Give $\mathbb{R}^I$ the uniform metric, where $I = [0, 1]$. Let $C(I, \mathbb{R})$ be the subspace consisting of continuous functions. Show that $C(I, \mathbb{R})$ has a countable dense subset, and therefore a countable basis.

**Proof.** For $f, g \in \mathbb{R}^I$, denote the uniform metric

$$
\rho(f, g) = \sup\{d(f(\alpha), g(\alpha)) : \alpha \in I\},
$$

where $d(f(\alpha), g(\alpha)) = \min\{|f(\alpha) - g(\alpha)|, 1\}$ is the standard bounded metric. For $n \geq 1$, let

$$
A_n = \bigcup_{0 = p_0 < p_1 < \ldots < p_n = 1, p_i \in \mathbb{Q}} \left[ \left( \prod_{i=0}^{n} p_i \right) \times \left( \prod_{j=0}^{n} q_j \right) \right].
$$

Then $A_n \subseteq (\mathbb{Q} \cap [0, 1])^{n+1} \times \mathbb{Q}^{n+1}$. Since the finite product of countable sets is countable, and a subset of a countable set is countable, it follows that $A_n$ is countable (Munkres Theorem 7.6, Corollary 7.3). Let

$$
A = \bigcup_{n=1}^{\infty} A_n.
$$

Then $A$ is the countable union of countable sets, hence is countable (Munkres, Theorem 7.5). Let $D$ denote the collection of continuous functions whose graphs consist of finitely many line segments with rational end points. Define $h : A \to D$, where for $(p_0, p_1, \ldots, p_n, q_0, q_1, \ldots, q_n) \in A$,

$$
h((p_0, p_1, \ldots, p_n, q_0, q_1, \ldots, q_n))(x) = \frac{q_{j+1} - q_j}{p_{j+1} - p_j} (x - p_j) + q_j
$$

for $x \in I$ with $p_j \leq x \leq p_{j+1}$. It follows that $h$ is surjective, hence $D$ is countable.

To see that $D$ is dense, let $0 < \epsilon < 1$ and $f \in C(I, \mathbb{R})$. Since $I$ is compact, $f$ is uniformly continuous so $\exists \delta > 0$ where $\forall x, y \in I$

$$
|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \frac{\epsilon}{5}.
$$

Let $\frac{1}{n} < \delta$, $n \in \mathbb{N}$. By $\mathbb{Q}$ dense in $\mathbb{R}$, let $q_k \in \mathbb{Q}$, $0 \leq k \leq n$ be such that

$$
|q_k - f\left(\frac{k}{n}\right)| < \frac{\epsilon}{5}.
$$
153. Let \( \{U_k\}_{k=1}^n \) be a finite open cover of \( X \) and \( f_k : U_k \to Y \) be continuous for each \( k = 1, \ldots, n \). Show that if \( f_k(x) = f_j(x) \) for all \( x \in U_k \cap U_j \), then the function \( F : X \to Y \) defined by \[
F(x) = f_k(x) \quad \text{for all } x \in U_k
\]
is continuous.

Proof. First of all, \( F \) is well defined on \( X \). Now for any fixed \( x \in X \), there exists some \( U_K, 1 \leq K \leq n \), such that \( x \in U_K \). Note that \( f_K \) is continuous over \( U_K \). Then for any neighborhood \( V \subset Y \) of \( y = F(x) = f_K(x) \), there is some neighborhood \( W \subset U_K \) of \( x \) such that \( f_K(W) \subset V \). Note that \( W \) is open in \( U_K \) and that \( U_K \) is open in \( X \), so \( W \) is open in \( X \), i.e., \( W \) is a neighborhood of \( x \) in \( X \). But \( F(W) = f_K(W) \subset V \). Since \( x \) and \( V \) were arbitrary, we conclude that \( F \) is continuous. \( \Box \)
Consider $a = (0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1, q_0, q_1, \ldots, q_m) \in A$ and let $g = h(a)$. Then $\exists k$, where $0 \leq k \leq n-1$ and $\frac{k}{n} \leq x < \frac{k+1}{n}$. In particular,

