psi(x) \) as

\[
L = \begin{pmatrix}
0 & -I & 0 \\
-\Delta_D & 0 & \alpha'(\psi) \frac{\partial}{\partial x} + \alpha'(\psi) I + \alpha''(\psi) \frac{\partial^2}{\partial x^2} \\
-\frac{1}{\beta} \alpha'(\psi) \frac{\partial}{\partial x} & 0 & -\frac{2}{\beta} \Delta_N + \frac{1}{\beta} f''(\psi) I - \frac{1}{\beta} \alpha''(\psi)(\alpha(\psi) + \Lambda + P - \frac{1}{2}) I
\end{pmatrix}.
\]

**Lemma 5.2.3** \( L \) has no essential spectrum, i.e. \( \sigma(L) \) contains only isolated eigenvalues of finite multiplicity, provided that the resolvent set \( \rho(L) \) is non-empty.

**Proof:** We write \( L = A + B \) with

\[
A = \begin{pmatrix}
0 & -I & 0 \\
-\Delta_D & 0 & 0 \\
0 & 0 & -\frac{2}{\beta} \Delta_N + I
\end{pmatrix},
\]

and

\[
B = -DF = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \alpha'(\psi) \frac{\partial}{\partial x} + \alpha'(\psi) I + \alpha''(\psi) \frac{\partial^2}{\partial x^2} \\
-\frac{1}{\beta} \alpha'(\psi) \frac{\partial}{\partial x} & 0 & R(\psi) I
\end{pmatrix},
\]

where \( R(\psi) = f''(\psi)/\beta - \alpha''(\psi)(\alpha(\psi) + \Lambda + P - 1/2)/\beta - 1. \)

In this case, \( A \) and \( B \) are unbounded linear operators on \( Y \) with \( D(A) \subset D(B) \), and \( A \) is closed and invertible with a compact inverse

\[
A^{-1} = \begin{pmatrix}
0 & -\Delta_D^{-1} & 0 \\
-1 & 0 & 0 \\
0 & 0 & (-\frac{2}{\beta} \Delta_N + I)^{-1}
\end{pmatrix}.
\]

So, we compute
\[ BA^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha'(\psi)(-\frac{\gamma}{\beta} \Delta_N + I)^{-1} + \alpha'(\psi) \frac{\partial}{\partial x}(-\frac{\gamma}{\beta} \Delta_N + I)^{-1} \\ 0 & \frac{1}{\beta} \alpha'(\psi) \frac{\partial}{\partial x} \Delta_D^{-1} & R(\psi)(-\frac{\gamma}{\beta} \Delta_N + I)^{-1} \end{pmatrix}. \]

Since the maps \( \frac{\partial}{\partial x}(-\frac{\gamma}{\beta} \Delta_N + I)^{-1} : L^2 \to H^1 \) and \( \frac{\partial}{\partial x}(-\Delta_D)^{-1} : L^2 \to H^1_0 \) are continuous, they are compact as maps from \( L^2 \) to itself by Rellich's lemma. Also, the factors \( \alpha'(\psi)x, \alpha'(\psi), \) and \( R(\psi) \) are all bounded in \( L^2 \). Therefore, \( BA^{-1} : Y \to Y \) is compact. Applying Theorem A.1 of Henry [1981], we conclude that either \( L \) has no essential spectrum or every point in \( C \) is an eigenvalue, since \( \sigma(A) \) has only isolated eigenvalues. \( \square \)

As a practical matter, computing the eigenvalues of \( L \) for a non-constant equilibrium seems difficult; however, we can gain some information about the constant equilibria and the local bifurcations near them. If \((\omega, 0, \psi)\) is a constant equilibrium for which \( \alpha'(\psi) > 0 \) (the case \( \alpha'(\psi) = 0 \) is Lemma 5.2.1), it is easy to see that for each \( n \geq 0 \), the vector \((A \sin n\pi x, B \sin n\pi x, C \cos n\pi x) \in L^2 \times L^2 \times L^2 \) is an eigenvector of \( L \) whenever \((A, B, C) \in C^3 \) is an eigenvector of the \( 3 \times 3 \) matrix

\[
M = \begin{pmatrix} 0 & -1 & 0 \\ -\beta^2 \alpha'(\psi) n \pi & 0 & -\alpha'(\psi) n \pi \\ -\frac{1}{\beta} \alpha'(\psi) n \pi & 0 & \frac{\gamma}{\beta} n^2 \pi^2 + \frac{1}{\beta} \delta \end{pmatrix},
\]

where \( \delta = f''(\psi) - \alpha''(\psi)(\Lambda - \frac{1}{2}) \). The corresponding eigenvalues \( \lambda \) satisfy the cubic equation

\[
(\lambda^2 + n^2 \pi^2)(-\lambda + \gamma n^2 \pi^2 - \frac{\delta}{\beta}) - \frac{1}{\beta} \alpha'(\psi)^2 n^2 \pi^2 = 0. \tag{5.10}
\]

**Lemma 5.2.4** The middle constant phase equilibrium is unstable in both the (H1) and (H2) cases.
Proof: For the middle constant phase equilibrium, \( \delta < 0 \), and hence \((0,0,1)\) is an eigenvector of \( L \) with eigenvalue \( \lambda = \delta/\beta < 0 \). Therefore, this equilibrium has a growing mode and is unstable. □

Using the Routh-Hurwitz test, it is a simple calculation to show that for the outer equilibria, all of the eigenvalues which are solutions of equation (5.10) have positive real parts. We would like to use this information about the spectrum of \( L \) to prove the asymptotic stability of the two outer equilibria in the (H2) case and to study the bifurcations from the middle constant phase equilibrium in both the (H1) and (H2) cases. However, as noted in Chapter 1, spectral stability does not imply stability in general, and the eigenvectors found above do not form a Riesz basis of \( L^2 \times L^2 \times L^2 \). Therefore, we have not been able to do a complete stability and local bifurcation analysis of the constant equilibria via linearization, but we do discuss the bifurcation of symmetric equilibria for a particular choice of parameters in §6.2.

5.3 Wave Ringing

We now consider a special class of time-dependent solutions which appear in numerical simulations to have a profound effect on the dynamics of the phase-field system. In the case that \( \alpha(\varphi) \) satisfies the (H1) condition, there exist periodic solutions to (5.2) due to the decoupling of the system on intervals where \( \alpha'(\varphi) \equiv 0 \), i.e. where \( \varphi \leq a \) or \( \varphi \geq b \). Simple examples of such periodic solutions are those constructed in the proof of Lemma 5.2.1 which have the form \((w,v,\varphi) = (w(x,t), w_t(x,t), 0)\) or \((w(x,t), w_t(x,t), 1)\) where \( w(x,t) \) is any solution to the wave equation, \( w_{tt} = w_{xx} \). In this case, the solution remains in a single phase, and undamped linear acoustic waves propagate in the bar.
In general, we can construct periodic solutions starting with any equilibrium which has intervals on which \( \alpha'(\varphi) \equiv 0 \), as follows. Suppose \((\omega(x), 0, \psi(x))\) is an equilibrium of (5.2) with \( \alpha'(\psi) \equiv 0 \) on the disjoint intervals \( I_1, I_2, \ldots, I_n \) in \([0, 1]\). On each \( I_k \), let \( w^k \) be a standing wave solution to the linear wave equation on \( I_k \) with boundary conditions \( w_x^k = 0 \) at the endpoints of \( I_k \), except when the left endpoint of \( I_1 \) or the right endpoint of \( I_n \) coincide with an end of the bar in which case the boundary conditions are \( w^1(0) = 0 \) or \( w^n(1) = 0 \). We would like to superpose the functions \( w^k(x, t) \) with the original equilibrium \((\omega(x), 0, \psi(x))\), but to maintain the fixed total displacement of the bar, we cannot choose the functions \( w^k(x, t) \) arbitrarily. The condition

