Definition

Let $\mathcal{S}$ and $\mathcal{T}$ be topological spaces. A transformation $f: \mathcal{S} \to \mathcal{T}$ is **continuous** if for any point $P$ of $\mathcal{S}$ and subset $A$ of $\mathcal{S}$, $P \in A$ implies $f(P) \in f(A)$. A **topological transformation** is a continuous invertible transformation whose inverse is also continuous. Two sets are **topologically equivalent** if there is a topological transformation between them. **Topology** is the study of properties of sets that are preserved when the sets are subjected to topological transformations.

Exercises

10. Prove that the composition of two continuous transformations is continuous.
11. Let $\mathcal{S}$ be a discrete space. Show that every function from $\mathcal{S}$ to itself is continuous. What is the corresponding result for the indiscrete topology?
12. Prove that compactness and connectedness are topological properties.
13. Prove that Example 7 is topologically equivalent to a sphere with the usual topology.
14. Let $A$ be a subset of a topological space $\mathcal{S}$. A point $P$ of $\mathcal{S}$ is called a **limit point** of $A$ if every neighborhood of $P$ contains a point of $A$ other than $P$ itself. Show that if $P$ is a limit point of $A$, then $P$ is near $A$, but conversely a point near $A$ need not be a limit point of $A$.

**Notes.** The informal discussion of combinatorial topology begun in this chapter will be gradually supplanted by a more and more formal treatment over the course of this book. Naturally it is important to begin with an informal understanding. There are many excellent treatments of topology from an informal point of view. The chapters on topology in the books by Courant and Robbins [6] and Kasner and Newman [15] are good examples. Especially fine is the picture essay in the Life Science Library [3]. These references are concerned with the popular aspects of topology. More formal treatments comparable with this book can be found in the books of Blackett [4] and Frechet and Fan [8]. For point set topology a standard reference is Kelley [16]. Our treatment based on nearness was suggested by Cameron, Hocking, and Naimpally [5].

§5 A LINK BETWEEN ANALYSIS AND TOPOLOGY

Poincaré's first pioneering work in topology grew out of his research into systems of differential equations. Ever since, applications to differential equations have played an important role in topology, especially algebraic and combinatorial topology. It may seem surprising at first that such superficially different subjects as topology and differential equations should be related, but in mathematics such startling connections are common. A link between the two is the concept of a vector field. A **vector field** $V$ on a subset $D$ of the plane is a function assigning to each point $P$ of $D$ a vector in the plane with its tail at $P$. Intuitively we can think of $V$ as giving the velocity of some substance that is presently in $D$ but in a state of agitation, like water in a bathtub.

Placement of the vector $V(P)$ with its tail at $P$ is mainly for dramatic effect, to aid in visualizing the vector field. As usual with vectors, the only
essential qualities of the vector $V(P)$ are its length and direction. For practical purposes it is usually more convenient to place all vectors with their tails at the origin. Then $V(P)$ may be described by the coordinates of its head,

$$V(P) = (F(x, y), G(x, y))$$

(1)

where $F$ and $G$ are real-valued functions of $P = (x, y)$ (see Figure 5.1). The vector field $V$ is called continuous when this function (1) is continuous as defined in Chapter One. If we let $f(P)$ stand for the point at the head of the vector $V(P)$ when it is placed with its tail at $P$, we obtain a transformation on $D$, the vector sum of $P$ and $V(P)$:

$$f(P) = P + V(P) = (x + F(x, y), y + G(x, y))$$

(2)

In view of the theorem that the sum of continuous transformations is continuous, $f$ will be continuous if $V$ is a continuous vector field. Conversely, supposing that a transformation $f$ is given on $D$, then the vector field $V$ can be defined by letting $V(P)$ be the vector from the point $P$ to the point $f(P)$. The vector field $V$ will be continuous if the transformation $f$ is continuous. Clearly the study of vector fields on a set $D$ coincides with the study of continuous transformations of the set.

Vector fields have many important applications. The force fields arising from gravitation and electromagnetism are vector fields; the velocity vectors of a fluid in motion, such as the atmosphere (wind vectors), form a vector field; and gradients, such as the pressure gradient on a weather map or the height gradient on a relief chart, are vector fields. These examples are usually studied from the point of view of differential equations. A vector field (1) determines a system of differential equations in the two unknowns $x$ and $y$.

These variables are taken to represent the position of a moving point in the plane dependent on a third variable, the time $t$. The system of differential equations takes the form

$$\begin{cases}
x' = F(x, y) \\
y' = G(x, y)
\end{cases}$$

(3)

where the differentiation is with respect to $t$. Such a system is called autonomous because the right-hand sides are independent of time. Autonomous systems are of particular interest because the fundamental laws of nature are believed to be independent of time. A solution of the system (3) consists of two functions expressing $x$ and $y$ in terms of $t$. These may be considered the parametric equations of a path in the plane: the path of a molecule of gas or liquid, the orbit of a planet or an electron, or the trajectory of a marble rolling down a hill, depending on the application. The original vector field $V(P)$ gives the tangent vector to the path of motion at the point $P = (x, y)$.

**Examples**

1. Consider the vector field $V(x, y) = (2, 1)$. All the vectors are equal, so the corresponding transformation, $f(x, y) = (x + 2, y + 1)$, is a translation (see Figure 5.2a). The resulting system of differential equations, $x' = 2, y' = 1$, has solutions $x = 2t + h, y = t + k$. The solution paths (dashed in Figure 5.2a) are the family of straight lines of slope $\frac{1}{2}$.

2. Consider the vector field $V(x, y) = (-y, x)$. Some of the vectors are sketched in Figure 5.2b. The corresponding transformation $f(x, y) = (x - y, x + y)$ is a rotation of $45^\circ$ counterclockwise combined with a stretching from

---

\[ \text{(a)} \quad \text{(b)} \]

*Figure 5.2*
the origin by a factor of $\sqrt{2}$ (see Example 4 of §2). The system of differential equations, $x' = -y$, $y' = x$, has the solutions $x = k \cos(t)$, $y = k \sin(t)$. The solution paths are the circles centered at the origin. They are tangent at every point to the vector field $V$.

These three mathematical objects—transformation, vector field, and system of differential equations—are essentially the same. Any one determines the other two, and all are continuous if any one is. Thus vector fields link topology and analysis. The relationship can be exploited in two ways: vector fields and differential equations can be used as tools in topology, and topological ideas enter the field of differential equations. In the next section we see an application of the first sort. The Brouwer fixed point theorem, a theorem of topology, will be proven using vector fields. The remainder of the chapter, on the other hand, is devoted to the topological aspects of differential equations.

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Exercises

1. Draw the vector fields corresponding to the examples of continuous transformations given in §2.
2. For each of the following vector fields sketch some of the vectors and describe roughly the corresponding transformation. If possible, solve the corresponding system of differential equations. The solutions will usually be simple exponential, trigonometric, or hyperbolic functions in these examples. Sketch some of the solution paths.

   (a) $V(x, y) = (1, y)$
   (b) $V(x, y) = (x, y)$
   (c) $V(x, y) = (x, -y)$
   (d) $V(x, y) = (4x, y)$
   (e) $V(x, y) = (-y, 1)$
   (f) $V(x, y) = (y, x)$
   (g) $V(x, y) = (-y, 4x)$

---

§6 SPERNER’S LEMMA AND THE BROUWER FIXED POINT THEOREM

Let $f$ be a continuous transformation of a set $D$ into itself. Intuitively $f$ is some sort of twisting, stretching, and crumpling in which the transformed set is eventually replaced within the original set (Figure 6.1). It may happen that some point $P$ is brought back to its original position. In effect $P$ is not moved, $f(P) = P$, and $P$ is called a fixed point of $f$. If it happens that every continuous transformation, no matter how violent, of $D$ into $D$ has a fixed point, then we say that $D$ has the fixed point property. The fixed point property is a topological property. To prove this, let $u: D \rightarrow R$ be a topological transformation from a set $D$ with the fixed point property to a set $R$. Let $g$ be a continuous transformation of $R$ into $R$. To find a fixed point for $g$, consider the composition $f = v \circ g \circ u$, where $v$ is the inverse transformation $u: R \rightarrow D$. This composition is a continuous transformation of $D$ to $D$. By hypothesis, $f$ has a fixed point $P$ in $D$, $v(g(u(P))) = f(P) = P$. This means that $g(u(P)) = u(P)$, so that $u(P)$ is a fixed point for $g$. This proves that $R$ has the fixed point property.

