101. (Section 26, #12) Let \( p : X \to Y \) be a closed continuous surjective map such that \( p^{-1}(y) \) is compact, for each \( y \in Y \). (Such a map is called a perfect map.) Show that if \( Y \) is compact, then \( X \) is compact. [Hint: If \( U \) is an open set containing \( p^{-1}(y) \), there is a neighborhood \( W \) of \( y \) such that \( p^{-1}(W) \) is contained in \( U \].]

**Proof.** Suppose that \( Y \) is compact and there exists a perfect map \( p : X \to Y \). Let \( \{U_\alpha\}_{\alpha \in I} \) be an open cover of \( X \). Fix an arbitrary point \( y \in Y \). Since \( p \) is a perfect map, then \( p^{-1}(y) \) is a nonempty compact subspace of \( X \). By Lemma 26.1 (on page 164), there is a finite subcover \( \{U_i\}_{i=1}^N \subseteq \{U_\alpha\}_{\alpha \in I} \) of \( p^{-1}(y) \). Let \( U^y = \bigcup_{i=1}^N U_i \). Then \( U^y \) is open in \( X \), and \( p^{-1}(y) \subseteq U^y \). Thus \( X \setminus U^y \) is closed in \( X \), and \( p^{-1}(y) \cap (X \setminus U^y) = \emptyset \), so \( y \notin p(X \setminus U^y) \). Since \( p \) is a closed map, \( p(X \setminus U^y) \) is closed in \( Y \). Consider

\[
V^y = Y \setminus [p(X \setminus U^y)],
\]

then \( V^y \) is open in \( Y \) and \( y \in V^y \). Thus, \( \{V^y\}_{y \in Y} \) forms an open cover for \( Y \), and since \( Y \) is compact, there exists a finite subcover \( \{V^{y_i}\}_{i=1}^N \) of \( Y \). We want to show that \( \{U^{y_i}\}_{i=1}^N \) covers \( X \). Consider any subset \( A \subseteq X \). For any \( b \in Y \setminus p(A) \), we have \( p^{-1}(b) \cap A = \emptyset \), so \( p^{-1}(b) \subseteq X \setminus A \). Then we have

\[
p^{-1}(Y \setminus p(A)) \subseteq X \setminus A.
\]

Take \( A = X \setminus U^y \), we have

\[
p^{-1}(V^y) = p^{-1}(Y \setminus p(A)) \subseteq X \setminus A = U^y,
\]

for any \( y \in Y \). Thus we have

\[
X = p^{-1}(Y) \subseteq p^{-1}\left( \bigcup_{i=1}^N V^y_i \right) = \bigcup_{i=1}^N p^{-1}(V^y_i) \subseteq \bigcup_{i=1}^N U_i.
\]

Furthermore,

\[
X \subseteq \bigcup_{i=1}^N U_i = \bigcup_{i=1}^N \left( \bigcup_{j=1}^{N_i} U_j^{y_i} \right).
\]

Therefore, \( \bigcup_{i=1}^N \bigcup_{j=1}^{N_i} U_j^{y_i} \) is a finite subcover of \( \{U_\alpha\}_{\alpha \in I} \) for \( X \). \( \square \)
Let $X$ be a metric space and $f : X \to X$ be a homeomorphism. Given $N \subset X$, define the maximal invariant set in $N$ by

$$\text{Inv}(N) = \cap_{k=-\infty}^\infty f^k(N) = \{x \in N : f^k(x) \in N, \forall k \in \mathbb{Z}\}.$$ 

**Definition:** $N \subset X$ is a trapping region for $f$ if

1. $N$ is compact and forward invariant, i.e. $f(N) \subset N$, and

2. there exists $K > 0$ so that $f^K(N) \subset \text{int}(N)$.

(a) Show that if $N$ is a trapping region, then

$$\text{Inv}(N) = \cap_{k=0}^\infty \text{cl}\left(\bigcup_{j=k}^\infty f^j(N)\right) = \{x \in N : \exists x_n \in N, k_n > 0, k_n \to \infty, \text{ such that } f^{k_n}(x_n) \to x \text{ as } n \to \infty\}.$$ 

(b) Show that if $N$ is a trapping region and nonempty, then Inv($N$) is compact and nonempty.

(c) Show that if $N$ is a trapping region, Inv($N$) \subset \text{int}(N).

**Proof.** (a) Since $X$ is a metric space, let $\rho$ denote the metric. Since $N$ is a trapping region and $X$ is Hausdorff, $N$ is closed. Let $A = \cap_{k=0}^\infty \text{cl}\left(\bigcup_{j=k}^\infty f^j(N)\right)$ and $B = \{x \in N : \exists x_n \in N, k_n > 0, k_n \to \infty \text{ such that } f^{k_n}(x_n) \to x \text{ as } n \to \infty\}$. We will show that $B \subset \text{Inv}(N) \subset A \subset B$.

Let $x \in B$. Then

$$\exists x_n \in N, k_n > 0, k_n \to \infty \text{ such that } f^{k_n}(x_n) \to x \text{ as } n \to \infty.$$ 

Let $k \in \mathbb{Z}$. Since $k_n \to \infty$, $\exists M \in \mathbb{N}$ so that $m \geq M$ implies $k_m + k > 0$. By forward invariance of $N$, $f^{k+k_m}(x_m) \in N$. Since $f$ is a homeomorphism, $f^k$ is continuous, being the composition of continuous functions. Then $f^{k+k_m}(x_n) \to f^k(x) \in N$ as $N$ is closed. As $k$ was arbitrary, this proves $B \subset \text{Inv}(N)$.

Let us see that Inv($N$) \subset $A$. For any $k \in \mathbb{Z}$

$$f^k(N) \subset \bigcup_{j=k}^\infty f^j(N) \subset \text{cl}\left(\bigcup_{j=k}^\infty f^j(N)\right).$$
Then
\[
\text{Inv}(N) = \bigcap_{k=-\infty}^{\infty} f^k(N) \subset \bigcap_{k=0}^{\infty} f^k(N) \subset \bigcap_{k=0}^{\infty} \text{cl} \left( \bigcup_{j=k}^{\infty} f^j(N) \right) = A.
\]

Now we show \(A \subset B\) by first showing \(A \subset N\). \(\forall k \geq 0,\)
\[
f^k(N) \subset N \Rightarrow \bigcup_{j=k}^{\infty} f^j(N) \subset \text{cl} \left( \bigcup_{j=k}^{\infty} f^j(N) \right) \subset N,
\]

since \(N\) is a trapping region (closed and forward invariant). Thus \(A \subset N\). Let \(x \in A\) and \(n \geq 0\). Since \(x \in \text{cl} \left( \bigcup_{j=n}^{\infty} f^j(N) \right)\),
\[
\exists x_n \in N, \exists k_n \geq n, \text{ so that } \rho \left( f^{k_n}(x_n), x \right) < \frac{1}{n}, \quad \text{(Munkres, Lemma 21.2)}.
\]
Then \(f^{k_n}(x_n) \to x\) and \(k_n \to \infty\) as \(n \to \infty\) showing that \(A \subset B\).