\[
\sum_{k=1}^{n} \int_{I_k} w^k_x(x, t) \, dx \equiv 0 \quad \text{for all } t \geq 0
\]

must also be satisfied. Let

\[
w(x, t) = \int_{0}^{x} [\alpha(\psi(y)) + P + \sum_{k=1}^{n} w^k_x(y, t)] \, dy.
\]  (5.11)

Then, \((w(x, t), w_t(x, t), \psi(x))\) is a \( C^1 \) periodic function satisfying the boundary conditions \( w(0, t) = w(1, t) = 0 \). At the endpoints of \( I_k \), \( \omega_{xx} = 0 \), and so for \( w \) to be \( C^2 \), we must have \( w^k_{xx} = 0 \) at the endpoints of \( I_k \) as well. However, there are no standing wave solutions to the linear wave equation which satisfy all of these boundary conditions, and hence \( w(x, t) \) defined by equation (5.11) is not a classical solution to (5.2). If we change to the \((p, q, \varphi)\) coordinates introduced in Chapter 4, then the corresponding functions \((p(x, t), q(x, t), \varphi(x, t))\) are periodic solutions to the first-order system (4.3) in \( C^1 \times C^1 \times C^2 \). Therefore, these solutions are mild solutions to the phase-field system.

In these multiphase periodic solutions, standing acoustic waves vibrate between stationary phase boundaries. In the numerical simulations of the next chapter, we
will show that orbits of (5.1) appear to approach this type of ringing solution from certain initial data. In the (H2) case, one expects that orbits can approach a similar ringing solution with stationary phase boundaries, but that then the acoustic vibrations would be slowly damped out at a rate depending on the form of the volume fraction $\alpha(\varphi)$.

5.4 Gradient Dynamics

As in the case of the viscoelastic model in Chapter 3, we would like to apply the results of Hale [1988] concerning gradient dynamical systems to study the asymptotic behavior of solutions to the phase-field system as $t \to \infty$. As indicated previously, an essential ingredient in the analysis of such systems is the asymptotic smoothness of the solution semigroup and the precompactness of individual orbits. Because of the strong coupling in the phase-field equations, we have not been able to prove the precompactness of orbits and have only partial results about their asymptotic behavior in the strong topology. However, following the approach of Ball and Slemrod [1979] and Artstein and Slemrod [1982], we can analyze the asymptotic behavior of solutions in the weak topology.

First, we show that the equilibrium set $\mathcal{E} = \{(w, \varphi) \in (H^2 \cap H^1_0) \times H^1 : (w(t), 0, \varphi(t)) \equiv (w, 0, \varphi) \text{ for all } t \geq 0\}$ is bounded by arguing as in §3.2 that the equilibria must be critical points of the potential energy.

**Lemma 5.4.1** The equilibrium set $\mathcal{E}$ is bounded in $\hat{X} = (H^2 \cap H^1_0) \times H^1$.

**Proof:** Consider the potential energy,

$$V[w, \varphi] = \frac{1}{2} \int_0^1 \left[ (w_x + \Lambda - \alpha(\varphi))^2 + H(\varphi) + \gamma \varphi_x^2 \right] \, dx,$$
where $H(\varphi) = \alpha(\varphi)(1 - \alpha(\varphi)) + 2f(\varphi)$. Any equilibrium $(w, \varphi) \in \mathcal{E}$ must be a critical point of $V$ and so must satisfy

$$
\int_0^1 \left[(w_x + \Lambda - \alpha(\varphi))(v_x - \alpha'(\varphi)\psi) + \frac{1}{2}H'(\varphi)\psi + \gamma \varphi_x \psi_x \right] dx = 0 \quad \forall (v, \psi) \in \hat{X}.
$$

In particular, we take $(v, \psi) = (w, \varphi)$. Recall that for equilibria, we have $w_x = \alpha(\varphi) - \int_0^1 \alpha(\varphi) dx$. Substituting into the above equation and rearranging terms, we obtain

$$
\int_0^1 \left[\mu \varphi^2 + \gamma \varphi_x^2 \right] dx = -\int_0^1 w_x^2 dx + \int_0^1 g(\varphi)\varphi dx + \int_0^1 h(\varphi) dx - \int_0^1 \mu \varphi^2(2\varphi^2 - 3\varphi) dx,
$$

where $g(\varphi) = \alpha'(\varphi)(\Lambda - \int_0^1 \alpha(\varphi) dx - 1 + 2\alpha(\varphi))$ and $h(\varphi) = (\Lambda - \alpha(\varphi))(\alpha(\varphi) - \int_0^1 \alpha(\varphi) dx)$. Since $g(\varphi)$ and $h(\varphi)$ are uniformly bounded, for every $\epsilon > 0$ we can find an $M > 0$ such that $|\varphi| \geq M$ implies $|g(\varphi)/\varphi| < \epsilon$ and $2\varphi^2 - 3\varphi > 0$. Let $J_1 = \{ x : |\varphi(x)| \geq M \}$ and $J_2 = [0, 1] \setminus J_1$. Then, we have

$$
\int_0^1 \left[\mu \varphi^2 + \gamma \varphi_x^2 \right] dx \leq \int_{J_1} \left| \frac{g(\varphi)}{\varphi} \right| \varphi^2 dx + \int_{J_2} |g(\varphi)| \varphi dx
$$

$$
+ \int_0^1 |h(\varphi)| dx + \int_{J_2} |\mu \varphi^2(2\varphi^2 - 3\varphi)| dx
$$

$$
\mu \|\varphi\|^2 + \gamma \|\varphi_x\|^2 \leq \epsilon \|\varphi\|^2 + C.
$$

Therefore, $\|\varphi\|^2_1 \leq C/(\delta - \epsilon)$ where $\delta = \min(\mu, \gamma)$ and we choose $\epsilon < \delta$. Hence $\varphi$ is bounded in $H^1$. Also, from the relation, $w_x = \alpha(\varphi) + P$, we see that $w$ is bounded in $H^2 \cap H^1_0$. \hfill \Box

Because the phase-field equation (5.2b) is parabolic, the phase variable $\varphi$ is well-behaved as described in the following lemma.
Lemma 5.4.2 Any strong solution \( z(t) = (w(t), v(t), \varphi(t)) \), has the following properties,

1. \( \|\varphi\|_1 \) is uniformly bounded, and \( \{ \varphi(t) : t > 0 \} \) is precompact in \( H^1 \),
2. \( \|\varphi_t\|_1 \) is uniformly bounded, and hence \( \{ \varphi(t) : t > 0 \} \) is precompact in \( L^2 \),
3. for any sequence \( t_n \to \infty \), there is subsequence \( t_{n_k} \) and a function \( \varphi(t) \in C([0, \infty, H^1]) \) such that \( \varphi(t_{n_k} + t) \to \varphi(t) \) in \( H^1 \) as \( k \to \infty \) uniformly on compact subsets of \( [0, \infty) \).