$$\left| g \left( \frac{k}{n} \right) - g(x) \right| = g \left( \frac{k}{n} \right) - g \left( \frac{k+1}{n} \right) - g \left( \frac{k}{n} \right) \left( x - \frac{k}{n} \right) + g \left( \frac{k}{n} \right) \leq g \left( \frac{k+1}{n} \right) - g \left( \frac{k}{n} \right) \left( x - \frac{k}{n} \right) \leq g \left( \frac{k+1}{n} \right) - g \left( \frac{k}{n} \right) + f \left( \frac{k+1}{n} \right) - f \left( \frac{k}{n} \right) + f \left( \frac{k}{n} \right) - g \left( \frac{k}{n} \right) \left( x - \frac{k}{n} \right) \leq \frac{3\varepsilon}{5}.$$ 

Then

$$|f(x) - g(x)| \leq f \left( \frac{k}{n} \right) + f \left( \frac{k+1}{n} \right) - g \left( \frac{k}{n} \right) + g \left( \frac{k}{n} \right) \left( x - \frac{k}{n} \right) \leq \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{3\varepsilon}{5} = \varepsilon < 1.$$ 

As $x$ was arbitrary,

$$\rho(f, g) = \sup \{d(f(x), g(x)) : x \in I\} = \sup \{|f(x) - g(x)| : x \in I\} \leq \varepsilon$$

proves that $D$ is dense in $\mathcal{C}(I, \mathbb{R})$. 

$\square$
154. (a) (Section 31, #1) Show that if $X$ is regular, every pair of points of $X$ have neighborhoods whose closures are disjoint.

Proof. Recall the following:

**Definition 1.** Suppose that one-point sets are closed in $X$. Then $X$ is said to be **regular** if for every pair of a point $x \in X$ and a closed set $A \subset X$ disjoint from $x$, there exist disjoint open sets containing $x$ and $A$, respectively. The space $X$ is said to be **normal** if for each pair $A, B$ of disjoint closed sets of $X$, there exist disjoint open sets containing $A$ and $B$, respectively.

**Lemma 2.** Let $X$ be a topological space. Let one-point sets in $X$ be closed.

(i) $X$ is regular if and only if given a point $x \in X$ and a neighborhood $U$ of $x$, there is a neighborhood $V$ of $x$ such that $\text{Cl}(V) \subset U$.

(ii) $X$ is normal if and only if given a closed set $A$ and an open set $U$ containing $A$, there is an open set $V$ containing $A$ such that $\text{Cl}(V) \subset U$.

Then, for any pair of points $x, y \in X$, by regularity of $X$, there are disjoint open sets $U, V$ containing $x, y$, respectively. However, by (i) of Lemma 2, there are neighborhoods $U'$ of $x$ and $V'$ of $y$ such that $\text{Cl}(U') \subset U$ and $\text{Cl}(V') \subset V$. Since $U, V$ are disjoint, so are $\text{Cl}(U')$ and $\text{Cl}(V')$.

(b) (Section 31, #2) Show that if $X$ is normal, every pair of disjoint closed sets have neighborhoods whose closures are disjoint.

Proof. Let $A, B \subset X$ be disjoint closed sets. By normality of $X$, there are disjoint open sets $U \supset A$ and $V \supset B$. However, by (ii) of Lemma 2, there are open sets $U' \supset A$ and $V' \supset B$ such that $\text{Cl}(U') \subset U$ and $\text{Cl}(V') \subset V$. Since $U, V$ are disjoint, so are $\text{Cl}(U')$ and $\text{Cl}(V')$.  

\[\square\]
Let \( f, g : X \to Y \) be continuous.
Assume that \( Y \) is Hausdorff.

To prove \( A = \{ x \mid f(x) = g(x) \} \) is closed in \( X \),
we need to prove \( X \setminus A \) is open in \( X \).