The following theorem is the most important result on fixed points in the plane

**Brouwer’s Fixed Point Theorem**

*Cells have the fixed point property.*

Thus the transformation of Figure 6.1, as well as every other continuous transformation of a cell, has a fixed point! The proof is quite long but involves many important ideas. It divides neatly into a combinatorial part (Sperner’s lemma) and a point set part (topological lemma), a pattern of proof already mentioned in §3. We begin with the combinatorial part.

Consider a cell in the form of a Euclidean triangle, and let this triangle be further divided into subtriangles (Figure 6.2). Later we will study much more complicated situations, but for the moment we may as well assume that all the triangles involved are Euclidean triangles with straight sides. Then we further assume this subdivision of the cell to be a triangulation in the following technical sense.

**Definition**

A triangulation of a triangle $D$ (or any other polygon) is a division of $D$ into a finite number of triangles so that each boundary edge of $D$ is the edge of just one
triangle of the subdivision, and each edge in the interior of \( D \) is an edge of exactly two triangles of the subdivision.

A triangulation is a type of complex (see §1). The vertices of the triangulation are now to be labeled with the letters \( A \), \( B \), and \( C \). First the three vertices of the original triangle are labeled, each with a different label. Then the vertices on the boundary of the original triangle are labeled (Figure 6.2), subject to the restriction that along the side of the triangle already labeled \( AB \) only \( A \)'s and \( B \)'s may be used, only \( B \)'s and \( C \)'s are to be used along the side already labeled \( BC \), and only \( A \)'s and \( C \)'s are to be used along the remaining side (labeled \( AC \)). Finally the vertices inside the triangle can be labeled any old way. A labeling satisfying these conditions is called a Sperner labeling. The combinatorial part of the proof of the Brouwer fixed point theorem is contained in the following lemma.

**Sperner’s Lemma**

At least one subtriangle in a Sperner labeling receives all three labels: \( A \), \( B \), and \( C \).

This mysterious result may seem far from fixed points, but combined with an application of compactness it will yield Brouwer’s theorem. The triangles that receive all three labels are called complete triangles. We shall actually prove that the number of complete triangles is odd. In Figure 6.2 the three complete triangles are shaded.

Consider first the analogous problem in one dimension: a single edge labeled \( AB \), subdivided into segments, and the vertices labeled with \( A \)'s and \( B \)'s only. The question here is whether there are any complete segments: segments with both labels \( A \) and \( B \) (see Figure 6.3). We shall prove that the number \( b \) of complete segments is odd. To do this let us count the number of vertices labeled \( A \) in the following manner: let \( a \) be the number of segments labeled \( AA \). Now the segments of this type have two \( A \) vertices, while the complete segments have only one \( A \) vertex. Other types of segment have no \( A \) vertices, so the total number of \( A \) vertices, counted segment by segment, is \( 2a + b \). But wait! In this total, the \( A \) vertices inside the original segment have been counted twice since they belong to two subsegments. Letting \( c \) be the number of internal \( A \) vertices, we have actually counted \( 2c + 1 \) vertices. Thus \( 2a + b = 2c + 1 \). This proves that \( b \) is odd.

Returning to two dimensions, let \( b \) be the number of complete triangles in a given Sperner labeling. To prove that \( b \) is odd, we count the number of edges labeled \( AB \) inside and on the triangle in the following way: let \( a \) be the number of triangles whose labels read \( ABA \) or \( BAB \). Now the triangles of these two types have two edges labeled \( AB \), while the complete triangles have one edge labeled \( AB \). Other types of triangles have no edges labeled \( AB \), so the total number of these edges, counted triangle by triangle, is \( 2a + b \). But wait! In this total, the edges inside the original triangle are counted twice since they belong to two triangles. Letting \( c \) be the number of edges labeled \( AB \) inside the triangle, we have actually counted \( 2c + d \) edges, where \( d \) is the number of edges labeled \( AB \) on the outside of the triangle. Thus \( 2a + b = 2c + d \). According to the one-dimensional result of the preceding paragraph, \( d \) is odd, and therefore \( b \) is odd as well. This completes the proof.

**Exercises**

1. Investigate the examples of continuous transformations given in §2 for fixed points.
2. Find a plane set that does not have the fixed point property.
3. Prove Sperner’s lemma in three dimensions: let a tetrahedron, whose vertices have been labeled \( A \), \( B \), \( C \), and \( D \) be divided into subtetrahedra with vertices labeled so that only three labels appear on each face of the original tetrahedron. Then at least one subtetrahedron has all four labels.
For the proof of Brouwer's theorem, consider the triangle $\mathcal{D}$ shown in Figure 6.4. We shall show that this triangle has the fixed point property. Since the triangle is a cell and the fixed point property is a topological property, it will follow immediately that all cells have this property. Basing the proof on a particular figure is a well-established geometric tradition. In Euclidean geometry, for example, one always chooses the figure in some convenient relation to the coordinate axes so that some special property of the figure can be used in the proof. In this case the proof will depend not only on the particular orientation of $\mathcal{D}$ to the coordinate axes but also on the fact that $\mathcal{D}$ is a triangle.

Let $f$ be a continuous transformation of $\mathcal{D}$ into itself, and let $V$ be the corresponding vector field. In vector field terms, we seek a point $P$ where $V(P) = 0$. This will be a fixed point for $f$. Intuitively here is the argument: since $f$ maps $\mathcal{D}$ into itself, the vectors $V$ all lie entirely inside $\mathcal{D}$. On this account these vectors must point in all kinds of different directions. For example, the vector at the vertex $C$ of Figure 6.4 must point south (more or less), while those at $A$ and $B$ point northeast and northwest, respectively. In the course of the proof we will find many more examples of triangles inside $\mathcal{D}$ with this same property: at the vertexes of these triangles one vector points south, one northeast, and one northwest. By producing an infinite number of such triangles and applying compactness, we will find a point $P$ near the set of points with south-pointing vectors. At the same time $P$ will be near the set of points with northeast-pointing vectors and near the set of points with northwest-pointing vectors. By continuity, the vector $V(P)$ will point all three directions simultaneously. It will follow that $V(P) = 0$, completing the proof. We now have a few details to fill in!

For the purposes of this proof we decide that there shall be only three directions: south, northeast, and northwest determined by the lower half plane and first and second quadrants, respectively. A given vector is placed with its tail at the origin and its direction determined by the location of its head. Figure 6.5 illustrates these three directions. Vectors pointing south in this sense actually range from east to west in the usual sense. Every nonzero vector points at least one of these directions. There are three situations where vectors point two different directions: vectors pointing due east, north, or west in the usual sense. For example, a vector pointing due north is considered to point both northeast and northwest in our sense. Note, however, that only the zero vector points all three directions simultaneously.

Let $\mathcal{D}$ be triangulated, and label each vertex of the triangulation according to the direction of the vector $V$ at that vertex: $A$ for points with northeast vectors, $B$ for points with northwest vectors, and $C$ for points with south-pointing vectors. To the points with vectors of ambiguous direction (due north, east, and west) we assign the label $A$ for definiteness in the first two cases and the label $B$ in the last case. The result is a Sperner labeling. Therefore by Sperner's lemma, one of the triangles of the triangulation has a complete set of labels, and the vectors of this triangle point all three directions (see Figure 6.6).

It is important to point out that $\mathcal{D}$ contains complete triangles with arbitrarily small sides. Figure 6.7 shows a sequence of triangulations of $\mathcal{D}$ in which the size of the triangles decreases toward zero. Each of these triangulations contains a complete triangle. All these ingredients can now be assembled in a complete proof. The remaining argument is purely topological. Because of its importance it is stated as a separate lemma.
V(P) is northeast-pointing too. Similarly, V(P) is northwest-pointing and south-pointing. Since the zero vector is the only vector pointing all three directions, V(P) = 0, completing the proof of Brouwer's theorem and the lemma.