Notation: In the rest of this this chapter, for any function \( h(t) \) and any sequence \( t_n \), we will use the notation \( h^n(t) = h(t_n + t) \).

Proof: The parabolic part of the phase-field system (5.2b) can be written as \( \varphi_t + B\varphi = G(\varphi, t) \) where \( G(\varphi, t) = (1/\beta)[-\varphi + \mu \varphi(\varphi - 1)(1 - 2\varphi) + \alpha(\varphi)(w_x(t) + \Lambda - 1/2)] \), and \( B = (1/\beta)[-\gamma \Delta \varphi + \varphi] \). Throughout the rest of this chapter, we will use this notation (i.e. \( B, G, \) and \( \varphi \)) to refer to just the parabolic part of the phase-field equations. When referring to the entire system, we will use the notation at the end of §5.1 (i.e. \( A, F, \) and \( z \)). We showed in Lemma 4.4.1 that \( \varphi(t) \) and \( G(\varphi(t), t) \) are uniformly bounded in \( H^1 \) and \( L^2 \) respectively using the Lyapunov functional. Since \( B \) has a compact resolvent and is sectorial, \( e^{-Bt} \) is a compact semigroup (cf. Lemma 4.2.3 of Hale [1988]). The precompactness of \( \{ \varphi(t) : t > 0 \} \) follows from the boundedness of \( \{ G(\varphi(t), t) : t > 0 \} \) (cf. Lemma 8.2.4 of Pazy [1983]).

To prove part (2) we argue exactly as in Lemma 3.2.1. For any \( T > 0 \), there exists a constant \( C(T) \) such that for any \( t, t_0 \) with \( 0 < t - t_0 < T \), the following estimate holds,

\[
\| \varphi_t(t) \|_1 \leq C(T) \left[ (t - t_0)^{-1} \| \varphi(t_0) \|_1 + (t - t_0)^{-\frac{1}{2}} \sup_{t_0 < s < t_0 + T} \| G(\varphi(s), s) \| \right].
\]

Here, we use the fact that, for strong solutions \( w_x(t) \) is Hölder continuous into
$L^2$. Therefore, setting $t - t_0 = 1$ and $T = 2$, part (2) follows from the uniform boundedness of $z(t)$ in $X$ (Lemma 4.4.1). Precompactness in $L^2$ follows from Rellich’s lemma.

Finally, to show part (3), we can choose a sequence $t_n \to \infty$ such that $\varphi(t_n) \to \bar{\varphi}$ in $H^1$. For $t \in [0, \infty)$, the functions $\{\varphi^n(t)\}$ are uniformly bounded in $H^1$ by part (1). We claim that for any $T > 0$, this sequence is also equicontinuous on $[0, T]$. In particular, $\varphi^n(t)$ satisfies the integral equation

$$\varphi^n(t) = e^{-Bt}\varphi(t_n) + \int_0^t e^{-B(t-s)}G(\varphi^n(s), w^n_x(s))\,ds,$$

which we can estimate using the standard estimates for sectorial operators (cf. Lemma 3.3.2 and Theorem 3.3.4 of Henry [1981]). Therefore,

$$\|\varphi^n(t + h) - \varphi^n(t)\|_1 \leq \|(e^{-Bh} - I)e^{-Bt}\varphi^n(t)\|_1 + \int_0^h \|e^{-B(t+h-s)}G(\varphi^n(s), w^n_x(s))\|_1\,ds \tag{5.12}$$

$$\leq C(T)h^{\delta}(\sup_{\tau}\|\varphi(\tau)\|_{1+2\delta} + \sup_{\tau}\|G(\varphi(\tau), w_x(\tau))\|)$$

$$\leq C(T)h^{\delta},$$

for any $\delta < 1/2$, and $C(T)$ is independent of $n$. Hence $\{\varphi^n(t)\}$ are equicontinuous on $[0, T]$, and applying the Arzela-Ascoli theorem, we obtain (3).

Since $E$ is a Lyapunov function, by the LaSalle invariance principle, we can characterize the $\omega$-limit sets of orbits in terms of the largest invariant set, $M$, contained in $\{z = (w, v, \varphi) \in X : \dot{E}(z) = 0\}$.

**Lemma 5.4.3** For any orbit, $\omega(z_0) \subset M$.

**Proof:** The argument is a standard one and is essentially the same as the finite-dimensional case. For completeness, we will give a brief sketch (cf. Lemma 3.8.2
of Hale [1988]).

Choose a sequence \( t_n \) such that \( t_n - t_{n-1} \geq 1 \) and \( z(t_n, z_0) \to \bar{z} \). Then, for all \( t \in (0, 1) \), we have

\[
E[z(t_n, z_0)] \leq E[z(t_{n-1} + t, z_0)] \leq E[z(t_{n-1}, z_0)].
\]

Therefore, by the continuity of \( E \), we see that

\[
E[z(t_{n-1} + t, z_0)] \to E[\bar{z}] \quad \text{as} \quad n \to \infty \quad \text{for all} \quad t \in [0, 1].
\]

Therefore, by the continuity of the solutions, \( \dot{E}(\bar{z}) = 0. \quad \Box \)

When the volume fraction \( \alpha(\varphi) \) satisfies the condition (H2), the largest invariant set \( M \) in \( \{ \dot{E} = 0 \} \) is the equilibrium set \( \mathcal{E} \). In the (H1) case, \( M \) also contains periodic, ringing solutions as described in §5.3. The former assertion follows easily from the fact that \( \dot{E} = 0 \) implies that \( \varphi_t = 0 \). By solving for \( w_x \) in (5.2 b),

\[
w_x = \frac{1}{2} - \Lambda + \frac{\gamma \varphi_{xx} - f(\varphi)}{\alpha'(\varphi)},
\]

integrating with respect to \( x \), and differentiating in time, we also obtain \( w_t = 0 \).

However, the above results do not tell us that every orbit approaches an equilibrium in the (H2) case (or a periodic orbit in the (H1) case), because we have not shown that all orbits are precompact in \( X \), only that they are bounded, and the \( \omega \)-limit set may be empty.

The fact that orbits are bounded in \( X = H_0^1 \times L^2 \times H^1 \) leads naturally to the study of the asymptotic behavior of solutions in the weak topology on \( X \), because a set is weakly compact iff it is weakly closed and bounded. This approach is taken by Ball [1978], Ball and Slemrod [1979], and Artstein and Slemrod [1982], who investigate systems similar to ours in the sense that they lack sufficient smoothness of the semigroup or the nonlinearity to determine the behavior in the strong
topology. However, since the Lyapunov functional, $E$, is not weakly continuous, the invariance principle (Lemma 5.4.3) no longer applies, and different techniques must be used.