(b) \(f\) is a bijection, therefore \(f^k(N) \neq \emptyset, \forall k \in \mathbb{Z}\). By (a), \(\text{Inv}(N) = A\) implies \(\text{Inv}(N)\) is closed as \(A\) is the intersection of closed sets. Since \(\text{Inv}(N) \subset N\) and \(N\) is compact, \(\text{Inv}(N)\) is compact. Let \(C_k = \text{cl} \left( \bigcup_{j=k}^{\infty} f^j(N) \right)\) for \(k \geq 0\). Since \(N\) is a trapping region, \(N \supset C_0 \supset C_1 \supset \ldots \supset C_k \ldots\) The \(C_k\) are non-empty since \(f\) is a bijection and \(N \neq \emptyset\) imply \(f^j(N) \neq \emptyset, \forall j \in \mathbb{Z}\). The \(C_k\) are nested because \(a \leq b \Rightarrow \bigcup_{j=a}^{\infty} f^j(N) \subset \bigcup_{j=b}^{\infty} f^j(N)\). The \(C_k\) are compact, being closed subsets of \(N\). By Munkres, Theorem 26.9, p.170 and the nested sequence property, it follows that \(\text{Inv}(N) \neq \emptyset\).

(c) Let \(N\) be a trapping region. Then \(\exists K > 0\) so that \(f^K(N) \subset \text{int}(N)\). Then
\[
\text{Inv}(N) = \bigcap_{k=0}^{\infty} f^k(N) \subset f^K(N) \subset \text{int}(N).
\]
Problem 104. Recall that $\mathbb{R}_K$ denotes $\mathbb{R}$ in the $K$-topology.

(a) Show that $[0, 1]$ is not compact as a subspace of $\mathbb{R}_K$.

(b) Show that $\mathbb{R}_K$ is connected [Hint: $(-\infty, 0)$ and $(0, \infty)$ inherit their usual topologies as subspaces of $\mathbb{R}_K$.]

(c) Show that $\mathbb{R}_K$ is not path connected.

Proof. The $K$ topology on $\mathbb{R}$ is described in Munkres: let $K$ denote the set of all numbers of the form $1/n$, for $n \in \mathbb{Z}_+$, and let $B''$ be the collection of all open intervals $(a, b)$, along with all sets of the form $(a, b) - K$. The topology generated by $B''$ will be called the $K$-topology on $\mathbb{R}$.

The open sets
\[
\left\{\left(\frac{1}{n+1}, 1 + \frac{1}{n}\right)\right\}_{n=1}^\infty \text{ and } (-1, 2) - K
\]
form an open cover of $[0, 1]$ with no possible finite subcover.

For part (b) the hint states that $(-\infty, 0)$ and $(0, \infty)$ inherit their usual topologies as a subspace of $\mathbb{R}_K$. Therefore both of these intervals are connected as they are in $\mathbb{R}$. Let $U, V$ be open sets of $\mathbb{R}_K$ such that $U \cup V = \mathbb{R}_K$ and $U \cap V = \emptyset$. Then
\[
(1) \quad (-\infty, 0) = (U \cap (-\infty, 0)) \cup (V \cap (-\infty, 0))
\]
\[
(2) \quad (0, \infty) = (U \cap (0, \infty)) \cup (V \cap (-\infty, 0)).
\]
Since (1) holds, and $(-\infty, 0)$ is connected, it must be true that $U \cap (-\infty, 0) = (-\infty, 0)$ or $V \cap (-\infty, 0) = (-\infty, 0)$. A similar situation occurs in (2). Without loss of generality suppose that $U \cap (-\infty, 0) = (-\infty, 0)$ which implies $(-\infty, 0) \subseteq U$ and similarly $(0, \infty) \cap V = V$ which implies $(0, \infty) \subseteq V$. It must be the case that $U = (-\infty, 0]$ and $V = (0, \infty)$ or $U = (-\infty, 0)$ and $V = [0, \infty)$. In either case, $V$ or $U$ fails to be open. Therefore $\mathbb{R}_K$ is connected.

$\mathbb{R}_K$ is strictly finer than the standard topology on $\mathbb{R}$. Then a continuous function $f : \mathbb{R} \to Y$, when $Y = \mathbb{R}_K$ is still continuous when $Y = \mathbb{R}$. Thus $f$ satisfies the conditions for the Intermediate Value Theorem to hold. Let $f$ be a path from 0 to 1 in $\mathbb{R}_K$ where $f(0) = 0$ and $f(1) = 1$. The Intermediate Value Theorem holds so it must be the case that
\[
[0, 1] \subseteq f([0, 1]) \subseteq \mathbb{R}_K.
\]
$f$ is continuous and $[0, 1] \subseteq \mathbb{R}$ is compact which implies $f([0, 1])$ must also be compact. By Theorem 27.1, $[0, 1] \subseteq f([0, 1]) \subseteq \mathbb{R}_K$ must also be compact because $[0, 1] \subset \mathbb{R}_K$ is a closed interval. But this is impossible since (a) showed $[0, 1] \subset \mathbb{R}_K$ is not compact. Therefore $\mathbb{R}_K$ is not path-connected.

$\square$
Show that a connected metric space having more than one point is uncountable.

Let $X$ be a connected space with metric $d$ having at least two points, say $a$ and $b$. Suppose there is some number $c$ between 0 and $d(a, b)$ such that no point $x \in X$ satisfies $d(a, x) = c$. Then the sets $\{ x \in X \mid d(a, x) < c \}$ and $\{ x \in X \mid d(a, x) > c \}$ form a separation of $X$, which is a contradiction since $X$ is assumed to be connected. Then for every real number $c$ between 0 and $d(a, b)$ there is a point $x \in X$ such that $d(a, x) = c$. Then the image of the function $f : X \to \mathbb{R}$ defined by $f(x) = d(a, x)$ must contain the entire interval $[0, d(a, b)]$, and thus $X$ must be uncountable.
Let $X$ be a compact Hausdorff space. Let $\{A_n\}$ be a countable collection of closed sets of $X$ with empty interior. Then the interior of $A = \bigcup A_n = A$ is empty.

**Proof.** Let $U_0$ be a nonempty open subset of $X$. Then $U_0$ is not contained in $A_1$ as the only open set contained in $A_1$ is the empty set. So $U_0 \setminus A_1 \neq \emptyset$. So there is an $x_0$ in $U_0 \setminus A_1$. Then $x_0 \notin X \setminus (U_0 \setminus A_1)$. By Lemma 26.4, there are disjoint open sets $U_1$ and $V_0$ containing $x_0$ and $X \setminus (U_0 \setminus A_1)$ respectively.

Let $y \in X \setminus (U_0 \setminus A_1)$. Then $V_0$ is an open nbhd of $y$ with $V_0 \cap U_1 = \emptyset$. So $y \notin cl(U_1)$. Thus $U_1 \subset cl(U_1) \subset U_0 \setminus A_1 \subset U_0$.