It is worth remarking that the topological portion of the proof of Brouwer's theorem depends only on the compactness of the set $\mathcal{D}$ and continuity of the vector field. This is in contrast with the combinatorial portion, where the triangular nature of $\mathcal{D}$ is crucial. To emphasize the generality of the topological argument, we restate it in a form suitable for application later.

**Topological Lemma**

Let $\mathcal{D}$ be any compact set with a continuous vector field $V$. If $V$ is never zero on $\mathcal{D}$, then there is a constant $\varepsilon > 0$ such that every complete triangle with vertexes in $\mathcal{D}$ has a side of length greater than $\varepsilon$.

---

**Exercises**

4. Explain why the labeling used in the proof of Brouwer's theorem is a Sperner labeling.
5. Outline a proof of Brouwer's fixed point theorem in three dimensions: the solid enclosed by a tetrahedron has the fixed point property.
6. Justify the restatement of the topological lemma given at the end of this section.
7. Prove that a triangle is a cell.

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**§7 PHASE PORTRAITS AND THE INDEX LEMMA**

Consider the system of differential equations

\[
\begin{align*}
\frac{dx}{dt} &= F(x, y) \\
\frac{dy}{dt} &= G(x, y)
\end{align*}
\]  

(1)

determined by a continuous vector field $V(x, y) = (F(x, y), G(x, y))$ in some region $\mathcal{D}$ of the plane. Regarding the independent variable $t$ as a parameter,
the solutions form a family of directed paths in the plane, called integral paths of the system, which are tangent at each point \( P \), through which they pass, to the vector \( V(P) \) at that point. According to the basic existence and uniqueness theorems for this system (which we shall not prove), exactly one of these integral paths passes through each point \( P \) at which \( V(P) \) is not zero. The picture formed by these paths is called the phase portrait of the system of differential equations. An example is given in Figure 7.1. Poincaré discovered that the nature of the phase portrait is determined by the exceptional points \( P \), called critical points, where \( V(P) = 0 \) and around which the integral paths gather. In Figure 7.1 there are four critical points. At the top is a center characterized by the closed integral curves that swirl around it. No integral path passes through the center. At the bottom are two critical points called nodes characterized by the fact that all the integral paths near these points end there. The difference between the two nodes can be expressed by saying that one is stable and the other is unstable. In general, a critical point is called stable if one can find a cell surrounding the point from which no integral paths exit. The idea behind this definition is that a point put down near a stable critical point is constrained to follow the integral paths will stay near the critical point. Thus the node where the arrows are directed toward the point is the stable one. The center is also a stable critical point. In between the center and the nodes is a saddle point, where exactly four integral paths meet, two beginning and two ending. The saddle point is unstable.

The most important characteristics of the phase portrait are the number and arrangement of the critical points, the pattern of the integral paths about each point, and the stability or instability of the critical points. These properties are part of a purely topological theory of differential equations developed by Poincaré. For if we imagine the portion of the plane represented in Figure 7.1 cut out of a sheet of rubber as in §1 and subjected to a continuous transformation, then, although the shape and length of the paths could change, the number and nature of the critical points cannot change; nodes remain nodes (stable or unstable); saddle points remain saddle points; and so forth. These properties, to which the rest of this chapter is devoted, are topological properties of the phase portrait, topological properties of the system of differential equations.

**Examples**

In a few cases the integral paths of a vector field (or corresponding system of differential equations) may be determined by solving the single differential equation

\[
\frac{dy}{dx} = \frac{G(x, y)}{F(x, y)}
\]

for \( y \) as a function of \( x \), instead of solving the system (1) for \( x \) and \( y \) in terms of \( t \). For example, consider the vector field \( V(x, y) = (x, 2y) \). In this case equation (2), \( \frac{dy}{dx} = \frac{2y}{x} \), is separable. Separating and integrating we obtain the solutions \( y = Kx^2 \). These integral curves (parabolas) are plotted for a few values of \( K \) in Figure 7.2 along with a few of the vectors \( V \) themselves from which the correct direction of motion along the integral paths can be determined. The only critical point is an unstable node at the origin.

![Figure 7.1](image1.png)

**Figure 7.1** A phase portrait.

![Figure 7.2](image2.png)
Exercises

1. For each of the following vector fields use the differential equation (2) to sketch the phase portrait, and classify the critical points by stability and type.

(a) \( V(x, y) = (x, y) \)  
(b) \( V(x, y) = (y, -x) \)  
(c) \( V(x, y) = (2y, -x) \)  
(d) \( V(x, y) = (y, x) \)  
(e) \( V(x, y) = (x, -y) \)  
(f) \( V(x, y) = (x^2 - 1, 2xy) \)  
(g) \( V(x, y) = (1 - x^2, 2xy) \)  
(h) \( V(x, y) = (x^2 - 1, y) \)  
(i) \( V(x, y) = (y, 1 - x^2) \)  
(j) \( V(x, y) = (x^2 - y^2, 2xy) \)  
(k) \( V(x, y) = (x, x^2 - y) \)

2. Let the continuous vector field \( V \) be defined on the cell \( \mathcal{D} \). Supposing that \( V \) has the property that all its vectors lie inside \( \mathcal{D} \), show that \( V \) must have a critical point in \( \mathcal{D} \).

Developments in succeeding sections are based on a generalization of Sperner's lemma: the index lemma. Consider a cell in the form of a polygon of any number of sides. Let the cell be triangulated and the vertexes labeled in any manner whatsoever with the labels \( A, B, \) and \( C \). Figure 7.3 gives an example. Two quantities are associated with the resulting figure. The content \( C \) is defined to be the number of complete triangles counted by orientation. This means that each triangle counts plus one if its labels read \( ABC \) in a counterclockwise direction around the triangle, but counts minus one if the labels read \( ABC \) clockwise around the triangle. This is in accord with a general convention in mathematics that counterclockwise means plus, clockwise minus. The index \( I \) is defined to be the number of edges labeled \( AB \) around the boundary of the polygon counted by orientation, meaning that an edge counts plus one if it reads \( AB \) counterclockwise around the polygon, and minus one if it reads \( AB \) clockwise around the polygon. In Figure 7.3 both index and content are minus one.

**Index Lemma**

The index equals the content.

For the proof let \( S \) be the number of edges labeled \( AB \) on and inside the polygon counted in the following way: each triangle is considered apart from all the others and its edges \( AB \) counted plus or minus by orientation. The proof is completed by showing that \( C = S = I \). To prove the first equality, consider a complete triangle of type \( A \)
\[
\begin{array}{c}
A \\
\text{C} \\
B
\end{array}
\]
This has just one edge of positive orientation and so contributes one to \( S \). At the same time as a complete triangle it counts one as part of \( C \). As a second example consider a triangle of type \( A \)
\[
\begin{array}{c}
A \\
\text{B} \\
A
\end{array}
\]
This has two edges \( AB \), one each of positive and negative orientation, and so contributes a total of zero to \( S \). It also contributes zero to \( C \), since it is not complete. By surveying all the possible types of triangle, one verifies that each triangle is counted in the same way in \( S \) as in \( C \); therefore \( C = S \).

On the other hand, consider an individual edge labeled \( AB \). If this edge is inside the polygon, then it is an edge in two triangles in one of which it counts plus one while in the other it counts minus one. These edges then contribute nothing to \( S \). But edges on the boundary of the polygon are counted only once in \( S \), and then just as they are for the index \( I \). Therefore \( S = I \).

Exercises

3. Deduce Sperner's lemma from the index lemma.
4. In the definition of the index, a special role is played by the edges \( AB \). Show that the same values would result if the edges \( BC \) or \( CA \) were used instead.
5. Consider an annulus triangulated and labeled as in Figure 7.4. The content may be defined as for cells, while there are now two indexes: one for the outside boundary $I_1$ and one for the inside $I_2$. Prove that $C = I_1 - I_2$.