As in the introduction to this chapter, we write the phase-field system in the form $z_t + Az = F(z)$ where $z = (w, v, \varphi) \in X$. We consider the mild solutions to the system, i.e. solutions to the integral equation obtained from the variation of constants formula (5.4).

Note that strong solutions are also mild solutions. The existence theory of Chapter 4 shows that for every $z_0 \in X$, there exists a unique mild solution with initial data $z_0$ which is globally defined and bounded in $X$ and will be denoted by $z(t, z_0)$. Moreover, for every bounded set $B \subset X$, the set $\{z(t, z_0) : z_0 \in B, t \geq 0\}$ is bounded because of the existence of a Lyapunov functional. Also, $F : X \to Y = H_0^1 \times L^2 \times L^2$ is sequentially weakly continuous, i.e. $z_n \to z$ in $X$ implies $F(z_n) \to F(z)$ in $Y$ as $n \to \infty$. By a modification of the proof of Theorem 2.3 of Ball and Slemrod [1979] to incorporate standard estimates of fractional powers of sectorial operators and using the estimate (5.12), the mild solutions are sequentially weakly continuous, i.e. $z_n \to z_0$ implies $z(t, z_n) \to z(t, z_0)$ in $X$.

The following lemma is an immediate consequence of the continuity of solutions in the weak topology and their boundedness (cf. Lemma 2.1 of Slemrod [1970]).

**Lemma 5.4.4** The weak $\omega$-limit set, $\omega_W(z_0)$, of any orbit, $z(t, z_0)$, is nonempty, invariant, sequentially weakly compact, and weakly connected.

The main difficulty in using this approach to study the asymptotic behavior of a system is determining which mild solutions are contained in the weak $\omega$-limit sets. In the following, we will call a solution to the integral equation (5.4) with initial data in $X$, a mild solution, and we will refer to a solution to the differential
equations (5.2) in $D(A)$ as a strong solution. One difficulty with the mild solutions is that they need not be Hölder continuous (strong solutions are Hölder continuous from the results of Chapter 4). In that case, the nonlinear term in the parabolic equation, $G(\varphi, t)$, is not Hölder continuous, and hence $\varphi_t$ may not exist in $H^1$. Therefore, we will only study the weak $\omega$-limits sets of strong solutions.

**Theorem 5.4.1** Let $z(t)$ be any strong solution and $\bar{z} \in \omega_W(z(t))$. Then, there exists a sequence $t_n \to \infty$ such that for each $t \geq 0$, $z^n(t) \to \bar{z}(t)$ weakly in $X$, and $\varphi^n(t) \to \bar{\varphi}(t)$ strongly in $H^1$, where $\bar{z}(t) = (\bar{w}(t), \bar{v}(t), \bar{\varphi}(t))$ denotes the mild solution with initial data $\bar{z}$. Moreover, $\varphi_t(t) \to 0$ in $L^2$ as $t \to \infty$, and for each $t > 0$, $\bar{\varphi}(t) \in D(B)$ with $B\bar{\varphi}(t) = G(\bar{\varphi}(t), \bar{w}_x(t))$, and $\varphi(t) \in C^1((0, \infty), L^2)$ with $\bar{\varphi}_t \equiv 0$ in $L^2$.

**Proof:** Many of the central ideas of this proof are contained in Ball [1977], Ball [1978], and Ball and Slemrod [1979].

By definition, there is a sequence $t_n \to \infty$ such that $z(t_n) \to \bar{z}$. From Lemma 5.4.2 and passing to a subsequence if necessary, we can choose $t_n$ such that $\varphi(t_n) \to \bar{\varphi}$ in $H^1$ and $\varphi_t(t_n) \to \psi$ in $L^2$. As stated above, $z^n(t) \to \bar{z}(t)$ for each $t \geq 0$ since $F$ is sequentially weakly continuous and $z(t)$ is uniformly bounded in $X$ (cf. Theorem 2.3 of Ball and Slemrod [1979]). Also, $\varphi^n(t) \to \bar{\varphi}(t)$ strongly in $H^1$ by Lemma 5.4.2.

Fix $t > 0$. Since $z(t)$ is a strong solution, $\varphi^n_t(t)$ satisfies the integral equation,

$$\varphi^n_t(t) = -B e^{-Bt} \varphi(t_n) + G(\varphi^n(t), w^n_x(t)) - \int_0^t B e^{-B(t-s)} G(\varphi^n(s), w^n_x(s)) \, ds.$$ 

So, for all $y \in D(B^*)$, we have

$$\langle \varphi^n_t(t), y \rangle = \langle -e^{-Bt} \varphi(t_n), B^* y \rangle + \langle G(\varphi^n(t), w^n_x(t)), y \rangle$$
\[
- \left( \int_0^t e^{-B(t-s)}G(\varphi^n(s), w^n_x(s)) \, ds, B^*y \right). \tag{5.13}
\]

Taking a subsequence if necessary, \( \varphi^n(t) \to \psi(t) \) for some \( \psi(t) \in L^2 \) by Lemma 5.4.2, and so passing to the limit in (5.13), we obtain
\[
\langle \psi(t), y \rangle = \langle -e^{-Bt} \bar{\varphi}, B^*y \rangle + \langle G(\bar{\varphi}(t), \bar{w}_x(t)), y \rangle
- \left( \int_0^t e^{-B(t-s)}G(\bar{\varphi}(s), \bar{w}_x(s)) \, ds, B^*y \right),
\]
by the dominated convergence theorem and the sequentially weak continuity of \( G \) and \( e^{-Bt} \). Therefore, by the lemma in Ball [1977], \( \int_0^t e^{-B(t-s)}G(\bar{\varphi}(s), \bar{w}_x(s)) \, ds \in D(B) \), and
\[
\psi(t) = -Be^{-Bt}\bar{\varphi} + G(\bar{\varphi}(t), \bar{w}_x(t)) + B \int_0^t e^{-B(t-s)}G(\bar{\varphi}(s), \bar{w}_x(s)) \, ds. \tag{5.14}
\]

Now, the right side of (5.13) converges without passing to a subsequence and so \( \varphi^n(t) \to \psi(t) \) in \( L^2 \) for each \( t \geq 0 \).

We claim that \( \psi(t) \equiv 0 \). Let \( T > 0 \). Since the norm is sequentially weakly lower semicontinuous, for \( t \in [0, T] \) we have
\[
\|\psi(t)\|^2 \leq \liminf_{n \to \infty} \|\varphi^n(t)\|^2.
\]
So, by Fatou's lemma, we obtain
\[
\int_0^T \|\psi(t)\|^2 \, dt \leq \liminf_{n \to \infty} \int_0^T \|\varphi^n(t)\|^2 \, dt
= \liminf_{n \to \infty} \int_{t_n}^{T+t_n} \|\varphi(t)\|^2 \, dt
= 0,
\]
The last equality follows from
\[
\beta \int_0^\infty \|\varphi(t)\|^2 \, dt = E(0) - E(t) < \infty,
\]
and hence $\psi(t) \equiv 0$ for all $t \geq 0$. The above argument shows that any sequence $t_n$ has a subsequence $t_{n_k}$ such that $\varphi(t_{n_k}) \to 0$ strongly in $L^2$ as $k \to \infty$, and hence $\varphi(t) \to 0$ as $t \to \infty$.