Recursively define the descending chain of sets $U_n$ by

$$U_n \subset cl(U_n) \subset U_{n-1} \setminus A_n \subset U_{n-1}.$$

Since $cl(U_n) \subset U_0 - A_n$ for all $n$, we have $\bigcap cl(U_n) \subset U_0 - A$. By Theorem 26.9, $\bigcap cl(U_n) \neq \emptyset$. So there is a point in $U_0 - A$. Thus given a nonempty open set $U_0$ in $X$ we can find a point of $U_0$ not contained in $A$. Therefore $A$ does not contain a nonempty open subset. Hence $int(A) = \emptyset$. Q.E.D.
Problem 108. Show that $[0, 1]$ is not limit point compact as a subspace of $\mathbb{R}_l$.

Proof. It is necessary and sufficient to find an infinite subset of $[0, 1]$ that does not have a limit point under the lower limit topology. The basis of the lower limit topology on $\mathbb{R}$ is the collection of all half open intervals of the form

$$[a, b) = \{x | a \leq x < b\}.$$  

Consider the infinite subset

$$S = \{1 - 1/n | n \in \mathbb{N} \text{ and } n > 1\}$$

Claim: no element of $S$ is a limit point of $S$. Let $s \in S$. Then $s = 1 - 1/n$ for some $n \in \mathbb{N}$, $n > 1$. There exists an open (in the lower limit topology) interval

$$[1 - 1/n, 1 - \frac{1}{n+1})$$

containing $s$ and no other element of $S$. Thus $s$ is not a limit point.

Claim: $S$ has no limit point in $[0, 1]$. First consider the case when $x = 1$. The neighborhood $[1, 2)$ in $\mathbb{R}_l$ contains $x$ and no point of $S$. Now suppose $x = 0$. $[0, 1/2)$ is a neighborhood of 0 and contains no elements in $S$. Let $x \in (0, 1) - S$. Then $1 - \frac{1}{n} < x < 1 - \frac{1}{n+1}$ (with the possibility that $n = 1$). $\mathbb{R}_l$ is strictly finer than the standard topology on $\mathbb{R}$ so all the open sets in the standard topology are open in the lower limit topology. $U = (1 - \frac{1}{n}, 1 - \frac{1}{n+1})$ is also open in $\mathbb{R}_l$ as a result. $x \in U$ and $U$ does not contain any elements of $S$. Therefore we conclude that $S$ has no limit point.
Let \((X, d)\) be a metric space and \(f : X \rightarrow X\) that satisfies
\[d(x, y) = d(f(x), f(y))\]
for all \(x, y \in X\). We say \(f\) is an isometry. Show that if \(X\) is compact, then \(f\) is bijective.

It follows from the fact that \(f\) is an isometry that \(f\) is injective since for \(x \neq y\), then \(d(x, y) \neq 0\) and
\[d(f(x), f(y)) = d(x, y) \neq 0\]
so \(f(x) \neq f(y)\). It also follows that \(f\) is continuous, using the \(\epsilon - \delta\) criterion with \(\epsilon = \delta\). It remains to show that \(f\) is surjective. Assume for a contradiction that \(f\) is not surjective, so there exists \(a \in X\) with \(a \notin f(X)\). Since \(X\) is compact, \(f(X)\) is compact and it is closed since every metric space are Hausdorff. So \(f(X)'\) is open, thus there exists \(\epsilon > 0\) such that
\[B_{\epsilon}(a) \cap f(X) = \emptyset\]  \(\text{(1)}\)

We set \(x_1 = a\) and for all \(n \geq 1\) we set \(x_{n+1} = f(x_n)\). So that
\[d(x_k, x_1) > \epsilon,\]
for all \(k \geq 2\) since \(x_k \in f(X)\) and we have (1). More generally, for \(n \neq m\) with \(n > m\), we use the inequality with \(k = n - m + 1\) and apply \(f\) \(m - 1\) times, thus
\[d(x_n, x_m) > \epsilon,\]
since \(f\) is an isometry. Thus, this sequence does not have a converging subsequence, since every pair of point are at least \(\epsilon\) away from each other. This is a contradiction since in metric spaces compact spaces are also sequentially compact and the sequence \(\{x_k\}_{k \geq 1}\) is contained in the compact space \(X\). Therefore, \(f\) is surjective and a bijection. It also follows that \(f\) is an homeomorphism since the inverse is also an isometry, thus continuous.
Let \((X, d)\) be a metric space, and \(f : X \to X\).

(a) Suppose that \(f\) is a contraction and \(X\) is compact.

Then \(\exists \alpha < 1\) s.t. \(d(f(x), f(y)) \leq \alpha d(x, y)\).

So, \(\forall \varepsilon > 0\) \(\exists \delta = \varepsilon / \alpha > 0\) s.t. \(\forall x, y \in X\)

\[ d(x, y) < \delta \implies d(f(x), f(y)) < \varepsilon \]

i.e. \(f\) is continuous.

Hence \(f(X)\) is compact.

Note that \(f(X) \subseteq X\) and by induction we get \(\forall n \in \mathbb{N}\),

\[ f^{n+1}(X) \subseteq f^n(X), \text{ where } f' = f \text{ and } f^{n+1} = f \circ f^n. \]

Also \(f^n(X)\) is compact \(\forall n \in \mathbb{N}\).

Now by Theorem 26.9,

\[ A = \bigcap_{n=1}^{\infty} f^n(X) \neq \emptyset. \]

Since \(X\) is closed and bounded (\(\therefore X\) is compact),

\[ \text{diam } X = \max_{x, y \in X} \{d(x, y)\} = M < \infty. \]

Now \(\text{diam } f^n(X) \leq \alpha^n \text{ diam } X = \alpha^n M, \forall n \in \mathbb{N}. \)

so, \(\text{diam } A = \lim_{n \to \infty} \text{diam } (f^n(X)) \leq \lim_{n \to \infty} \alpha^n M = 0. \)

\(\therefore f\) is continuous and \(\alpha < 1\).

Hence \(\text{diam } A = 0. \)

Since \(A \neq \emptyset\) and \(\text{diam } A = 0\), \(A = \{a\}, a \in X.\)

so, \(a \in A\) and \(f(a) \in A. \)

\[ \implies f(a) = a \quad \text{and } a \text{ is a fixed point.} \]

If \(x \in X\) is a fixed point of \(f\), then \(x \in A\). So, \(x = a. \)

\(\therefore f\) has a unique fixed point.
(b) Suppose that \( f \) is a shrinking map and \( X \) is compact.

Then \( d(f(x), f(y)) < d(x, y) \forall x, y \in X \) with \( x \neq y \).

So, \( f \) is continuous.

Let \( A \) as before. Then \( A \neq \emptyset \) and \( \text{diam} \, A = M < \infty \).

Note that \( A \) is closed as it is the countable intersection of closed sets.

By the definition of \( A \), we have \( f(A) \subseteq A \).

Now let \( x \in A \). Then \( x \in f^n(x), \forall n \in \mathbb{N} \).

So, we can take a sequence \( x_1, x_2, \ldots, x_n, \ldots \) such that \( f^{n+1}(x_n) = x \forall n \in \mathbb{N} \).

Since compactness \( \Rightarrow \) sequential compactness in a metric space,

\( \exists \) a subsequence \( \{x_{n_k}\}_{k=1}^{\infty} \) that converges to some point \( a \in A \). (\( \because \) \( A \) is closed).