§8 WINDING NUMBERS

Consider a continuous vector field $V$ on a closed path $\gamma$. Suppose that $V$ is never zero on $\gamma$. Starting at a fixed point $Q$ on $\gamma$, imagine a variable point $P$ traversing the path $\gamma$ once in the counterclockwise direction returning to $Q$. Two examples are given in Figure 8.1. The vector $V(P)$, which is what we are interested in, starting at $V(Q)$ will wiggle about during the trip around $\gamma$ and return to the position $V(Q)$. During the journey $V(P)$ will make some whole number of revolutions. For example, the vector field $V$ makes a single clockwise revolution on the curve $\gamma_1$ of Figure 8.1. Counting these revolutions positively if they are counterclockwise, negatively if they are clockwise, the resulting algebraic sum of the number of revolutions is called the winding number of $V$ on $\gamma$ and denoted $W(\gamma)$. Thus, for example, $W(\gamma_1) = -1$. In contrast, on $\gamma_2$, after some preliminary hesitation (between $Q$ and $T$) the vector $V(P)$ makes two counterclockwise revolutions (between $T$ and $V$) followed by a clockwise revolution before returning to $V(Q)$. Therefore $W(\gamma_2) = +1$. The figure shows only an occasional vector, of course. You must imagine the remaining vectors filled in in a continuous manner. Intuitively the winding number $W(\gamma)$ may be thought of as a measure, although a very crude one, of the activity of the vector field on $\gamma$. The winding number will be our main tool in the study of differential equations. As we shall see, it also contains information on the activity of the vector field inside $\gamma$.

This definition of winding number will have to be made more precise, however, before it becomes useful. Here is an alternative method for computing the winding number, suggested by Poincaré himself. Instead of keeping track of the vector $V(P)$ as $P$ traverses the path $\gamma$, we watch just one particular direction of our choosing and record only the times that the vector $V(P)$ points in that direction. If the vector $V(P)$ passes through the direction going counterclockwise, we count plus one; if it passes through the direction going clockwise, we count minus one; and we count zero if the vector only comes up to our chosen direction and then retreats back the way it came. For the distinguished direction we choose due north. In Figure 8.1 the places have been marked where the vector $V(P)$ points due north. On $\gamma_1$ this occurs twice at $Q$ and at $R$. At $R$ the vector $V(P)$ is passing through the vertical clockwise, while at $Q$ it approaches the vertical from the right but reverses direction at $Q$ and retreats on the same side. Counting minus one at $R$, zero at $Q$, the total winding number is $-1$. On $\gamma_2$ there are six occasions when the vector $V(P)$ points north: $R$, $T$, and $U$ each count plus one, $Q$ and $S$ count minus one, and $V$ counts zero. The total winding number is $+1$. Note that trouble develops if the vector field oscillates infinitely often about due north. Something will be done about this difficulty shortly.

This method of reckoning the winding number can be refined still further as follows. First partition $\gamma$ by choosing a finite number of points $\{P_i\}$
dividing $\gamma$ into a number of edges. The points of subdivision $P_i$ are then labeled according to the direction of the vectors $V(P_i)$ using the same conventions regarding the labels $ABC$ as in the proof of the Brouwer fixed point theorem. Now, if the points $\{P_i\}$ have been chosen sufficiently close together, when the vector $V(P)$ passes through the vertical in a counterclockwise direction an edge labeled $AB$ will appear, and when the vector passes through the vertical going clockwise an edge labeled $BA$ will appear. The winding number can therefore be computed just like the index in §7. The cases where the vector reaches the vertical and then retreats are also counted properly this way: they will correspond to a sequence of points labeled $AAA$ or $BAB$ and will contribute zero to the index. To illustrate, Figure 8.2 repeats the curves of Figure 8.1 with the winding numbers worked out in this way. For convenience the subdivision points are chosen to be the same as the points at which vectors were drawn in Figure 8.1.

The index could now be adopted as the definition of the winding number except for the problem of deciding how close together the points of the subdivision $\{P_i\}$ should be in order that no circuit of the vector $V(P)$ be missed. In order to attack this problem, let us say that a subdivision $\{P_i\}$ of $\gamma$ is $\varepsilon$-dense if any point inserted between two points of the subdivision will be within a distance $\varepsilon$ of the points on either side (see Figure 8.3). Intuitively it is clear that the more dense the subdivision (i.e., the smaller the $\varepsilon$), the more certain that the index will equal the winding number. The question is, How dense a subdivision is needed? An answer of a sort is given in the following theorem.

**Theorem**

Let the continuous vector field $V$ be defined on the closed path $\gamma$, and assume that $V$ is never zero on $\gamma$. For any subdivision $\mathcal{P} = \{P_i\}$ of $\gamma$, let $I(\mathcal{P})$ be the index of the polygon $\mathcal{P}$ labeled according to the direction of the vectors $V$ at the vertexes of $\mathcal{P}$. Then there exists a constant $\varepsilon > 0$ such that if $\mathcal{P}$ and $\mathcal{Q} = \{Q_i\}$ are any two $\varepsilon$-dense subdivisions of $\gamma$, then $I(\mathcal{P}) = I(\mathcal{Q})$.

In other words, the indexes obtained from the subdivisions of $\gamma$ of $\varepsilon$-density or finer all agree. The number upon which they all agree must be the winding number. This leads to the following definition.

**Definition**

Given a closed path $\gamma$ and a continuous vector field $V$ that is never zero on $\gamma$, the winding number of $V$ on $\gamma$, notated $W(\gamma)$, is the index of the labeled polygon obtained from any $\varepsilon$-dense subdivision of $\gamma$, where $\varepsilon$ is the constant supplied by the previous theorem.

This definition deserves some comment. We have now given three definitions of winding number, each more elaborate and more precise than its predecessor. The first definition, which involves watching the vector $V(P)$ while the point $P$ traverses $\gamma$, is the intuitive essence of winding numbers but is difficult to apply. The second definition, involving watching only one direction, is the easiest to apply in examples and will be used to actually compute winding numbers in preference to the other definitions (see the example below). The third definition, which is clearly equivalent to the second, is the most combinatorial of the definitions and therefore the most appropriate for theoretical use. Thus it has been adopted officially and will appear consistently in the theory ahead. It can be used even if the vector field oscillates infinitely often about due north; all that is necessary is the continuity of the vector field. (It is also important to point out that the winding number is independent of the chosen direction and the labeling conventions used in computing the index. Although this is intuitively obvious, a proof is quite technical. A simple proof will be given in Chapter Six.) The one disadvantage of our official definition of winding numbers is that it is next to impossible to apply in examples because the theorem upon which it is based only half answers the question, How dense should the subdivisions be? The theorem only asserts that there is some measure $\varepsilon$ of density beyond
which all subdivisions have the same index. But the proof, to which we now turn, gives no means of finding this $\varepsilon$.

To begin the proof, recall the topological lemma of §6. This applies to $\gamma$ because closed paths are compact. Therefore there is a constant $\varepsilon > 0$ such that every complete triangle with vertexes in $\gamma$ has a side greater than $\varepsilon$. This $\varepsilon$ will be the constant $\varepsilon$ referred to in the present theorem. Let $\mathcal{P}$ be an $\varepsilon$-dense subdivision of $\gamma$. First examine what happens when one point is added to $\mathcal{P}$; say the point $Q$ is inserted between the points $P_t$ and $P_{t+1}$ (Figure 8.3). The points $P_t$, $Q$, and $P_{t+1}$ are the vertexes of a triangle, all of whose sides are less than $\varepsilon$. It follows that the labels for these vertexes cannot be a complete set. Now by examining all the possibilities one can verify that the insertion of $Q$ cannot alter the index $I(\mathcal{P})$. For example, suppose the labels on $P_t$ and $P_{t+1}$ are $AB$. Then the label on $Q$ must be an $A$ or a $B$, and the index is unchanged by the insertion. If the labels read $AA$ before insertion, the label on $Q$ could be a $B$. Then an edge of type $AB$ is added by the insertion of $Q$, but also an edge of type $BA$, so that the index is unchanged. What is specifically prevented by our choice of $\varepsilon$ is that the labels read $AC$ before insertion and $ABC$ afterward. This would lead to an additional $AB$ edge but also leads to a complete triangle, and so would contradict the topological lemma. The complete analysis of all possibilities is left to you.