Since $\bar{\varphi}(t)$ satisfies the integral equation

$$\bar{\varphi}(t) = e^{-Bt}\bar{\varphi}(0) + \int_0^t e^{-B(t-s)}G(\bar{\varphi}(s), \bar{w}_x(s)) \, ds,$$

equation (5.14) implies that $\bar{\varphi} \in D(B)$ and $B\bar{\varphi}(t) = G(\bar{\varphi}(t), \bar{w}_x(t))$. By Theorem 4.2.4 of Pazy [1983], $\bar{\varphi}(t) \in C^1((0, \infty), L^2)$ is a strong solution (in $L^2$) to

$$\varphi_t = B\bar{\varphi} + G(\bar{\varphi}(t), \bar{w}_x(t)),$$

since $B$ is sectorial, and $G$ and $\bar{w}_x(t)$ are continuous. Therefore, we conclude that $\bar{\varphi}_t(t) \equiv 0$ in $L^2$. \qed

**Corollary 5.4.1** If the volume fraction $\alpha(\varphi)$ satisfies the condition (H2), then the weak $\omega$-limit set of any strong solution contains only equilibria. Moreover, if the equilibria are isolated, then every strong solution tends weakly to an equilibrium with the phase-field converging strongly.

**Proof:** From Theorem 5.4.1, every orbit in the weak $\omega$-limit set satisfies the equation $B\bar{\varphi} = G(\bar{\varphi}, \bar{w}_x(t))$ in $L^2$, or equivalently

$$\gamma \bar{\varphi}_{xx} = \mu \bar{\varphi} (\bar{\varphi} - 1)(1 - 2\bar{\varphi}) + \alpha'(\bar{\varphi})(\bar{w}_x(t)) + \Lambda - \frac{1}{2}.$$

Note that $\varphi(t) \equiv 0$ in $L^2$ so that $\bar{\varphi}(t)$ is independent of $t$. We claim that $\bar{w}_x(t)$ is also independent of $t$ by simply solving for $\bar{w}_x(t)$ in terms of $\bar{\varphi}$, since $\alpha'(\bar{\varphi}) > 0$. Since $\bar{w} \in C([0, \infty), \mathcal{H}_0^1) \cap C^1((0, \infty), L^2)$, $\bar{w}_t \equiv 0$, and $(\bar{w}, 0, \bar{\varphi})$ is an equilibrium. The corollary follows from Lemma 5.4.4. \qed
Remark: In case (H1), it is possible that while $\alpha'(\varphi) \neq 0$ does not hold everywhere, for some solution we do have $\alpha'(\varphi) > 0$ on the phase-fields of the weak $\omega$-limit set. In that case, Corollary 5.4.1 holds for that solution. See the numerical simulations of Chapter 6.

More generally, in the case that the condition (H1) is satisfied, the weak $\omega$-limit set may contain time-dependent solutions in which the phase-field is time-independent, and the displacement is a weak solution to $\bar{\omega}_{tt} = \bar{w}_{xx} - \alpha(\varphi)_x$. By taking $\omega = \bar{w} + \int_0^x \alpha(\varphi) \, dy$, $\omega$ is a weak solution of the wave equation $\omega_{tt} = \omega_{xx}$. Hence, if $\alpha'(\varphi(x)) = 0$ on some part of the unit interval, there exists a continuum of mild solutions which can be contained in the weak $\omega$-limit set. These solutions are all periodic in time with periods no greater than two, as demonstrated in §5.3.
Chapter 6

Numerical Simulations of the Phase-Field Model

6.1 Introduction

In this chapter, we illustrate numerically some of the dynamical behavior of the phase-field model discussed in the previous chapter. In particular, we investigate the interaction of acoustic waves and phase boundaries, the decay of energy and the stabilization of phase boundaries, and wave ringing. We experiment with the problem described in §5.1 in which an acoustic pulse is added to one of the pure phases of an equilibrium with a single phase boundary and study the re-stabilization of the system. Since there are many possible parameter regimes and choices for the volume fraction $\alpha(\varphi)$, we focus on a particular volume fraction for which we can find equilibria with a single phase boundary.

First, we describe the numerical integration method to be used. Recall the phase-field system (4.3) expressed in the $(p, q, \varphi)$ variables,

\[ p_t = p_x - \alpha(\varphi), \]
\[ q_i = -q_x + \alpha(\varphi), \quad (6.1) \]

\[ \beta \varphi_t = \gamma \varphi_{xx} - f'(\varphi) + \alpha'(\varphi)(p_x + q_x - 1/2), \]

with \( f(\varphi) = \mu \varphi^2(1 - \varphi^2)/2 \) and “hard” boundary conditions

\[ p(0) = -q(0) \quad \text{and} \quad p(1) = 2\Lambda - q(1), \]

and \( \varphi_x(0) = \varphi_x(1) = 0. \)

The numerical integration scheme we employ to solve (6.1) involves integrating the parabolic equation (6.1 c) by finite differences and then integrating the hyperbolic equations (6.1 a, b) along characteristics.

Let \( \{x_i : i = 0, \ldots, N\} \) be a regular mesh on \([0, 1]\) with mesh size \( h = 1/N \), and take the time step \( k = h \), (see below). Let \( F_i^n \) denote the approximate value of the function \( F \) at the grid point \((x_i, t_n) = (hi, kn)\), \( i = 0, \ldots, N, n \geq 0 \). The first step in the numerical integration of (6.1) is to apply an implicit finite difference scheme to the parabolic equation in \( \varphi \) using the current values of \( p \) and \( q \). We will use the Backward Euler method

\[
\varphi_i^{n+1} = k \varphi_i^n + \frac{k}{h^2} (\varphi_{i+1}^{n+1} - 2\varphi_i^{n+1} + \varphi_{i-1}^{n+1}) - k f'(\varphi_i^n) \]

\[ + \frac{k}{h} (\varphi_i^n) \left( \frac{1}{2h} (p_i^{n+1} - p_i^n - q_{i+1}^{n+1} - q_{i-1}^{n+1}) - \frac{1}{2} \right), \]

with \( \varphi_{-1} = \varphi_1^{n+1} \) and \( \varphi_{N+1}^{n+1} = \varphi_{N-1} \) to implement the homogeneous Neumann boundary conditions.