Since \( f \) is continuous,

\[
  f(a) = f\left(\lim_{k \to \infty} x_{n_k}\right) = \lim_{k \to \infty} f(x_{n_k}) = \lim_{k \to \infty} x = x.
\]

\( \Rightarrow \) \( x \in f(A) \)

So, \( A \subseteq f(A) \).

Hence, \( A = f(A) \).

Note that \( A \times A \) is compact as \( A \) is compact.

Now, \( d : A \times A \rightarrow \mathbb{R} \) is continuous and by the Theorem 27.4 ,

\( \exists (x_1, x_2) \in A \times A \) s.t. \( d(y_1, y_2) \leq d(x_1, x_2) \)

for all \( (y_1, y_2) \in A \times A \).
Therefore, \( \text{diam } A = M = d(x_1, x_2) \).

Since \( A = f(A) \),
\[ x_1 = f(\overline{x}_1) \quad \text{and} \quad x_2 = f(\overline{x}_2) \quad \text{for some} \]
\[ \overline{x}_1, \overline{x}_2 \in A \, . \]

Then, if \( \overline{x}_1 \neq \overline{x}_2 \), we have
\[ d(f(\overline{x}_1), f(\overline{x}_2)) < d(\overline{x}_1, \overline{x}_2) \]
\[ \text{i.e.} \quad \text{diam } A = M = d(x_1, x_2) = d(f(\overline{x}_1), f(\overline{x}_2)) < d(\overline{x}_1, \overline{x}_2) \]

\[ \rightarrow \quad \leftarrow \, . \]
Hence \( \overline{x}_1 = \overline{x}_2 \)
\[ \Rightarrow \quad x_1 = x_2 \]
\[ \Rightarrow \quad \text{diam } A = 0 \, . \]

So, \( A \) has a one point and \( A = f(A) \).
\[ \text{i.e.} \quad f(x) = x \]

\[ : \text{ } f \text{ has a unique fixed point.} \]
Let \( X = [0,1] \), and \( f(x) = x - x^{2/2} \)

since, \( 0 \leq x \left( 1 - \frac{x}{2} \right) < 1 \) for \( x \in [0,1] \).

\( f \) maps \( X \) into \( X \).

Let \( x, y \in X \) with \( x \neq y \). Then,

\[
|f(x) - f(y)| = \left| (x - x^{2/2}) - (y - y^{2/2}) \right|
\]

\[
= \left| (x-y) - (x^{2}-y^{2})/2 \right|
\]

\[
= |x-y| \left| 1 - (x+y)/2 \right|
\]

\[
< |x-y| \quad (\because x+y \Rightarrow \frac{x+y}{2} < 1 \text{ for } x, y \in X)
\]

Hence \( f \) is a shrinking map.

If we choose \( x, y < 1 - \alpha \), where \( \alpha < 1 \).

Then,

\[
\frac{x+y}{2} < \frac{(1-\alpha) + (1-\alpha)}{2} = 1 - \alpha.
\]

\[
\Rightarrow - \left( \frac{x+y}{2} \right) > \alpha - 1
\]

\[
\Rightarrow 1 - \left( \frac{x+y}{2} \right) > \alpha.
\]

Now,

\[
|f(x) - f(y)| = |x-y| \left| 1 - (x+y)/2 \right|
\]

\[
> \alpha |x-y| \quad \text{for } \alpha < 1.
\]

Hence it is not a contraction.
(d) Let $f: \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \frac{[x + (x^2+1)^{1/2}]}{2}.$$ 

Let $x, y \in \mathbb{R}$ with $x \neq y$,

$$|f(x) - f(y)| = \frac{|x + \sqrt{x^2+1} - y - \sqrt{y^2+1}|}{2}.$$

$$= \frac{1}{2} \left| (x - y) + \frac{\sqrt{x^2+1} - \sqrt{y^2+1}}{\sqrt{x^2+1} + \sqrt{y^2+1}} \right|.$$

$$= \frac{1}{2} \left| (x - y) + \frac{(x-y)(x+y)}{\sqrt{x^2+1} + \sqrt{y^2+1}} \right|.$$

$$= \frac{1}{2} |x-y| \left( 1 + \frac{x+y}{\sqrt{x^2+1} + \sqrt{y^2+1}} \right).$$

$$\leq \frac{1}{2} |x-y| \left( 1 + \frac{|x|+|y|}{\sqrt{x^2+1} + \sqrt{y^2+1}} \right).$$

$$< \frac{1}{2} |x-y| \left( 1 + \frac{|x|+|y|}{\sqrt{x^2} + \sqrt{y^2}} \right).$$

$$= \frac{1}{2} |x-y| \left( 1 + \frac{|x|+|y|}{|x|+|y|} \right).$$

$$= \frac{1}{2} |x-y| \cdot 2 = |x-y|.$$

So, $|f(x) - f(y)| < |x-y|$ for all $x, y \in \mathbb{R}$ with $x \neq y$.

Hence $f$ is a shrinking map.
To show $f$ is not a contraction,

Let $y = x - 1$. Then,

$$|f(x) - f(y)| = \frac{1}{2} \left| 1 + \frac{2x - 1}{\sqrt{x^2 + 1} + \sqrt{(x-1)^2 + 1}} \right|$$

as $x \to \infty$, \( \frac{2x - 1}{\sqrt{x^2 + 1} + \sqrt{(x-1)^2 + 1}} \to 1 \).

So, for large enough $x \in \mathbb{R}$,

$$|f(x) - f(y)| \to 1 = |x - y|.$$

Hence $f$ is not a contraction.

Suppose $f$ has a fixed point.

Then, $f(x) = x$, $x \in \mathbb{R}$.

\[ \Rightarrow \quad \frac{x + (x^2 + 1)^{\frac{1}{2}}}{2} = x \]

\[ \Rightarrow \quad (x^2 + 1)^{\frac{1}{2}} = x \]

\[ \Rightarrow \quad \frac{x^2 + 1}{2} > x \quad \text{for all} \quad x \in \mathbb{R}. \]

So, $f$ has no fixed point.
Show that the rationals $\mathbb{Q}$ are not locally compact.

Let $U$ be a neighborhood of some $x \in \mathbb{Q}$. So there exists some $q \in \mathbb{Q}$ such that $(x-q, x+q) \cap \mathbb{Q} \subset U$. Thinking of $(x-q, x+q)$ as a subset of the real numbers, there exists some irrational number, $y$, in $(x-q, x+q)$. Now $(x-q, y) \cup (y, x+q) = (x-q, x+q)$, and let $\ell_1 =$length$(x-q, y)$ and $\ell_2 =$length$(y, x+q)$. So

$$\mathbb{Q} \cap \left\{ \mathbb{Q} \cap \left( (x-q, y-\frac{\ell_1}{n}) \cup (y+\frac{\ell_2}{n}, x+q) \right) \right\}$$

is an open cover of $(x-q, x+q) \cap \mathbb{Q}$, with no finite subcover. So $(x-q, x+q) \cap \mathbb{Q}$, and by extension $U$, is not compact. Thus $\mathbb{Q}$ is not locally compact.
118. Prove the following lemma. Use the lemma to prove the proposition.