Let \( x \in X \setminus A \). Then \( f(x) \neq g(x) \).

So, \( \exists \ U, V \) open in \( Y \) s.t. \( U \cap V = \emptyset \) with
\( f(x) \in U \) and \( g(x) \in V \).

Since \( f, g \) are continuous, \( f^{-1}(U) \) and \( g^{-1}(V) \)
are open with \( x \in f^{-1}(U) \) and \( x \in g^{-1}(V) \).

\[ \Rightarrow x \in f^{-1}(U) \cap g^{-1}(V) = W \text{ is open, since finite}
\text{intersection of open sets are open.} \]

Now to show \( W \cap A = \emptyset \), suppose \( W \cap A \neq \emptyset \).

Then, \( f(x) = g(x) \) for \( x \in W \).

i.e., \( f(x) \in U \) and \( g(x) \in V \)
\[ \Rightarrow f(x) \in U \text{ and } f(x) \in V \] \( \quad ( \because f(x) = g(x) ) \)
\[ \Rightarrow f(x) \in U \cap V \]
\[ \Rightarrow \quad \text{as } U \cap V = \emptyset. \]

Hence \( x \in W \subseteq X \setminus A \).

Since \( x \) was arbitrary,
\[ \forall x \in X \setminus A \exists W \text{ open s.t. } x \in W \subseteq X \setminus A \]
i.e. \( X \setminus A \) is open.

so, \( A = \{ x \mid f(x) = g(x) \} \) is closed in \( X \).
Let $p : X \to Y$ be a closed continuous surjective map. Show that if $X$ is normal, then so is $Y$.

Let $A$ and $B$ be disjoint closed sets in $Y$. By continuity, $p^{-1}(A)$ and $p^{-1}(B)$ are disjoint closed in $X$. Since $X$ is normal, there exist disjoint open sets $U_1$ and $U_2$ in $X$ such that $p^{-1}(A) \subset U_1$ and $p^{-1}(B) \subset U_2$. Now as $U_1$ and $U_2$ are open, then $X - U_1$ and $X - U_2$ are closed. And as $p$ is a closed map, then $p(X - U_1)$ and $p(X - U_2)$ are also closed. Thus $V_1 = Y - p(X - U_1)$ and $V_2 = Y - p(X - U_2)$ are open with $A \subset V_1$ and $B \subset V_2$. Now

$$V_1 \cap V_2 = (Y - p(X - U_1)) \cap (Y - p(X - U_2)) = Y - (p(X - U_1) \cup (X - U_2)).$$

We claim now that $Y = p(X - U_1) \cup (X - U_2)$. Suppose $y \in Y$. Then since $p$ is surjective, there is some $x \in X$ such that $p(x) = y$. Since $U_1 \cap U_2 = \emptyset$, then $x \in X - U_1$ or $x \in X - U_2$. Thus $y \in p(X - U_1) \cup (X - U_2)$, implying $Y = p(X - U_1) \cup (X - U_2)$. Therefore, $V_1 \cap V_2 = \emptyset$. Thus $Y$ is normal.
159. Show that a closed subspace of a normal space is normal.

P: Let $Y$ be a closed subspace of the normal space $X$. Then $Y$ is Hausdorff by Thm 17.11. Let $A$ and $B$ be disjoint closed subspaces of $Y$. Since $A$ and $B$ are closed also in $X$, they can be separated in $X$ by disjoint open sets $U$ and $V$. Then $U \cap Y$ and $V \cap Y$ are open sets in $Y$ separating $A$ and $B$. $Y$ is normal. ■
Show that every locally compact Hausdorff space is regular.

Since $X$ is locally compact and Hausdorff, by theorem 29.1 there exists a compact Hausdorff space $Y$ such that $X$ is a subspace of $Y$ and $Y \setminus X$ is a single point. Moreover, by theorem 32.3 $Y$ is normal since it is compact and Hausdorff. But every normal space is regular, thus $Y$ is regular. Finally, by theorem 31.2 a subspace of a regular space is also regular. Thus the space $X$ is regular, as desired.
162. (Section 32, #4) Show that every regular Lindelöf space is normal.