![Figure 8.3](image)

Now consider two $\varepsilon$-dense subdivisions $\mathcal{P} = \{P_i\}$ and $\mathcal{Q} = \{Q_j\}$. Starting with $\mathcal{P}$, we add the points of $\mathcal{Q}$ one at a time to obtain a new subdivision $\mathcal{R}$ consisting of the points of $\mathcal{P}$ and $\mathcal{Q}$ together. Since at each step the index is unaltered, $I(\mathcal{P}) = I(\mathcal{R})$. Similarly, $I(\mathcal{R}) = I(\mathcal{S})$, and this completes the proof.

**Example**

Let us find the winding number of the vector field $V(x, y) = (2xy, y^2 - x^2)$ on the unit circle $x^2 + y^2 = 1$. First by means of a rough sketch (Figure 8.4) and the first definition of winding number, we conclude that $W(y) = 2$. Next applying the second definition, we find the points on the circle where the vector $V$ points north by solving simultaneously the equations

\[
\begin{align*}
x^2 + y^2 &= 1 & \text{the circle} \\
2xy &= 0 & \text{the vector $V$ points north}
\end{align*}
\]

The solutions are the points $(0, 1)$ and $(0, -1)$. At these points the $x$-coordinate of $V$ tells whether the vector is moving counterclockwise or clockwise. In each case $2xy$ changes sign from $+$ to $-$, indicating that $V$ moves from the first to the second quadrant or from direction $A$ to direction $B$. Therefore each solution counts $+1$ as part of the winding number, and so we confirm that $W(y) = +2$.

**Exercises**

1. Compute the winding number of $V(x, y) = (y, 1 - x^2)$ on the following curves.
   
   (a) $x^2 + y^2 = 2x$  
   (b) $x^2 + y^2 = -2x$  
   (c) $x^2 + y^2 = 2y$  
   (d) $x^2 + y^2 = -2y$
2. Compute the winding number of \( V(x, y) = (y(x^2 - 1), x(y^2 - 1)) \) on the following curves.

(a) \( x^2 + y^2 - 2x - 2y + 1 = 0 \)  
(b) \( x^2 + y^2 + x + y = \frac{1}{2} \)  
(c) \( x^2 + y^2 = 1 \)  
(d) \( x^2 + y^2 = 4 \)

The official definition of winding number, combined with the index lemma, yields the following theorem.

**The Fundamental Theorem on Winding Numbers**

Let \( \mathcal{D} \) be a cell with the closed path \( \gamma \) as boundary. If the continuous vector field \( V \) is never zero on \( \mathcal{D} \), then \( W(\gamma) = 0 \).

In other words, if \( W(\gamma) \neq 0 \), then \( V \) has a critical point in \( \mathcal{D} \). This theorem (due to Poincaré) is a first hint that the winding number can give information on the vector field inside the path. In the following sections many applications of this idea will appear. For the proof we apply the topological lemma to \( \mathcal{D} \). Since \( V \) is never zero on \( \mathcal{D} \), there is an \( \varepsilon > 0 \) such that every complete triangle with vertexes in \( \mathcal{D} \) has a side of length greater than \( \varepsilon \). Choose an \( \varepsilon \)-dense subdivision of \( \gamma \). By definition, the index of this subdivision is the winding number of \( V \) on \( \gamma \). Incorporating this subdivision into a triangulation of \( \mathcal{D} \), by the index lemma the winding number equals the content of this triangulation. Now if enough vertexes are added to the triangulation so that the distance between adjacent vertexes is always less than \( \varepsilon \), the triangulation will have no complete triangles. Then the content is zero, and this proves the theorem.

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**§9 ISOLATED CRITICAL POINTS**

Let \( V \) be a continuous vector field and \( P \) a point. We suppose that \( V \) is defined not only at \( P \) but in some neighborhood of \( P \). If \( P \) is an ordinary point, then \( V(P) \) is not zero and it is possible to choose the neighborhood so that \( V \) is never zero there. If \( P \) is a critical point, then it may still be possible to choose a neighborhood so that \( V \) vanishes only at \( P \) in the neighborhood. Then \( P \) is called an isolated critical point. For example, all the critical points in Figure 7.1 are isolated.

Winding numbers become a tool for the study of isolated critical points in the following way: choose a circle \( \gamma \) about the critical point \( P \) so that within and on \( \gamma \) the vector field \( V \) never vanishes except at \( P \). The index of \( V \) at \( P \), denoted \( I(P) \), is defined as the winding number \( W(\gamma) \) of \( V \) on \( \gamma \). Clearly there are many circles of this description about \( P \). What makes the index useful is that it does not depend on which circle is chosen. Supposing that \( \gamma \) and \( \mu \) are two such circles (see Figure 9.1), we will prove that \( W(\gamma) = W(\mu) \). On the annulus \( A \) between the two circles, \( V \) is never zero. Adding two paths \( c_1 \) and \( c_2 \) cutting across the annulus, we divide the annulus and the two circles into halves. Let the halves of the two circles be called \( \gamma_1 \) and \( \gamma_2 \) for \( \gamma \), and \( \mu_1 \) and \( \mu_2 \) for \( \mu \). These names are chosen so that if \( A_1 \) and \( A_2 \) are the two halves of the annulus \( A \), then the boundary of \( A_1 \) consists of \( \gamma_1 \), \( \mu_1 \), and the two cuts, while the boundary of \( A_2 \) consists of \( \gamma_2 \), \( \mu_2 \), and the two cuts. By the fundamental theorem on winding numbers, the winding numbers of \( V \) on these boundary curves are zero. In the sum of these two winding numbers the contributions from the two cuts \( c_1 \) and \( c_2 \) cancel out because the edges labeled \( AB \) along them are counted both plus one and minus one. All that remains are the contributions from the parts of the two circles. The halves of the outside circle are taken counterclockwise as parts of the boundary of \( A_1 \) and \( A_2 \). Therefore the sum of the numbers of \( AB \) edges on \( \gamma_1 \) and \( \gamma_2 \) equals \( W(\gamma) \); however, the parts of the inner circle are taken clockwise as parts of the boundary of \( A_1 \) and \( A_2 \), and therefore their contributions amount to minus \( W(\mu) \). Thus \( W(\gamma) = W(\mu) \).

Since the winding number \( W(\gamma) \) is the same no matter how small the circle \( \gamma \), it follows that the index measures something about the vector field \( V \) that depends only on the behavior of \( V \) in an arbitrarily small area around \( P \). Figure 9.2 gives some examples of critical points. In each case we give a picture of the integral paths in the vicinity of the point and a picture of the behavior of the vector field on a circle drawn around the critical point. The index is only a crude means of classifying critical points, since points of widely different types may have the same index. Nonetheless, the index turns out to be an extremely useful theoretical tool.
3. Use the result of Exercise 5 of §7 plus the topological lemma to give another proof that the index of a critical point does not depend on the choice of the circle around P.

We now give a survey of the different types of isolated critical points. First there is the center, which we introduced earlier, and a close relative the focus (Figure 9.3). The integral paths of a focus never reach the critical point but spiral endlessly about it. Foci may be stable or unstable.

Exercises

1. Find the indexes for the critical points of the vector fields in Exercise 1 of §7.
2. Find the indexes for the following critical points.

All other critical points, although we shall not prove this, are made up of sectors of the three types shown in Figure 9.4: elliptic sectors, where all paths begin and end at the critical point; parabolic sectors, where just one end of the path is at the critical point; and hyperbolic sectors, where the paths do not reach the critical point at all. A typical critical point might have sectors of all three types (Figure 9.5). The paths that divide each sector from the next are called separatrices. It may happen that a critical point has only one type of
sector. Those with only parabolic sectors are called nodes. Since a parabolic sector can be stable or unstable, nodes can be stable or unstable. The other types of sectors are unstable so the only stable critical points are the stable focus, node, and center. A critical point with only elliptic sectors is called a rose. An example is the dipole (Figure 9.2). A critical point with only hyperbolic sectors is called a cross point. Saddle points are cross points with four sectors. Still more complicated types are possible with an infinite number of sectors, and there may be nonisolated critical points. We shall always assume that critical points are isolated and have a finite number of sectors.