**Lemma 1.** A space $X$ is first-countable if and only if for each $x \in X$ there exist nested open neighborhoods $V_n$ of $x$ for $n \in \mathbb{N}$ (i.e., $V_1 \supset V_2 \supset V_3 \supset \cdots$) such that every neighborhood of $x$ contains $V_n$ for at least one $n \in \mathbb{N}$.

**Proof.** The $\iff$ direction is trivial by the definition of first-countable. Now we show the $\implies$ direction. Suppose $X$ is first-countable, then for any fixed $x \in X$, there is a countable base $\{U_n\}_{n=1}^\infty$ at $x$. We define $V_1 = U_1$ and $V_n = V_{n-1} \cap U_n$ for $n = 2, 3, \ldots$. We show by induction that $\{V_n\}_{n=1}^\infty$ is a countable collection of nested open neighborhoods of $x$. Note that $V_1 = U_1$ is an open neighborhood of $x$, $V_2 = V_1 \cap U_2$ is open, $x \in V_1$, and $x \in U_2$, so $V_2 \subset V_1$ is an open neighborhood of $x$. Now assume that $\{V_n\}_{n=1}^k$ is a collection of nested open neighborhoods of $x$ for some $k \in \mathbb{N}$. Similarly, $V_{k+1} = V_k \cap U_{k+1} \subset V_k$ is open and contains $x$, so $\{V_n\}_{n=1}^{k+1}$ is a collection of nested open neighborhoods of $x$. Thus the induction is complete. Now for any open neighborhood $U$ of $x$, there is some $N \in \mathbb{N}$ so that $U_N \subset U$. By construction, we have $V_N \subset U_N \subset U$. \hfill $\Box$

**Proposition 2.** If $X$ is limit point compact, Hausdorff, and first countable, then $X$ is sequentially compact.

**Proof.** Let $A = \{a_n\}_{n=1}^\infty$ be any sequence in $X$. If $A$ is finite, then there exists a subsequence $\{a_{n_k}\}_{k=1}^\infty$ such that $a_{n_k} = a$ for some $a \in A$, for all $k \in \mathbb{N}$. So $\{a_{n_k}\}_{k=1}^\infty$ converges to this $a \in X$. Thus we may assume that $A$ is infinite. Since $X$ is limit point compact, there exists a limit point $a \in X$ of $A$. Note that $X$ is first countable, then by Lemma 1, there exists a nested countable base $\{V_n\}_{n=1}^\infty$ at $a$. Also note that $X$ is Hausdorff, so $X$ satisfies the $T_1$ axiom. Then by Theorem 17.9 (page 99), every neighborhood of $a$ contains infinitely many point of $A$. Then we can construct a subsequence $\{a_{n_k}\}_{k=1}^\infty$ as follows: Pick $a_{n_1} \in V_1 \cap A$ with $a_{n_1} \neq a$. For $k = 2, 3, \ldots$, pick $\{a_{n_k}\} \in V_k \cap A$ such that $a_{n_k} \neq a$ and $n_k > n_{k-1}$. Now we show $a_{n_k} \to a$ as $k \to \infty$. For any neighborhood $U$ of $a$, there is some $K \in \mathbb{N}$ so that $V_K \subset U$ (since $\{V_n\}_{n=1}^\infty$ is a countable base). Then we have $a_{n_K} \in V_K \subset U$ and, for any $k > K$, $a_{n_k} \in V_k \subset V_K \subset U$ (since $\{V_n\}_{n=1}^\infty$ is nested). Thus $a_{n_k} \in U$ for all $k \geq K$, and since $U$ was arbitrary, we have that $\{a_{n_k}\}_{k=1}^\infty$ converges to $a \in X$. Therefore, $X$ is sequentially compact. \hfill $\Box$
a) Suppose $A, B$ are nonempty, disjoint, closed subsets of a metric space $X$. Show that the function $f : X \to [0,1]$ defined by
\[ f(x) = \frac{\text{dist}(x, A)}{\text{dist}(x, A) + \text{dist}(x, B)} \]
is continuous with $f(x) = 0$ for all $x \in A$, $f(x) = 1$ for all $x \in B$, and $0 < f(x) < 1$ for all $x \in X \setminus (A \cup B)$.

b) Show that if $X$ is a connected metric space with at least two distinct points, then $X$ is uncountable.

a) First note that the distance function is continuous on $X$. Thus $f(x)$ is continuous on $X$ except for the points $x \in X$ in which $\text{dist}(x, A) + \text{dist}(x, B) = 0$. Since $\text{dist}(x, C) = \inf_{y \in C} \{\text{dist}(x, y)\} \geq 0$ for all $C \subset X$, then $\text{dist}(x, A) + \text{dist}(x, B) = 0$ only when $\text{dist}(x, A) = 0$ and $\text{dist}(x, B) = 0$. This would imply that $x \in \text{cl}(A) \cap B = \emptyset$ or $x \in A \cap \text{cl}(B) = \emptyset$ since $A$ and $B$ are disjoint and closed. Thus $\text{dist}(x, A) + \text{dist}(x, B) > 0$ for all $x \in X$.

Let $x \in A$. Then $\text{dist}(x, A) = 0$ implies $f(x) = \frac{0}{\text{dist}(x, B)} = 0$, since $\text{dist}(x, B) > 0$ as $A$ and $B$ share no limit points. Similarly, let $x \in B$, so $\text{dist}(x, B) = 0$ and $\text{dist}(x, A) > 0$. Thus $f(x) = \frac{\text{dist}(x, A)}{\text{dist}(x, A)} = 1$. Now suppose $x \in X \setminus (A \cup B)$. Since $X$ being a metric space implies $X$ is regular, then $x$ and $A$ may be separated by disjoint open sets (similarly with $x$ and $B$). Thus $\text{dist}(x, A) > 0$ and $\text{dist}(x, B) > 0$. So $f(x) > 0$ since both the numerator and denominator are positive. Finally,
\[ f(x) = \frac{\text{dist}(x, A)}{\text{dist}(x, A) + \text{dist}(x, B)} < \frac{\text{dist}(x, A)}{\text{dist}(x, A)} = 1. \]

b) Let $X$ contain the distinct points $x_1$ and $x_2$. Letting $A = \{x_1\}$ and $B = \{x_2\}$, then using $f(x)$ as in part a, $f$ is a continuous function from $X \to [0,1]$ where $f(A) = f(x_1) = 0$ and $f(B) = f(x_2) = 1$. Since $f$ is continuous and $X$ is connected, then $f(X)$ is also connected in $[0,1]$. But the only connected subset of $[0,1]$ which also contains 0 and 1 is all of $[0,1]$. Since the image of $X$ under $f$ is uncountable, then so is $X$. 

120. Show that the set $A = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}$ is a compact, connected space of $\mathbb{R}^2$.

P: \{1\} \in \mathbb{R}$ is a singleton set, thus closed.

Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by $(x, y) \to x^2 + y^2$.

$f$ is continuous. $A = f^{-1}(\{1\})$ is closed in $\mathbb{R}^2$.

Let $g: [0, 2\pi] \to \mathbb{R}^2$ be defined by $x \to (\cos x, \sin x)$. $g$ is continuous. $[0, 2\pi]$ is connected. $g([0, 2\pi]) = A$. $A$ is connected. ■
Definition: A subset \( A \) of a metric space \((X, d)\) is precompact if its closure \( cl(A) \) is compact.