Then, to compute the new values of \( p \) and \( q \), we use the trapezoid rule to integrate along characteristics using the new \( \varphi \) values (see the integral equations (4.6 a, b)):

\[
p_i^{n+1} = p_i^n + \frac{1}{2h} (\alpha(\varphi_{i+1}) + \alpha(\varphi_i^{n+1})) \quad \text{for} \quad i = 0, \ldots, N - 1, \]

\[
q_i^{n+1} = q_i^n + \frac{1}{2h} (\alpha(\varphi_{i-1}) + \alpha(\varphi_i^{n+1})) \quad \text{for} \quad i = 1, \ldots, N, \]
and the reflection boundary conditions give $p(N) = 2\Lambda - q(N)$ and $q(0) = -p(0)$. Note that in order to integrate along the characteristics which have slope 1, we need to have the time step equal to the spacial mesh size. Hence, the finite difference method applied to the parabolic equation must be implicit since the CFL condition

$$\frac{\gamma k}{\beta h^2} < 1$$

will not be satisfied for small $k = h$, and any explicit scheme will be unstable.

### 6.2 Acoustic Wave and Phase Boundary Interaction

For our first set of numerical experiments, we will use a volume fraction $\alpha(\varphi)$ defined as follows. Let $a = (3 - \sqrt{3})/6$ and $b = (3 + \sqrt{3})/6$, and define

$$\alpha(\varphi) = \begin{cases} 0 & \text{for } -\infty < \varphi \leq a, \\ \frac{\exp(1/(1-a-\varphi))}{\exp(1/(1-a-\varphi)) + \exp(1/(b-\varphi))} & \text{for } a \leq \varphi \leq b, \\ 1 & \text{for } b \leq \varphi < \infty. \end{cases}$$

(6.2)

Note that the intervals $(-\infty, a]$ and $[b, \infty)$ correspond exactly to the pure phases as described in §1.4. In this case, $\alpha(\varphi)$ satisfies the condition (H1). We also choose the parameter values $\gamma = .1$, $\beta = 1$, and $\mu = 1$. These parameter values are not scaled according to the Fried and Gurtin scaling described in §5.1, but the results are indicative of the qualitative behavior of the phase-field system, as will be seen in §6.3.

In all of the following experiments, we will use the hard (displacement) boundary conditions of equation (5.1) with $\Lambda = 1/2$. This is the most natural choice for studying multiphase solutions. Since $\alpha(\varphi)$ is odd about the point $(1/2, 1/2)$, it follows that $\int_0^1 \alpha(\varphi(x)) \, dx = 1/2$ for any $\varphi(x)$ which is symmetric with respect
to $\left(1/2,1/2\right)$. Therefore, if we take $P = 0$ in the equilibrium equations (5.5) and (5.7), we obtain equilibria which satisfy

$$0.1 \varphi_{xx} + \varphi(\varphi - 1)(1 - 2\varphi) + \alpha'(\varphi)(\alpha(\varphi) - \frac{1}{2}) = 0,$$

$$\varphi_x(0) = \varphi_x(1) = 0 \quad \text{and} \quad \int_0^1 \alpha(\varphi) \, dx = \frac{1}{2},$$

where the last condition is automatically satisfied by any symmetric solution to (6.3). A bifurcation and phase-plane analysis similar to that in §3.3 and §3.4 shows that at each $\gamma_n = \left(1/2 + \alpha'(1/2)^2 \right)/n^2 \pi^2$, a pitchfork bifurcation occurs, creating a global branch of symmetric solutions to (6.3) with $n$ transition layers in the parameter range $0 < \gamma < \gamma_n$, (similar to Figure 3.1). These solutions are equilibria of the phase-field system, and with the parameters $\gamma$ and $\mu$ chosen as above, symmetric equilibria with up to six phase boundaries exist.

In the following, the four panels of each figure show the displacement field $u$, the phase field $\varphi$, the strain field $u_x$, and the volume fraction $\alpha(\varphi)$, respectively. These spatial fields are plotted at chosen time intervals as indicated in the caption. Note that all propagating acoustic disturbances move with speed $c = 1$. The scale of the axes in the strain field panel were chosen for easy comparison with the volume fraction, and hence the plots are sometimes clipped.

The figures in this section, Figures 6.1–6.13, show the results of two numerical experiments with the phase-field system (6.1) involving acoustic wave and phase boundary interaction. In both simulations, the initial data consists of a right-moving acoustic pulse, superposed with a two-phase equilibrium state. After propagating through the pure zero-phase end of the bar, it interacts with the phase boundary in the middle of the bar.

Figures 6.1–6.5 indicate the results for the relatively low energy acoustic pulse
as shown in Figure 6.1. As the acoustic wave passes through the phase boundary, most of the energy is transmitted, but a reflected wave of lower energy is produced as seen in Figure 6.2. The phase boundary itself is almost unaffected and does not move appreciably from its equilibrium position. When the acoustic wave reaches the end of the bar, it is reflected back and again passes through the phase boundary in the same manner as before. Some of the energy of the acoustic disturbance is dissipated during its interaction with the phase boundary as shown in Figure 6.3. This cycle repeats itself as the acoustic disturbances are damped out, and the system appears to re-stabilize and approach the original equilibrium as \( t \to \infty \) (see Figure 6.4). Of course, an analytic stability result is necessary to determine whether solutions can be expected to approach an equilibrium asymptotically; it is possible that this equilibrium is unstable (see §5.2). We will further investigate numerically the apparent stability properties of equilibria in §6.3. Since \( \alpha(\varphi) \) satisfies the condition (H1), there is no energy decay when the acoustic wave is near the ends of the bar, because the equations decouple in regions away from the phase boundary, i.e. where \( \alpha'(\varphi) \equiv 0 \). This leads to the stair-step type plot of the energy decay shown in Figure 6.5.

In the above experiment, the acoustic pulse did not possess sufficient energy to cause the phase boundary to move noticeably from its equilibrium position. In the simulation illustrated in Figures 6.6–6.13, the amplitude of the acoustic pulse is an order of magnitude greater than in the first experiment (see Figure 6.6). In this case, the acoustic wave causes the phase boundary to sharpen and then move with it to the right at near the speed of sound \( c = 1 \), expanding the zero-phase behind it, as shown in Figures 6.7–6.8. When the acoustic wave reaches the end of the bar, the bar is primarily in the zero-phase, but some of the one-phase remains.
Numerical studies (not illustrated here) reveal that it is possible for an acoustic pulse with sufficient energy to destroy the phase boundary causing the solution to approach a single-phase, time-periodic solution as described in Lemma 5.2.1. However, in the present case this does not occur.

After the acoustic wave is reflected back, it again causes the phase boundary to move to the left, expanding the one-phase. This process repeats, and each successive time this entrainment occurs, the phase boundary does not move as far. Eventually, as shown in Figures 6.8–6.10, the energy of the acoustic waves is sufficiently dissipated that the phase boundary stalls and remains in its original equilibrium position, and the behavior of the solution is similar to that of the first simulation. Note from Figure 6.10 that the solution essentially repeats itself every two time units, and the propagating acoustic disturbances slowly decay during each interaction with the phase boundary.