Exercises

4. Verify that the system of differential equations corresponding to the vector field \( V(x, y) = (x + y, -x + y) \) has solution \( x = Ke^t \sin(t), y = Ke^t \cos(t) \). What type of critical point does \( V \) have at the origin? Is it stable or unstable?

5. Analyze the examples of critical points in Exercise 2 into sectors of different types.

6. The number of elliptic sectors plus the number of hyperbolic sectors is always even. Verify this for the examples of critical points illustrated above, and then prove it.

Let \( P \) be an isolated critical point other than a center or a focus. The object of the next few exercises is to derive a formula for the index of \( V \) at \( P \) in terms of the numbers \( e, p, \) and \( h \) of elliptic, parabolic, and hyperbolic sectors, respectively. Assume for simplicity that the sectors are bounded by separatrices that enter \( P \) from a definite direction (in other words, have a tangent at \( P \)). Then each sector can be given an angle measure, namely the angle between the tangents to the separatrices of that sector. In the example illustrated in Figure 9.6 there are four sectors: two of parabolic type and one each of elliptic and hyperbolic type. Thus \( e = 1, p = 2, \) and \( h = 1 \).

In general let \( \alpha_1, \alpha_2, \ldots, \alpha_e \) be the angle measures of the elliptic sectors, \( \beta_1, \beta_2, \ldots, \beta_p \) the angle measures of the parabolic sectors, and \( \gamma_1, \gamma_2, \ldots, \gamma_h \) the angle measures of the hyperbolic sectors.

Exercises

7. Show that

\[
\alpha_1 + \alpha_2 + \cdots + \alpha_e + \beta_1 + \beta_2 + \cdots + \beta_p + \gamma_1 + \gamma_2 + \cdots + \gamma_h = 2\pi
\]

8. To compute the index, draw a circle about \( P \) and let the winding number of \( V \) be computed on this circle. Let \( a_1, a_2, \ldots, a_e, b_1, b_2, \ldots, b_p, c_1, c_2, \ldots, c_h \) be the angles through which the vector \( V \) revolves in each of the corresponding sectors about \( P \). Figure 9.6 gives an example. Explain why

\[
a_i = \alpha_i + \pi \quad i = 1, 2, \ldots, e
\]

\[
b_j = \beta_j \quad j = 1, 2, \ldots, p
\]

\[
c_k = \gamma_k - \pi \quad k = 1, 2, \ldots, h
\]
9. Show that

\[ I(P) = 1 + \left( e^{-\frac{h}{2}} \right) \]

This is the desired formula. Note that in the end the number of parabolic sectors is irrelevant. Does this formula agree with previous computations of indexes?

10. In what way is an ordinary point like an isolated critical point with just two hyperbolic sectors? In what way unlike?

11. Among critical points of 4 and fewer sectors, find all the topologically distinct ones of index 3, 2, and 1.

§10 THE Poincaré INDEX THEOREM

There is an amazing connection between the indexes of the critical points and the winding numbers of the vector field.

The Poincaré Index Theorem

Let \( V \) be a continuous vector field. Let \( \mathcal{D} \) be a cell and \( \gamma \) its boundary. Suppose that \( V \) is not zero on \( \gamma \), then

\[ W(\gamma) = I(P_1) + I(P_2) + \cdots + I(P_n) \]

where \( P_1, P_2, \ldots, P_n \) are the critical points of \( V \) inside \( \mathcal{D} \).

This is a remarkable theorem because it connects the behavior of the vector field inside the cell with its behavior on the boundary. For the proof we first point out that \( V \) can have only finitely many critical points in \( \mathcal{D} \). If \( V \) had an infinite set of critical points, then by compactness there would be a point \( P \) of \( \mathcal{D} \) near \( A \). By continuity \( V(P) \) would be zero, so that \( P \) would be a nonsolated critical point. Therefore \( V \) can have only a finite number of critical points in \( \mathcal{D} \). Let these points be \( P_1, P_2, \ldots, P_n \). Around each we construct a circle \( \gamma_i \) that encloses \( P_i \) and no other critical point (see Figure 10.1). Then drawing paths between \( \gamma \) and the circles \( \gamma_i \), and among the circles \( \gamma_i \), we divide the region in \( \mathcal{D} \) outside the circles into a number of cells \( D_k \) to each of which we may apply the fundamental theorem on winding numbers. Then, just as with Figure 9.1, we find that

\[ W(\gamma) = W(\gamma_1) + W(\gamma_2) + \cdots + W(\gamma_n) = I(P_1) + I(P_2) + \cdots + I(P_n) \]

EXAMPLES OF PHASE PORTRAITS

Let us now examine a few specific systems of differential equations. Using only a crude sketching technique, one can still investigate a number of interesting phase portraits.

Consider the vector field \( V(x, y) = (F, G) = (2xy, y^2 - x^2 - k^2) \), where \( k \) is a constant. There are two critical points at \((0, k)\) and \((0, -k)\), \( k \neq 0 \). They are found by solving simultaneously the equations \( F = 2xy = 0 \) and \( G = y^2 - x^2 - k^2 = 0 \). To sketch the phase portrait it suffices to draw the vector field along the curves given separately by the equations \( F = 0 \) and \( G = 0 \). These curves, called critical curves, intersect at the critical points. In this case the critical curves are the coordinate axes \( (F = 0) \) and a hyperbola \( (G = 0) \). Along the critical curves the vectors are either horizontal or vertical, and therefore particularly easy to draw. This has been done in Figure 10.2 (thick vectors). In between the critical curves there can be no vertical or horizontal vectors. Therefore in these regions determined by the critical curves all the vectors point in the direction of just one quadrant. The vector field can thus be filled in with roughly parallel vectors in these regions (thin vectors). The magnitude of the vectors is not important, since we are in-
interested in them only as the tangents of the integral curves. They can be drawn all of the same length. The integral paths are now easily sketched in (Figure 10.3). We find that the critical points are nodes, one stable and one unstable. The phase portrait represents a flow from one critical point to the other.

Now suppose the constant $k$ tends to zero. Then the two critical points move toward each other, meeting at the origin in the single critical point of the vector field $V(x, y) = (2xy, y^2 - x^2)$. The critical curves now include, in addition to the axes as before, the set of 45° lines, which come from the hyperbola. Using the same technique as before (Figures 10.4 and 10.5), we find that the critical point is a dipole. Thus the two nodes of the original

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**Figure 10.2** The vector field $V(k \neq 0)$.

**Figure 10.3** The phase portrait of $V(k \neq 0)$.

**Figure 10.4** The vector field $V(k = 0)$.

**Figure 10.5** The phase portrait of $V(k = 0)$. 

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§10 THE Poincaré INDEX THEOREM
vector field have joined to form a single critical point of index two! This is expressed by saying that the dipole is the confluence of the two nodes. You are urged to sketch these and other phase portraits personally, because no description can replace this experience.

THE VOLterra PREY-PREDATOR EQUATIONS

These concern the growth of two species, one the prey of the other—say foxes and rabbits. Supposing that the rabbits have abundant forage, we expect the rabbit population to grow at a rate simply proportional to the size of the population (exponential growth). In other words, if \( x(t) \) denotes the number of rabbits in the population at time \( t \), then \( x' = ax \), where the constant \( a \) represents the difference between the birth and death rates for rabbits. But what about the foxes? Supposing that some proportion of the encounters between foxes and rabbits results in the death of a rabbit, the foxes represent a check on the growth of the rabbit population. Assuming that the product \( xy \), where \( y(t) \) is the number of foxes at time \( t \), is a good measure of the number of encounters between foxes and rabbits, then the differential equation for rabbits becomes

\[
x' = ax - bxy
\]

where the constant \( b \) reflects the proportion of fox-rabbit encounters that end badly for the rabbit. On the other hand, the fox population will be governed by the equation

\[
y' = -cy + dxy
\]

The coefficient of \( y \) is the natural growth rate of the foxes ignoring the rabbits, by analogy with \( a \). This is negative, reflecting the fact that the foxes depend on the rabbits for prey. The last term is positive, since encounters with rabbits are good for foxes.