Show that if \( A \) is precompact, then for every \( \epsilon > 0 \) there exists a finite covering of \( A \) by open balls of radius \( \epsilon \) with centers in \( A \).

Proof. Let \( \epsilon > 0 \) and let \( O = \{ B(a, \epsilon) : a \in A \} \). Then \( O \) is an open cover of \( A \). Suppose \( x \) is a limit point of \( A \). Then \( \exists a \in A \) so that \( a \in B(x, \frac{\epsilon}{2}) \setminus \{x\} \). Then \( d(x, a) < \frac{\epsilon}{2} \) implies \( x \in B(a, \epsilon) \), so \( O \) is also an open cover of \( cl(A) \). Since \( A \) is precompact, there is a finite subcover of \( O \) that covers \( cl(A) \). Since \( A \subseteq cl(A) \), this subcover is also a finite covering of \( A \) by open balls of radius \( \epsilon \) with centers in \( A \).
Let \( f : X \to Y \) be a continuous bijection with \( Y \) hausdorff and \( X \) a closed subset of \( \mathbb{R}^n \). Show that if for all \( y \in Y \) there exists a neighborhood \( Z_y \) such that \( f^{-1}(Z_y) \) is bounded, then \( f \) is a homeomorphism. Prove that a continuous bijection \( f : \mathbb{R} \to \mathbb{R} \) is a homeomorphism.

To show that \( f : X \to Y \) is a homeomorphism we have to show that for all \( y \in Y \), \( f^{-1}(Z_y) \) is contained in a compact subset and the result will follow from exercise 124. Note that since \( X \) is a closed subset of \( \mathbb{R}^n \), we only need that \( f^{-1}(Z_y) \) is bounded, since its closure is closed and bounded, thus compact. It follows from assumption that for all \( y \in Y \) there exists a neighborhood \( Z_y \) such that \( f^{-1}(Z_y) \) is bounded and it is contained in its closure, which is compact by the previous argument. Therefore \( f \) is a homeomorphism by exercise 124.

To show that \( f : \mathbb{R} \to \mathbb{R} \) is a homeomorphism we can use this result since \( \mathbb{R} \) is Hausdorff. First, note that a continuous bijection from \( \mathbb{R} \) to \( \mathbb{R} \) need to be strictly monotone, otherwise it fails to be one to one. If a function is increasing then decreasing it reaches a relative maximum at \( x \) so for \( a < x < b \) we have that \( f(a) < f(x) \) and \( f(b) < f(x) \) and then one can use the intermediate value theorem to show that \( f \) will reach the value \( f(a) \) or \( f(b) \) twice. A similar argument can be made for the case decreasing then increasing with a relative minimum.

Assume without loss of generality that \( f \) is strictly increasing. Otherwise one could argue with \(-f\) but \( f \) is a homeomorphism if \(-f\) is. Let \( y \in \mathbb{R} \), we need to find a neighborhood of \( y \) with bounded inverse image. Let \( a, b \) such that \( y \in (a, b) \), since \( f \) is surjective there exists \( u, v \) such that \( f(u) = a \) and \( f(v) = b \). But \( f \) is strictly increasing so \( f^{-1}((a, b)) = (u, v) \) and the result follows since \((u, v)\) is bounded and \( y \) was arbitrary. Therefore a bijection \( f : \mathbb{R} \to \mathbb{R} \) is a homeomorphism.
Problem 129. Prove Sperner’s lemma in three dimensions: Let a tetrahedron, whose vertexes have been labeled $A, B, C,$ and $D$ be divided into subtetrahedra with vertexes labeled so that only three labels appear on each face of the original tetrahedron. Then at least one subtetrahedron has all four labels.

Proof. First consider the number of sides labeled $ABC$ in the division of the tetrahedron into subtetrahedron. The sides labeled $ABC$ on the inside of the tetrahedron belong to exactly two subtetrahedron. Thus the number of sides labeled $ABC$ on the inside of the tetrahedron must be even. The labeling of $ABC$ on the outside of the main tetrahedron only occurs at one side of the main tetrahedron (namely the side with vertexes $ABC$). From Sperner’s Lemma for the two dimensional case, we know the number of sides labeled $ABC$ on the outside must be odd. Therefore the total number of sides labeled $ABC$ must be odd. It is analogous to find the amount of sides labeled $ABD$, $BCD$, and $ACD$ are all odd, as well.

Let $d$ be the number of subtetrahedron with vertexes $ABCD$ and $a$ be the number of subtetrahedron with sides $ABC$ that do not have vertexes $ABCD$. The other subtetrahedra with sides $ABC$ are those with vertexes $ABCA$, $ABCB$, and $ABCC$. Each subtetrahedra of this form must have two sides with vertexes $ABC$ as opposed to $ABCD$ only having one side with vertexes $ABC$. Thus the total amount of sides with vertexes $ABC$ is $2a + d$ which is odd by the results of the previous paragraph. Therefore $d$ is odd. This is a stronger result compared to proving there is a single complete subtetrahedron.

\[\square\]
Consider an annulus triangulated with vertices labeled A, B, or C. The content may be defined as for cells, while there are now two indexes: one for the outside boundary, $I_1$, and one for the inside boundary, $I_2$. Prove that $C = I_1 - I_2$.

Let $X$ denote the triangulated annulus and $Y$ denote the inner complement complement of $X$ (see figure to right). Note that the boundary of $X_1$ is the inner boundary of $X$.

Using the labeling of the inner boundary of $X$, we can create a triangulation of $X_1$, in any manner we choose. So the triangulation of $X_1$ combined with the triangulation of $X$ gives a triangulation of $X \cup X_1$. Thus by the Index Lemma, $C_{X\cup X_1} = I_{X\cup X_1} = I_1$. But, as the content is defined as for cells, then $C_{X\cup X_1} = C_X + C_{X_1} = C_X + I_2$, again by the Index Lemma. So $I_1 = C_{X\cup X_1} = C_X + I_2$ implies $C_X = I_1 - I_2$. 

![Diagram of an annulus with labeled vertices and boundaries](image)
Let \( V(x,y) = (y, 1-x^2) \)

\[
\begin{align*}
(a) \quad & x^2 + y^2 = 2x \\
& x^2 - 2x + y^2 = 0 \\
& (x-1)^2 + y^2 = 1 \\
\end{align*}
\]

\( \sigma : (x-1)^2 + y^2 = 1 \)

Now, \( V : \sigma([0,1]) \to \mathbb{R}^2 \) is a continuous vector field with \( V \neq 0 \) on \( \sigma([0,1]) \).

\[
\begin{align*}
(y, 1-x^2) &= (0, c) \quad \text{North, } (c > 0) \\
y &= 0, \quad 1-x^2 = c \\
(x-1)^2 + y^2 &= 1 \\
(x-1)^2 &= 1 \\
x-1 &= \pm 1 \\
x &= 2, 0 \\
x = 2 \quad \Rightarrow \quad c = 1 - 2^2 = -3 \\
x = 0 \quad \Rightarrow \quad c = 1 \\
\end{align*}
\]

so, North is at \((0,0)\).