Proof. Recall:

Definition 3. A space for which every open covering contains a countable subcovering is called a Lindelöf space.

The idea of the proof is similar to proving Lemma 32.1 (on page 200). Let $X$ be regular and Lindelöf. Let $A, B \subset X$ be an arbitrary pair of disjoint closed sets. Then, for every $p = (x, y) \in C = A \times B$, there are disjoint open sets $U_p, V_p$ such that $x \in U_p, y \in V_p$. Since $A, B$ are closed, we may assume that each $U_p$ is disjoint from $B$ and that each $V_p$ is disjoint from $A$ (if not, subtract $A$ or $B$ from $V_p$ or $U_p$, respectively, and we still keep all the properties mentioned above). Moreover, by part (a) of Lemma 31.1 (on page 196), we may assume that

$$\text{Cl}(U_p) \cap B = \text{Cl}(V_p) \cap A = \emptyset.$$  \hspace{1cm} (1)

Let $W_p = U_p \cup V_p$. Note that $\{W_p\}_{p \in C} \cup \{X \setminus (A \cup B)\}$ forms an open covering of $X$. Then by the Lindelöf condition, there is a countable subcovering $\{W_n\}_{n=0}^{\infty}$ of $X$, where $W_0 = X \setminus (A \cup B)$. Thus $\{W_n\}_{n=1}^{\infty} \subset \{W_p\}_{p \in C}$ is a countable open covering of $A \cup B$. By earlier assumption, we have that $\{U_n\}_{n=1}^{\infty} \subset \{U_p\}_{p \in C}$ covers $A$ and that $\{V_n\}_{n=1}^{\infty} \subset \{V_p\}_{p \in C}$ covers $B$. The rest of the proof is identical to the proof of Lemma 32.1. For every $n$, define

$$U'_n = U_n - \bigcup_{i=1}^{n} \text{Cl}(V_i) \quad \text{and} \quad V'_n = V_n - \bigcup_{i=1}^{n} \text{Cl}(U_i).$$  \hspace{1cm} (2)

Note that by (1), $\{U'_n\}_{n=1}^{\infty}$ and $\{V'_n\}_{n=1}^{\infty}$ are open coverings of $A$ and $B$, respectively. Define

$$U' = \bigcup_{n=1}^{\infty} U'_n \supseteq A \quad \text{and} \quad V' = \bigcup_{n=1}^{\infty} V'_n \supseteq B,$$

both are open. Claim: $U' \cap V' = \emptyset$. Suppose not, then there is $z \in U' \cap V'$, so there are $i, j \in \mathbb{N}$ such that $z \in U'_i \cap V'_j$. Without loss of generality, assume that $i \leq j$. Then by (2), we have $z \in U_i$ but $z \notin \text{Cl}(U_i) \supseteq U_i$, which is a contradiction, completing the proof of the claim. Therefore $X$ is normal by definition. \hfill \square
**Theorem** (Strong form of the Urysohn Lemma). Let $X$ be a normal space. There is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ for $x \in A$, and $f(x) = 1$ for $x \in B$, and $0 < f(x) < 1$ otherwise, if and only if $A$ and $B$ are disjoint closed $G_\delta$ sets in $X$.

**Definition** $A$ is a $G_\delta$ set in $X$ if $A$ is the intersection of a countable collection of open sets of $X$.

**Proof.** ($\Rightarrow$) Let us see that

$$G_A := \bigcap_{n=1}^{\infty} f^{-1}\left(\left[0, \frac{1}{n}\right]\right) = A. \quad (1)$$