Thus we are led to consider the vector field \( U(x, y) = (ax - bxy, -cy + dxy) \). This vector field and the corresponding phase portrait are sketched in Figures 10.6 and 10.7. There are two critical points at \((0, 0)\) and \((c/d, a/b)\). The critical curves are the axes and the lines \( x = c/d \) and \( y = a/b \). The critical points turn out to be a center and a saddle point. You may well wonder how it was decided that the point \((c/d, a/b)\) was a center, since the sketched data must seem equally consistent with a focus. In this case there is luckily a clever technique for graphing the integral paths in the first quadrant (see Exercise 4) that reveals them as closed paths. In general, however, this is one of the most difficult problems in the qualitative theory of differential equations: identifying closed integral paths when they exist. Closed paths are important because they correspond to periodic solutions of the system of differential equations. In the next section we encounter a necessary condition for the existence of closed paths (Exercise 2 in that section), but no really comprehensive sufficient condition is known.

From the point of view of the foxes and rabbits, only the first quadrant portion of the phase portrait is of interest. The closed paths correspond to a cyclical growth and decay pattern for both fox and rabbit populations. The two cycles are out of phase with each other, as in Figure 10.8.
Exercises

1. Examine the results of Exercise 1 of §8. Do you find any evidence for the Poincaré index theorem?

2. Sketch the phase portraits of the following vector fields without solving the corresponding differential equations (* denotes a center, ** a focus).

   *(a) \( V(x, y) = (y, x^2 - 1) \)
   *(b) \( V(x, y) = (y, x^2) \)
   *(c) \( V(x, y) = (y, x - x^3) \)
   **(d) \( V(x, y) = (y - x, xy - 2x) \)
   (e) \( V(x, y) = (x^2 - y, x + 3y) \)
   (f) \( V(x, y) = (x^2 - y, xy - x) \)
   (g) \( V(x, y) = (y^2 - x, xy - y) \)
   (h) \( V(x, y) = (x(y^2 - 1), y(x^2 - 1)) \)

3. Using parts (a) and (b) of the previous problem, discuss the confluence of a saddle point and a center.

4. By solving equation (2) of §7, show that the integral paths of the prey-predator equations are given by the equation

   \[ y^a e^{-by} = Kx^{-c} e^{dx} \quad K \text{ constant} \]

This equation cannot be solved for \( y \) in terms of \( x \) but can be graphed by the following trick, due to Volterra. Introduce two new variables \( z \) and \( w \), related to \( x \) and \( y \) by the equations

   \[ z = y^a e^{-by} \quad w = Kx^{-c} e^{dx} \]

These two equations can be graphed simultaneously on a pair of axes in which all four axes represent different variables. This is demonstrated in Figure 10.9. In the second quadrant the equation of \( z \) and \( y \) is plotted, in the fourth quadrant the equation of \( w \) and \( x \) is plotted, while the line \( z = w \) is plotted in the third quadrant. Choosing a point on the line \( z = w \) and following around to the first quadrant will produce points on the integral path. Use this scheme to graph the whole path. Show how different values of \( K \) yield concentric integral paths.

5. Consider the following prey-predator relationship. The prey is the gypsy moth caterpillar, and the predator is one of several parasitic wasp larvae that attack the caterpillars. A spray program is to be instituted to control the caterpillars, but the spray kills wasp and caterpillar alike. Suppose first that it is a question of a single spraying and the result is the death of an equal proportion of the wasp larvae and caterpillar populations. Explain how the timing of the spraying is crucial if this is not to lead to an eventual increase in the number of caterpillars. Next suppose that a systematic program of repeated sprayings is instituted, resulting in an increase in the death rate for wasp larvae and caterpillars. Explain how this could also lead to an eventual increase in the number of caterpillars.

6. Develop an interaction theory of the prey-predator type for two species, each of which has a positive natural growth rate, but which compete with each other? What are some possible fields of application for this theory? What other hypothetical relationships between two species can you discuss?

§11 CLOSED INTEGRAL PATHS

Among the closed paths to which one might apply the Poincaré index theorem, none are more important than the closed integral paths, which represent periodic solutions of the corresponding system of differential equations. The winding number of the vector field on such curves is given by another theorem of Poincaré.
Theorem

The winding number of a vector field on a closed integral path is one.

The following proof is due to H. Hopf. Let $\gamma$ be a closed integral path for the continuous vector field $V$. Pick a point $P$ on $\gamma$. For every other point $S$ on $\gamma$, let $s$ be the arc length from $P$ to $S$ along the curve $\gamma$, taken in the direction indicated by the vectors $V$. Given the base point $P$, which will be fixed for the rest of the proof, the arc length $s$ uniquely determines the point $S$. If $L$ is the total length of $\gamma$, then for every number $s$, $0 < s < L$, there is exactly one point $S$ whose distance along $\gamma$ from $P$ is $s$ (see Figure 11.1a).

Consider next a new plane with coordinate axes $x$ and $y$. Within this plane, concentrate on the triangle $ABC$ consisting of the points $(x, y)$ such that $0 \leq x < y \leq L$ (see Figure 11.1b). To each point $(x, y)$ in this triangle there corresponds a pair of points $X$ and $Y$ on $\gamma$, such that the point $Y$ is further along $\gamma$ from $P$ than $X$. The points on the hypotenuse $(x, x)$ are exceptional and correspond only to a single point of $\gamma$. In addition, the corner point $(0, L)$ corresponds to a single point, the point $P$ itself.

Using this correspondence, a vector field can be defined on the triangle $ABC$ as follows: let $U(x, y)$ be the unit vector in the direction of the secant vector on $y$ from $X$ to $Y$. Figure 11.1 gives a few examples. Each secant drawn on Figure 11.1a corresponds to a vector of $U$ drawn on Figure 11.1b. The vector field $U$ is clearly continuous. At the exceptional points $(x, x)$, $U$ is defined to be the unit tangent vector pointing in the same direction as $V(X)$. This makes $U$ continuous on the whole triangle $ABC$, since these tangent vectors are the limits of the secant vectors of $\gamma$, which are the values of $U$ inside the triangle. Similarly, for reasons of continuity, $U(0, L)$ must be defined as the unit vector pointing in the opposite direction from $V(P)$.

To complete the proof, we compute the winding number of $U$ around the triangle $ABC$. This can be divided into two parts, the first arising from the hypotenuse, the second arising from the two legs of the triangle. On the hypotenuse, $U$ simply repeats the vectors of $V$ around $\gamma$, so the first part equals $W(\gamma)$. On the legs, $U$ makes a single clockwise revolution (half a revolution on each leg). Therefore the total winding number of $U$ is $W(\gamma) - 1$. On the other hand, since $U$ is never zero inside or on the triangle, by the fundamental theorem of winding numbers, this winding number must be zero. This completes the proof.

APPLICATION TO CONTOUR LINES

A continuous real valued function $\phi(x, y)$ defined on a region $\mathcal{D}$ of the plane defines a surface in space: the graph of the equation $z = \phi(x, y)$. An example is given in Figure 11.2a together with some of the contour lines or
lines of constant height: solutions of the equation
\[ \phi(x, y) = \text{a constant} \]

Figure 11.2b contains the corresponding contour map. Supposing \( \phi \) differentiable, and taking the derivative of the above equation, we obtain
\[ \frac{\partial \phi}{\partial x} \, dx + \frac{\partial \phi}{\partial y} \, dy = 0 \]

Thus the contour lines satisfy the differential equation
\[ \frac{dy}{dx} = -\frac{\phi_y}{\phi_x} \]

In other words, the contour lines are integral paths for the vector field
\[ V(x, y) = \left( \frac{\partial \phi}{\partial y}, -\frac{\partial \phi}{\partial x} \right) \quad (1) \]

By studying the vector field \( V \) we gain information on contour lines and vice versa. For example, it is clear that different contour lines cannot intersect or even have points near each other. Therefore \( V \) cannot have nodes or in general any critical point with elliptic or parabolic sectors. The most common critical points are centers (which correspond to peaks such as \( P \) and \( P' \) in Figure 11.2, and bottoms such as \( R \) and \( R' \)) and saddle points (which correspond to cols such as \( Q \) and \( Q' \)). Other cross points can also occur.