The principal difference between the first experiment and the latter stage of the second one is shown in Figure 6.11. After a sufficient time, the propagating acoustic waves have essentially disappeared, but in the second simulation, “standing” acoustic waves remain ringing in the bar. We discuss wave ringing in the next section and specifically the issue of whether or not these waves are slowly damped out. However, it appears from these experiments that there is sufficient dissipation to allow the phase-field to re-equilibrate, as suggested by the results of Chapter 5 (Theorem 5.4.1).
Figure 6.1: Initial data consisting of an acoustic pulse moving to the right with an amplitude of $u_x = 1$ added to the pure "zero-phase" end of a bar in a two-phase equilibrium state.
Figure 6.2: The solution at time $t = .6$ at which the acoustic wave has passed through the phase boundary producing a lower energy reflected wave. Note that the phase boundary has not moved noticeably from its original position.
Figure 6.3: The solution at time $t = 1.9$ at which the acoustic wave has reflected back and passed through the phase boundary a second time. The amplitude (and energy) of the acoustic pulse is smaller, and again the phase boundary has not moved.
Figure 6.4: The solution at time $t = 50$ in which the system has apparently re-stabilized to the original equilibrium.
Figure 6.5: This plot shows the “stair-step” manner in which the energy decays toward $E_1$, the energy of the original equilibrium. Energy losses occur due to interaction of the acoustic wave with the phase boundary.
Figure 6.6: Initial data with an acoustic pulse moving to the right with an amplitude of $u_z = 10$ added to a two-phase equilibrium.
Figure 6.7: The solution at time $t = .5$ at which the acoustic wave has interacted with the phase boundary causing it to sharpen.
Figure 6.8: The solution at time $t = 1$ shows the acoustic wave pushing the phase boundary to the right and expanding the zero-phase behind it.
Figure 6.9: The solution at time $t = 2$ at which the acoustic wave has reflected back, pushed the phase boundary to the left, and expanded the one-phase behind it.
Figure 6.10: The solution at time $t = 3$ shows that on the third pass the phase boundary stalls and the acoustic wave passes through it.
Figure 6.11: The solution at times $t = 20, 22, 24, 26, 28,$ and $30$ are pictured simultaneously to show that the phase boundary appears to no longer move appreciably from its equilibrium position. The fields almost repeat after each round trip with the propagating acoustic waves slowly dissipating each time they pass through the phase boundary.
Figure 6.12: The solution at times $t = 74$ and 75 are pictured simultaneously, which show "standing" acoustic waves continuing to ring.
Figure 6.13: This plot shows the decay of energy toward the energy of the single phase boundary equilibrium, $E_1$. 
6.3 Equilibrium States, Stability, and Wave Ringing

We now investigate numerically the stability properties of some of the equilibria of the phase-field system in relation to the analytic results of Chapter 5. In particular, we will use the specific parameter values in §6.2 for which the system has symmetric equilibrium states with up to six phase boundaries.

In Figure 6.14, a solution with initial data $\varphi = .2$ and an acoustic wave of amplitude $u_z = 1$ is shown at times $t = 0$ and $t = 2$ (the time of one round trip). Since $0 < a$, $\alpha'(\varphi) \equiv 0$ and the equations decouple as in Lemma 5.2.1. The phase, $\varphi$, approaches zero, during which the acoustic wave propagates without loss of energy. As proved in the lemma, this equilibrium is stable but not asymptotically stable.

Next, we consider equilibria with multiple phase boundaries. In the simulations of the previous section, the system appears to stabilize at a state with a single phase boundary, but symmetric equilibria with up to six boundaries exist. Of the equilibrium states we have found (constant phase and symmetric), the two states with a single phase boundary have the lowest energy (lower than the pure zero- and one-phases), and we conjecture that they are stable. From numerical experiments, the equilibria with multiple boundaries appear to be unstable, as is demonstrated in Figures 6.15–6.20.

In this experiment, the initial data consists of three transition layers and is symmetric about $1/2$ with no added acoustic disturbances, (cf. Figure 6.15). Figure 6.16 shows that on the time interval $t = 1$ to $t = 25$, the solution remains close to the equilibrium with three phase boundaries. No sizable acoustic waves seem to be generated, and the symmetry appears to be maintained. Near the time $t = 30,$
the solution moves rapidly toward a single phase boundary equilibrium, and the symmetry appears to be maintained as two of the transition layers are destroyed (cf. Figure 6.17). In this process, acoustic waves are generated as shown in Figure 6.18, and Figure 6.19 shows the ringing effect that remains after the propagating waves have disappeared. The energy plot of Figure 6.20 gives an overview of this process; notice that the energy due to the wave ringing is larger than was seen in previous experiments. It is also interesting to note that the middle phase boundary does not move throughout this simulation, although it does change orientation.

Since there are no symmetric equilibria with more than six phase boundaries, we use initial data with seven transition layers that is symmetric about 1/2 for the experiment shown in Figures 6.21–6.26. As expected, the solution initially approaches the middle constant phase equilibrium with small acoustic disturbances as shown in Figure 6.22. From Lemma 5.2.4, we know that the middle equilibrium is unstable, and after about time $t = 2$, the solution begins to approach a two-phase equilibrium (cf. Figure 6.23). Figure 6.24 shows the ringing which appears after the two-phase state is reached, and this effect is larger than we have seen in previous simulations. The energy plot in Figure 6.25 shows the large amount of energy which is due to the wave ringing, and its slow decay.

In the above experiments, the ringing effect that appears in two-phase solutions involves some interaction with the phase boundary, and the energy continues to decay slowly. In the second experiment, this interaction occurs near the edges of the transition layer (cf. Figure 6.12), and the energy due to the ringing is small. However, in the last simulation, the entire transition layer seems to move slightly (cf. Figure 6.24), and the energy difference is quite large. At first glance, these experiments seem to contradict the partial explanation of wave ringing given in
§5.3. Since the volume fraction satisfies the (H1) condition, one might expect to find standing waves which do not interact with the phase boundary nor dissipate energy. However, the two-phase equilibria have values inside the interval \((a, b)\), and hence \(\alpha'(\varphi) > 0\) for these equilibria. This situation is equivalent to being in the (H2) case (see §5.4), in which it is expected that the “standing” waves and the phase boundaries should interact, leading to slow decay. In fact, Figure 6.26 shows the solution in the last experiment at times \(t = 998.5\) and \(999.5\) in which the large ringing observed at times \(t = 6.4\) and \(t = 7.4\) has now been nearly damped out.

In contrast to the previous experiment, Figure 6.27 shows the solution at times \(t = 1999, 1999.5,\) and \(2000\) to the phase-field system (5.1) with parameter values \(\beta = \gamma = .1\) and \(\mu = 10\). These values correspond to the scaling described in §5.1. In this case, the two-phase equilibrium has \(\varphi\) values well outside the interval \((a, b)\), corresponding to the (H1) case as shown in Figure 6.27, and the ringing acoustic waves do not appear to be appreciably damped, as indicated by Figure 6.28 which shows the solution near \(t = 10,000\).