Note that $\left[0, \frac{1}{n}\right]$ are open in $[0, 1]$ so by continuity of $f$, $G_A$ is a $G_\delta$ set. Let $x \in G_A$ and suppose towards a contradiction $x \notin A$. Then $f(x) = p > 0$. Let $\frac{1}{n} < p$; but $x \in f^{-1}(\left[0, \frac{1}{n}\right])$ is a contradiction. Observing that $A = f^{-1}(\{0\}) \subseteq G_A$ proves $G_A = A$. Similarly let us see that

$$G_B := \bigcap_{n=1}^{\infty} f^{-1}\left(\left[1 - \frac{1}{n}, 1\right]\right) = B. \quad (2)$$

Note that $\left(1 - \frac{1}{n}, 1\right]$ are open in $[0, 1]$ so by continuity of $f$, $G_B$ is a $G_\delta$ set. Let $x \in G_B$ and suppose towards a contradiction $x \notin B$. Then $f(x) = p < 1$. Let $n$ be such that $p < 1 - \frac{1}{n}$; but $x \in f^{-1}(\left(1 - \frac{1}{n}, 1\right])$ is a contradiction. Observing that $B = f^{-1}(\{1\}) \subseteq G_B$ proves $G_B = B$. In summary, $G_A = f^{-1}(0) = A$ and $G_B = f^{-1}(1) = B$ are disjoint closed $G_\delta$ sets in $X$.

($\Leftarrow$) By Problem 4 (Munkres Section 33, page 213), let $f, g : X \to [0, 1]$ be a continuous functions so that

$$f(x) = \begin{cases} 0, & \text{if } x \in A \\ f(x) > 0, & x \notin A \end{cases} \quad g(x) = \begin{cases} 0, & \text{if } x \in B \\ g(x) > 0, & x \notin B \end{cases}$$

Let

$$h(x) = \frac{f(x)}{f(x) + g(x)}.$$ 

Then $h$ is continuous with

$$h(x) = \begin{cases} 0, & \text{if } x \in A \\ 1, & \text{if } x \in B \\ 0 < h(x) < 1 & \text{otherwise} \end{cases}$$
167. Show that every locally compact Hausdorff space is completely regular.

P: Let $Z$ be a locally compact Hausdorff space. By Corollary 29.4, $Z$ is homeomorphic to an open subspace of a compact Hausdorff space. Every compact Hausdorff space is normal (Thm 32.3). $Z$ is the subspace of a normal space. “A normal space is completely regular, by the Urysohn lemma...” (Textbook p. 211). $Z$ is the subspace of a completely regular space. A subspace of a completely regular space is completely regular (Thm 33.2) $Z$ is completely regular. $\blacksquare$
170. Let $X$ be a locally compact Hausdorff space. Is it true that if $X$ has a countable basis, then $X$ is metrizable? Is it true that if $X$ is metrizable, then $X$ has a countable basis?

P: Let $X$ be a locally compact Hausdorff space. $X$ 2nd countable $\Rightarrow$ $X$ metrizable.

$\Leftarrow$: Since a discrete uncountable space is metrizable but not 2nd countable.

$\Rightarrow$: Every locally compact space is regular (Ex #161). Every 2nd countable regular space is metrizable (Urysohn metrization Thm [34.1]) ■
Suppose that $h, h': X \to Y$ are homotopic and $k, k': Y \to Z$ are homotopic. Then there exist continuous maps $F_1$ and $F_2$ such that

$F_1 : X \times [0, 1] \to Y$ with $F_1(x, 0) = h(x), F_1(x, 1) = h'(x)$

and

$F_2 : Y \times [0, 1] \to Z$ with $F_2(x, 0) = k(x), F_2(x, 1) = k'(x)$

Now, $koh : k'oh' : X \to Z$ are continuous and define $F_3 : X \times [0, 1] \to Z$ by

$$F_3(x, t) = \begin{cases} k(F_1(x, 2t)) & 0 \leq t \leq \frac{1}{2} \\ F_2(h'(x), 2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

Then $k(F_1(x, 2t)) : X \times [0, \frac{1}{2}] \to Z$ and $F_2(h'(x), 2t-1) : X \times [\frac{1}{2}, 1] \to Z$ are continuous.