\( W_V(\sigma) = 1 \)
(b) \( x^2 + y^2 = -2x \)
\( x^2 + 2x + y^2 = 0 \)
\( (x+1)^2 + y^2 = 1 \)

\( \sigma : [0,1] \rightarrow \mathbb{R}^2 \) defined by \( \sigma(t) = (-1 + \cos(2\pi t), \sin(2\pi t)) \)

is a simple closed curve oriented counter clockwise.

So, \( \nabla : \sigma([0,1]) \rightarrow \mathbb{R}^2 \) is a continuous vector field with \( \nabla \neq 0 \) on \( \sigma([0,1]) \).

\((y, 1-x^2) = (0, c) \) North \((c > 0)\)
\[ y = 0, \quad 1-x^2 = c \]
\[ (x+1)^2 + y^2 = 1 \]
\[ (x+1)^2 = 1 \]
\[ x+1 = \pm 1 \]
\[ x = -2, 0 \]
\[ x = -2 \Rightarrow c = 1 - (-2)^2 = -3 \]
\[ x = 0 \Rightarrow c = 1 \]

So, North is at \((0,0)\)

\[ W_{\nabla}(\sigma) = -1 \]
\[ \mathbf{V} = (y, 1-x^2) \]

\[ \begin{align*}
(C) & \quad x^2 + y^2 = 2y \\
& \quad x^2 + y^2 - 2y = 0 \\
& \quad x^2 + (y-1)^2 = 1
\end{align*} \]

\[ \begin{align*}
(D) & \quad x^2 + y^2 = -2y \\
& \quad x^2 + y^2 + 2y = 0 \\
& \quad x^2 + (y+1)^2 = 1
\end{align*} \]

\[ \gamma(t) = (\cos(2\pi t), 1 + \sin(2\pi t)) \]

\[ \gamma : [0, 1] \rightarrow \mathbb{R}^2 \]

\[ \gamma([0,1]) = \partial \mathbb{D} \]

Now \( \partial \mathbb{D} = \partial \mathbb{D}^2 \) and \( \gamma \) is a counter clockwise parametrization \( \partial \mathbb{D} \).

Also, \( \mathbf{V} \) is continuous vector field on all of \( \partial \mathbb{D} \).

Since \( \mathbf{V} \) is non vanishing on \( \partial \mathbb{D} \) for both \((C)\) \& \((D)\),

\[ \text{W}_{\mathbf{V}}(\gamma) = 0. \]
Problem 135. Compute the winding number of \( V(x, y) = (y(x^2 - 1), x(y^2 - 1)) \) on the following curves.

\[
(a) \quad x^2 + y^2 - 2x - 2y + 1 = 0 \quad (b) \quad x^2 + y^2 + x + y = 1/2 \\
(c) \quad x^2 + y^2 = 1 \quad \quad (d) \quad x^2 + y^2 = 4
\]

Solution (a). Points that satisfy the following equations represent the points on the circle (of radius 1) where the vector \( V \) points north:

\[
\begin{cases}
(x - 1)^2 + (y - 1)^2 = 0 \\
y(x^2 - 1) = 0 \\
x(y^2 - 1) \geq 0
\end{cases}
\]

The only point that satisfies these equations is the point \((1, 2)\) on the circle. As one travels around the circle counterclockwise, the \(x\)-value of \( V, (y(x^2 - 1) \) goes from positive to negative which means \( V \) moves from quadrant one to quadrant two (so +1). Since this is the only solution, the winding number is equal to 1.

Solution (b): Points that satisfy the following equations represent the points on the circle where the vector \( V \) points north:

\[
\begin{cases}
x^2 + y^2 + x + y = \frac{1}{2} \\
y(x^2 - 1) = 0 \\
x(y^2 - 1) \geq 0
\end{cases}
\]

The two points \((-1, \frac{\sqrt{3} - 1}{2})\) and \((-\frac{1 - \sqrt{3}}{2}, 0)\) satisfy this system. As one travels around the circle counterclockwise, the \(x\)-value of \( V, (y(x^2 - 1) \) goes from negative to positive at the point \((-1, \frac{\sqrt{3} - 1}{2})\) (so -1). When considering the point \((-\frac{1 - \sqrt{3}}{2}, 0)\), \(y(x^2 - 1) \) goes from positive to negative (so +1). This results in a winding number of 0.

Solution (c): Points that satisfy the following equations represent the points on the circle where the vector \( V \) points north:

\[
\begin{cases}
x^2 + y^2 = 1 \\
y(x^2 - 1) = 0 \\
x(y^2 - 1) \geq 0
\end{cases}
\]

The only point that satisfies these conditions is \((-1, 0)\). When traveling around the unit circle counterclockwise, \(y(x^2 - 1) \) changes from negative to positive at this point resulting in a winding number of -1.
Solution (d): Points that satisfy the following equations represent the points on the circle where the vector $V$ points north:

\[
\begin{align*}
    x^2 + y^2 &= 4 \\
    y(x^2 - 1) &= 0 \\
    x(y^2 - 1) &\geq 0.
\end{align*}
\]

There are three points satisfying this system; namely $(-2, 0), (1, -\sqrt{3}), (1, \sqrt{3})$. At each one of these points, $y(x^2 - 1)$ goes from positive to negative resulting in a winding number of 3.
Let $\phi$ be the height function on a desert island. Let $V$ be the corresponding vector field. Let $P$ be the number of peaks, $C$ the number of cols, and $B$ the number of bottoms, and $L$ the number of lakes denote the only critical points of the island. Prove that $P - C + B = 1 - L$. Verify this by carefully identifying all the critical points on the island in Figure 11.4 (Henle, page 72).

**Fact** The winding number of $V$ along the shoreline is 1, and $P - C + B = 1$ for the same island with no lakes (from a previous problem).

**Proof.** Around each lake, the height of the shoreline must be higher than the lake. Then $V$ is non-zero on the shoreline. Let $e_0$ denote the counterclockwise curve around the island and $e_1, e_2, \ldots, e_L$ be the clockwise curves around each of the $L$ lakes. Then the winding number is 1 around $e_0$ and $-1$ about $e_i, 1 \leq i \leq L$. By the Poincaré Index Theorem $W_L(V) = P - C + B + L = 1$. \hfill $\square$
141. Show that if $X$ has a countable basis then every basis $\mathcal{C}$ for $X$ contains a countable basis for $X$. [Hint. For every pair of indices $n, m$ for which it is possible, choose $C_{n,m} \in \mathcal{C}$ such that $B_n \subset C_{n,m} \subset B_m$.]

P: Let $B = \{B_n\}_{n=1}^{\infty}$ be a countable basis. Pick $C_{n,m}$ (when possible). It is a countable sub-collection of $\mathcal{C}$ which is a (countable) basis; $\forall$ open $U \in X$ and $x \in U \exists$

- an open set $B_m \subset U$ containing $x$
- an open set $C \subset B_m$ containing $x$
- open set $B_n \subset C$ containing $x$

$\Rightarrow x \in C_{n,m}$. $\exists C_{n,m} \subset U$. $\{C_{n,m}\}$ is a countable basis by Lemma 13.2.■
Let $X$ have a countable basis and $A$ an uncountable subset of $X$. Show that uncountably many points of $A$ are limit points of $A$.