In all of the simulations in this chapter, the phase-field seems to converge to an equilibrium state as is proven in Theorem 5.4.1. However, we have seen that acoustic waves can continue to ring long after the phase arrangement has stabilized. These vibrations are slowly dissipated if \(\alpha(\varphi) > 0\) on the stationary phase-field as suggested by Corollary 5.4.1, but they persist undamped if \(\alpha(\varphi) = 0\) on some subinterval.
Figure 6.14: This plot shows the solution with initial data which has $\varphi \equiv .2$ and an acoustic wave of amplitude $u_x = 1$ at the times $t = 0$ and 2. Since $.2 < a$, the equations are decoupled. The phase approaches the constant zero-phase, and the acoustic wave propagates without any loss in energy.
Figure 6.15: Initial data with three phase boundaries which is odd about $\frac{1}{2}$.
Figure 6.16: The solution at times $t = 1$ and 25. On this time interval, the solution remains close to an equilibrium with three phase boundaries and maintains its symmetry about $\frac{1}{2}$. 
Figure 6.17: The solution at times $t = 30.6$ and $30.8$. On this time interval, two phase boundaries are destroyed, and the solution retains its symmetry.
Figure 6.18: The solution at times $t = 31$ and $31.2$. After the destruction of the phase boundaries, acoustic waves are produced which propagate symmetrically.
Figure 6.19: The solution at times $t = 41.4$ and $42.4$ shows that the solution continues to ring.
Figure 6.20: This plot shows the energy decaying toward the energy of the equilibrium with three phase boundaries. Near the time $t = 30$, the solution suddenly moves toward the single phase boundary equilibrium which has energy, $E_1$. 
Figure 6.21: Initial data with seven phase boundaries which is odd about $\frac{1}{2}$. 
Figure 6.22: The solution at time $t = 1.8$ has become close to the middle constant phase equilibrium with small acoustic disturbances.
Figure 6.23: The solution at time $t = 2.4$ is approaching the two-phase equilibrium.
Figure 6.24: The solution at times $t = 6.4$ and $7.4$ shows a large ringing effect.
Figure 6.25: This plot shows the energy decaying at first toward the energy of the middle equilibrium and then toward the energy of the single phase boundary equilibrium, \( E_1 \). However, there is a large energy difference due to the wave ringing.
Figure 6.26: The solution at times $t = 998.5$ and $t = 999.5$ at which the ringing waves have been nearly damped out.
Figure 6.27: The solution to the system with the Fried-Gurtin scaling at times $t = 1999$, $1999.5$, and $2000$ at which the ringing acoustic waves have not been damped appreciably.
Figure 6.28: Wave ringing in the system with the Fried-Gurtin scaling near the time $t = 10,000$. 
Chapter 7

Conclusions

In this thesis, we investigate two infinite-dimensional evolutionary systems which are models of dynamic phase transitions in multiphase elastic solids. Materials which undergo a transformation in phase are often modeled by nonconvex energy functionals. While variational methods have provided many results about the static problem, the dynamic equations suffer a change in type causing mathematical difficulties, and the nucleation and motion of phase boundaries also lead to computational difficulties, particularly concerning the formation of microstructure. The models analyzed in this thesis are regularized systems which include terms representing interfacial or phase boundary energies and which are also more amenable to mathematical analysis and numerical computation. We study these systems in one space dimension, but they can be regarded as prototypical models whose behavior provides insight into that of more complex higher-dimensional problems.

The first system is a viscoelastic model with an added capillarity term which incorporates the higher-order effects of interfacial energy. It also contains an energy
penalty term to promote the formation of fine structure. The viscous damping is already enough to provide a strong dissipation of energy, but without the capillarity term, no solution minimizes energy as the phase boundaries do not move once they are created. In fact, the energy has no global minimizers, but does possess minimizing sequences and a continuum of weak relative minima. With the additional capillarity term, phase boundaries can move, the energy possesses global minimizers, and there are only finitely many equilibrium states.

Our main contributions to the study of this type of viscoelastic model include proofs of (1) the existence of strong solutions (Theorem 2.2.1), (2) the precompactness of orbits and their asymptotic smoothness which implies that the system has a compact global attractor, none of which is known for the system without capillarity (Lemma 3.2.1 ff.). We also perform (3) a local bifurcation analysis near the trivial solution (Lemmas 3.3.1 and 3.3.2), and give (4) a description of the equilibria when the energy penalty is small (Theorem 3.4.1).

In the absence of the energy penalty term, the viscoelastic model with capillarity has a finite-dimensional attractor whose structure is completely known (at least in the case of traction boundary conditions), and the asymptotic approach to equilibrium involves the exponentially slow motion of phase boundaries just as in bistable reaction-diffusion equations. This is due to the fact that without the fine structure term, the equation can in some sense (i.e. assuming that accelerations are negligible) be integrated twice to obtain a bistable parabolic equation. However, when the energy penalty term is present, the system cannot be treated in this way, since the energy penalty involves the displacement, which is a non-local quantity with respect to the strain.

In this case, our results show that the attractor still consists of finitely many
equilibria and connecting orbits, but in practice the fourth-order equilibrium equation is difficult to solve. We begin the analysis of equilibria by considering local bifurcations from the trivial solution, but a complete description is an open problem. The characterization of the asymptotic behavior and motion of phase boundaries is also an interesting open problem. As explained in §3.5, it appears from preliminary numerical analysis that the phase boundary motion need not be exponentially slow, at least initially. In some cases, the phase boundaries seem to move to symmetrize or periodize their arrangement before stalling for a long time. This is due to the fact that the motion of phase boundaries no longer depends only on their number but also on their relative positions because of the non-local energy penalty.

The phase-field system regularizes the mechanical equation of elasticity indirectly by introducing an order parameter to determine the phase independently of the strain, (the phase is determined by the strain in the viscoelastic model). The mechanical equation no longer changes type and remains hyperbolic, but it is coupled to a parabolic equation in the phase-field. Another advantage to this system is that both equations are second-order, and so for example, the equilibria can be determined by a phase plane analysis. However, the primary disadvantage is that the coupling between the equations involves the strain and the gradient of the phase-field which makes it more difficult to determine the asymptotic behavior of solutions.

Our contributions to the study of the phase-field system are fourfold: (1) we establish the existence of classical solutions (Theorem 4.2.1), (2) we prove, under suitable assumptions, that all strong solutions approach the equilibrium set weakly with the phase-field converging strongly (Theorem 5.4.1 and Corollary
5.4.1), (3) we begin to investigate the equilibrium states and their stability (Lemmas 5.2.1, 5.2.2, and 5.2.4), and (4) we numerically study the interaction between acoustic waves and phase boundaries and the wave ringing which occurs in solutions for which the equations become decoupled or nearly decoupled on some subinterval.

In the analysis of Chapter 5, we were unable to determine whether solutions to the phase-field system are strongly precompact. This issue involves the analysis of an asymptotically autonomous wave equation of the form \( u_{tt} = u_{xx} + f(x, t) \) where \( f(x, t) \to \hat{f}(x) \) as \( t \to \infty \). It is an interesting question to ask under what conditions on \( f \) are \( u_{tt} \) and \( u_{xx} \) bounded, and from a limited literature search, this problem does not appear to have been studied much.

Of course, one primary outstanding issue in determining the dynamics of the phase-field system is a complete description of the equilibria and their stability types. In the numerical simulations of Chapter 6, only symmetric equilibria were observed, but it is not clear that these are the only equilibrium states. The difficulty is due to the fact that the equilibria depend on the volume fraction which can be chosen fairly arbitrarily.
Bibliography