(\because \text{composition of continuous functions})

Also $X \times [0, \frac{1}{2}]$ and $X \times [\frac{1}{2}, 1]$ are closed in $X \times [0, 1]$ with $X \times [0, 1] = X \times [0, \frac{1}{2}] \cup X \times [\frac{1}{2}, 1]$.

Since $k(F_1(x, 2t)) = F_2(h'(x), 2t-1) = koh'(x)$ in $X \times \{\frac{1}{2}\} = X \times [0, \frac{1}{2}] \cap X \times [\frac{1}{2}, 1]$, by the pasting lemma in page 108, $F_3(x, t)$ is a continuous map such that $F_3(x, 0) = (koh)(x)$ and $F_3(x, 1) = (k'oh')(x)$.

So, koh and k'oh' are homotopic.
Show that contractibility is a topological property, i.e. if $X \approx Y$, then $X$ is contractible if and only if $Y$ is contractible.

By assumption, there exists an homeomorphism $f : X \rightarrow Y$, so that $f^{-1} : Y \rightarrow X$ is an homeomorphism as well. Assume that $f$ is contractible, therefore there exists an homotopy $H : X \times I \rightarrow X$ and $x_0 \in X$ such that

$$H(x, 0) = x \quad \text{and} \quad H(x, 1) = x_0, \forall x \in X.$$  

We define $G(y, s) = f(H(f^{-1}(y), s))$, so that $G$ is continuous since it is the composition of continuous function and

$$G(y, 0) = f(H(f^{-1}(y), 0)) = f(f^{-1}(y)) = y \quad \text{and} \quad G(y, 1) = f(H(f^{-1}(y), 1)) = f(x_0).$$

Thus the identity on $Y$ is homotopic to the point $f(x_0)$ and it follow that $Y$ is contractible. The other direction is done the same way by switching the roles of $X$ and $Y$. Thus, contractibility is a topological property, as desired.
Let $p : E \to B$ be a covering map, with $E$ path connected. Show that if $B$ is simply connected, then $p$ is a homeomorphism.

Proof. Let $p(e_0) = b_0$ and $\phi$ denote the lifting correspondence. Since $B$ is simply connected, $\pi_1(B, b_0)$ is trivial. Since the lift of $b_0$ that begins at $e_0$ is $e_0$, for any $[f] \in \pi_1(B, b_0)$ with $\bar{f}$ the lift that begins at $e_0$,

$$\bar{f}(1) = \phi([f]) = \phi([b_0]) = e_0.$$  

Since $E$ is path connected, the lifting correspondence is surjective (Munkres, Theorem 54.4, page 345) hence $p^{-1}(b_0) = \{e_0\}$. This shows that $p$ is injective. Since $p$ is a covering map, $p$ is an open map (Munkres, page 336). We conclude that $p$ is a homeomorphism. \qed
Let $A$ be a retract of $X$, and $X$ is contractible, i.e. $A \subseteq X$, $\gamma : X \to A$ with $\gamma(a) = a \forall a \in A$ and $\text{id}_X : X \to X$ is nullhomotopic.

Let $H : X \times [0,1] \to X$ be the homotopy between $\text{id}_X$ and a constant map $f(x) = x_0$. Then $H(x,0) = x$ and $H(x,1) = x_0 \forall x \in X$.

Now consider the map $\gamma_0 H_A : A \times [0,1] \to A$.

This is continuous being a composition of continuous maps.

Also we have $(\gamma_0 H_A)(x,0) = \gamma(H_A(x,0)) = \gamma(x) = x \ (\because x \in A) = \gamma_A(x)$

and $(\gamma_0 H_A)(x,1) = \gamma(H_A(x,1)) = \gamma(x_0) = \xi_0$

Hence $\gamma_0 H_A$ is a homotopy between $\gamma_A(x)$ and a constant map $g(x) = \xi_0$.

But $\gamma_A(x)$ is the identity map of $A$.

So, $\text{id}_A : A \to A$ is nullhomotopic.

$\Rightarrow A$ is contractible.

Hence a retract of a contractible space is contractible.