Let $A'$ be the set of limit points of $A$. Assume for a contradiction that $A'$ is countable, so that $F = A \setminus A'$ is uncountable. Let $x \in F$, since $x$ is not a limit point of $A$ there exists a neighborhood $U_x$ of $x$ that does not intersect $A \setminus \{x\}$. Moreover, there exists a basis element $B_x$ such that $x \in B_x$ and $B_x \subset U$. Note that for $x \neq y \in F$ we have that $B_x$ and $B_y$ don’t intersect $A$ at a different point than $x$ and $y$ respectively, so $x \notin B_y$ and $y \notin B_x$ and it follows that $B_x \neq B_y$.

So each element of $F$ is contained in a distinct basis element. This is a contradiction since $F$ is uncountable and $X$ have a countable basis. Therefore $A$ as uncountably many limit points.
Let \( X \) be a compact metrizable space. 
so, let \( d \) be a metric on \( X \) that induces 
the topology of \( X \).

\[ \forall n \in \mathbb{N}, \text{ consider the open cover } A = \left\{ B_d(x, \frac{1}{n}) : x \in X \right\} \]
of \( X \).

Then by compactness, \( \exists \) a finite subset \( X_n \) of \( X \)
s.t. \( A_n = \left\{ B_d(x, \frac{1}{n}) : x \in X_n \right\} \) covers \( X \).

Now define \( A = \bigcup_{n \in \mathbb{N}} A_n \).

since countable union of finite sets are 
countable, \( A \) is countable.

Let \( U \) be an open set in \( X \) and \( x \in U \).
Then \( \exists \varepsilon > 0 \) s.t. \( B_d(x, \varepsilon) \subset U \).

Choose \( N \in \mathbb{N} \) s.t. \( \frac{1}{N} < \varepsilon \).

Since \( A_n \) covers \( X \), \( \exists \) \( B_d(y, \frac{1}{N}) \) s.t. \( x \in B_d(y, \frac{1}{N}) \)
Let \( z \in B_d(y, \frac{1}{N}) \). Then
\[ d(x, z) \leq d(x, y) + d(y, z) < \frac{1}{N} + \frac{1}{N} = \frac{2}{N} < \varepsilon \]
so, \( z \in B_d(x, \varepsilon) \)

\[ \Rightarrow B_d(y, \frac{1}{N}) \subset B_d(x, \varepsilon) \subset U \]
Hence \( U \) can be written as union of the 
elements of \( A \).
\[ \text{i.e. } A \text{ is a basis by Lemma 13.2 of Munkres} \]
\[ \therefore X \text{ has a countable basis.} \]
Every metrizable space with a countable dense subset has a countable basis.

Proof. Let $X$ be metrizable with metric $d$, and have a countable dense subset $A$. For every $x \in A$ and $n \in \mathbb{N}_{>0}$ there is $B(x, \frac{1}{n})$. Since $A$ is countable we have $A_n = \{B(x, \frac{1}{n})| x \in A\}$ is countable. Define $B = \cup A_n$. Then $B$ is the countable union of countable sets. Hence $B$ is countable.

Let $x \in X$ and $n > 0$. By density of $A$ in $X$, there is an $a$ in $A$ so that $d(x, a) < \frac{1}{n}$. So $x \in B(a, \frac{1}{n})$ which is in $B$.

Let $B(x, \frac{1}{n})$ and $B(y, \frac{1}{m})$ be in $B$ and suppose $B(x, \frac{1}{n}) \cap B(y, \frac{1}{m}) = B_3$. Let $b \in B_3$. Let

$$\frac{1}{k} < \frac{1}{2} \min\{\frac{1}{n} - d(b, x), \frac{1}{m} - d(b, y)\}.$$ 

Then by density there is an $a \in A$ so that $b \in B(a, \frac{1}{k})$.

Let $t \in B(a, \frac{1}{k})$. Then

$$d(t, x) < d(a, t) + d(a, x) < d(t, a) + d(a, b) + d(b, x)$$

$$< \frac{1}{k} + \frac{1}{k} + d(b, x) < \frac{2}{k} + d(b, x)$$

$$< \min\{\frac{1}{n} - d(x, b), \frac{1}{m} - d(y, b)\} + d(x, b) \leq \frac{1}{n}.$$

Hence $t \in B(x, \frac{1}{n})$. Also,

$$d(t, y) < d(a, t) + d(a, y) < d(t, a) + d(a, b) + d(b, y)$$

$$< \frac{1}{k} + \frac{1}{k} + d(b, y) < \frac{2}{k} + d(b, y)$$

$$< \min\{\frac{1}{n} - d(x, b), \frac{1}{m} - d(y, b)\} + d(y, b) \leq \frac{1}{m}.$$

Hence $t \in B(y, \frac{1}{m})$. Thus $t \in B_3$. So $B(a, \frac{1}{k}) \subseteq B_3$. Therefore $B$ is a countable basis for $X$. Q.E.D.

Every metrizable Lindelöf space has a countable basis.

Proof. Let $X$ be metrizable and Lindelöf with metric $d$. For every $x \in X$ and $n \in \mathbb{N}_{>0}$ there is $B(x, \frac{1}{n})$. For a given $n$, $X$ is covered by these balls.
Moreover, $X$ is Lindelöf, so there is a countable subcover, $A_n$, of $X$. Let $\mathcal{B} = \bigcup A_n$. Then $\mathcal{B}$ is a countable union of countable sets. So $\mathcal{B}$ is countable.

Let $x \in X$. For every $n$, $A_n$ covers $X$. So $x$ is in a ball in $A_n$. Thus $x$ is in a ball in $\mathcal{B}$.

Now let $B(x, \frac{1}{n})$ and $B(y, \frac{1}{m})$ be in $\mathcal{B}$ and suppose $B(x, \frac{1}{n}) \cap B(y, \frac{1}{m}) = B_3$. Let $b \in B_3$.

Let $\frac{1}{k} < \frac{1}{n} \min\{\frac{1}{n} - d(b, x), \frac{1}{m} - d(b, y)\}$. Let $t \in X$ such that $b \in B(t, \frac{1}{k})$. Let $z \in B(t, \frac{1}{k})$. Then we have

$$d(z, x) \leq d(z, t) + d(t, x) \leq d(z, t) + d(b, t) + d(b, x)$$

$$\leq \frac{2}{k} + d(b, x) \leq \frac{1}{n}.$$

Thus $B(t, \frac{1}{k}) \subset B(x, \frac{1}{n})$.

Similarly, let $t \in X$ such that $b \in B(t, \frac{1}{k})$. Let $z \in B(t, \frac{1}{k})$. Then we have

$$d(z, y) \leq d(z, t) + d(t, y) \leq d(z, t) + d(b, t) + d(b, y)$$

$$\leq \frac{2}{k} + d(b, y) \leq \frac{1}{m}.$$

So $B(t, \frac{1}{k}) \subset B(y, \frac{1}{m})$. Therefore $B(t, \frac{1}{k}) \subset B(x, \frac{1}{n}) \cap B(y, \frac{1}{m})$. Ergo $\mathcal{B}$ is a countable basis for $X$.

Q.E.D.