**Example**

Consider the vector field \( V(x, y) = (1 - x^2, 2xy) \). By inspection (try it!) we see that \( \phi(x, y) = y - x^2y \) is a height function for \( V \). There are two critical points, \((-1, 0)\) and \((1, 0)\). At the critical points \( \phi \) has the value zero, so that graphing the equation \( \phi(x, y) = y - x^2y = 0 \) gives the separatrices of the phase portrait. In this case there are the lines \( y = 0, x = 1, x = -1 \). The other integral paths are graphs of the equation \( \phi(x, y) = y - x^2y = k \), where \( k \) is any constant. The phase portrait is drawn in Figure 11.3. Both critical points are saddle points; the \( x \)-axis is a sort of ridge with two mountain ranges rising below it and one rising above it.

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**Exercises**

1. In the proof of Poincaré's theorem on closed integral paths, explain why the winding number of \( U \) on the two legs of \( ABC \) is minus one.
2. Prove that every closed integral path encloses a critical point. This is an important result. It implies that the search for closed integral paths must take place around the critical points of a vector field.
3. The following vector fields \( V \) have height functions \( \phi \) with which they are related by equation (1). Find these height functions and use them to sketch the phase portraits.

(a) \( V(x, y) = (y, -x) \)  
(b) \( V(x, y) = (x, -y) \)  
(c) \( V(x, y) = (x + y, -y) \)  
(d) \( V(x, y) = (-2xy, y^2 - x^2 - 1) \)  
(e) \( V(x, y) = (y^2 - x^2 - 1, 2xy) \)  
(f) \( V(x, y) = (x^3 - 3xy^2, y^3 - 3yx^2) \)

4. Let \( \phi \) be the height function on a desert island. Let \( V \) be the corresponding vector field. Prove that the winding number of \( V \) along the shoreline is 1. Then supposing that the only critical points are peaks, bottoms, and cols, prove that \( P - C + B = 1 \), where \( P \) is the number of peaks, \( C \) is the number of cols, and \( B \) is the number of bottoms. Of what other combinatorial result does this remind you?

5. Referring to the previous exercise, suppose that the island is no longer desert but has \( L \) lakes. Prove that \( P - C + B = 1 - L \). Verify this by carefully identifying and counting all the critical points on the island in Figure 11.4.
6. Cross points other than saddle points may occur in topography. Figure 11.5 shows a monkey saddle. Explain the derivation of this name, and draw a contour map of this surface. What is the index of the critical point? Modify the results of the two preceding exercises to apply to surfaces with monkey saddles.

![Figure 11.5 A monkey saddle.](image)

7. Draw contour lines on the map in Figure 11.6. The contour interval should be 200 feet. Use the elevations provided, and mark all the critical points.

![Figure 11.6 Treasure Island.](image)

8. Draw contour maps of islands with the following specifications:
   (a) one lake and one peak
   (b) two saddle points and two lakes
   (c) one saddle point and one monkey saddle

§12 FURTHER RESULTS AND APPLICATIONS

**Dual Vector Fields**

Given the vector field \( V(x, y) = (F(x, y), G(x, y)) \), the **dual vector field** is defined by

\[
V^*(x, y) = (-G(x, y), F(x, y))
\]
Exercises

1. Search among previous examples and exercises for pairs of vector fields and their duals. There are at least four such pairs. Draw their phase portraits together, and study the relationship between them.

2. Show that \( V^*(P) \) is the vector \( V(P) \) rotated 90° counterclockwise. Use this fact to explain that

(a) \( V \) and \( V^* \) have the same winding numbers on all closed paths

(b) \( V \) and \( V^* \) have the same critical points with the same indexes

(c) the integral paths of \( V^* \) are the **orthogonal trajectories** of the integral paths of \( V \); that is, the two families of curves always meet at right angles

3. Conduct a study of the behavior of critical points under dualization. This can be done systematically for complicated critical points by sketching the orthogonal trajectories sector by sector. Start, however, by simply examining the examples of dualization mentioned in Exercise 1. Is it possible to predict the number of sectors of each type in the dual critical point from these numbers for the original critical point? Is there any connection between stability of a critical point for a given field and for its dual?

4. The dual of the confluence of two critical points is the confluence of their duals. Use this principle plus the examples in §10 of the confluence of two nodes and the confluence of a center and a saddle point to discuss the confluence of two centers and the confluence of a node and a saddle point. Also determine the confluence of two saddle points (use Figure 11.3).

5. What is the relation of the double dual \( V^{**} \) to the original vector field \( V \)?

---

**GRADIENTS**

Consider a real valued function \( \phi(x, y) \) defined on a region of the plane. In §11 we considered the associated vector field

\[
V = \left( \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial x} \right)
\]

whose integral paths are the contour lines of \( \phi \). The dual

\[
V^* = \nabla \phi = \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right)
\]

is equally important and is called the **gradient** of \( \phi \). In this context \( \phi \) is called a potential function for \( V^* \). The integral curves of \( V^* \) are called **lines of steepest ascent**, because the vector \( V^* \) always points in the direction of steepest ascent on the surface defined by \( \phi \), in other words, in the direction of greatest increase of \( \phi \).

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Exercises

6. Draw the lines of steepest ascent for the potentials of Exercise 3 of §11. Do the same for the map in Figure 11.4. Don't forget that water runs downhill by a path of steepest descent.

7. What types of critical point are possible for gradient vector fields?

8. Barometric pressure is a potential function that plays a crucial role in determining the world's weather. The integral paths of \( V \) in this case are called **isobars**. Neglecting the effect of the earth's rotation (the Coriolis effect), the integral paths of \( V^* \) are the **streamlines** of the wind, which moves from high to low pressure. Fill in the isobars and streamlines on the weather map in Figure 12.1. Use a contour interval of 0.1, and remember that the prevailing winds especially over water (i.e., at the edges of the map) are from the west at these latitudes. Find the winding number of \( V^* \) around the edge of the map. Label all critical points, and verify that Poincaré's index theorem is satisfied. (Actually, the Coriolis effect produces a flow of air parallel to the isobars, the so-called **geostrophic flow**.)

9. Much of the terminology of gradients comes from electrostatics, where potential energy is in fact the potential function. The integral paths of \( V \) are called **equipotential lines**, and the integral paths of \( V^* \) are the **field lines**.
along which the electrostatic force acts. Figure 12.2 shows the field lines produced by a pair of opposite charges. Topologically this is equivalent to the phase portrait of Figure 10.3, although there the integral paths are circles. This field is called the dipole field. Imagine viewing this field from a great distance. The two critical points would flow together and appear like a single critical point: the dipole.

Draw the quadrupole field, the field associated with the following arrangement of four charges. There will be a fifth critical point in the center where the charges balance.

\[ \begin{array}{cccc}
  - & + & - & + \\
  + & - & + & - \\
  - & + & - & + \\
  + & - & + & - \\
  - & + & - & + 
\end{array} \]

What single critical point will the quadrupole resemble when seen from a great distance? Draw the field associated with a crystalline substance composed of dipoles as shown below. This is called an electret.

\[ \begin{array}{cccc}
  - & + & - & + \\
  - & + & - & + \\
  - & + & - & + \\
  - & + & - & + \\
  - & + & - & + 
\end{array} \]

**FLUID FLOW**

Consider a spring-fed lake as in Figure 12.3. The velocity vectors for the water (ignoring vertical motion) form a continuous vector field. Fluid flows like this provide some of the most interesting and important examples of vector fields. They do not generally arise from potential functions. In Figure 12.3 the integral paths, called streamlines, spiral out toward the shore, which on this account is called a limit cycle.

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**Exercises**

10. Describe the nature of the critical points associated with the following common phenomena:

(a) springs
(b) sinks (subterranean exits for the water)
(c) eddies
Cross points in this context are called stagnation points. Are other types of critical points possible?

11. Let $S_1$ be the number of springs, $S_2$ the number of sinks, $S_3$ the number of saddle points, and $I$ the number of islands. Assuming that these are the only critical points, show that

$$S_1 + S_2 - S_3 = 1 - I$$

12. Draw streamlines for lakes with the following features:
   (a) one island, one spring, and one eddy
   (b) a spring and two eddies
   (c) a spring, a sink, and one stagnation point
   (d) two islands and a spring
   (e) one island and no critical points

Notes. The qualitative theory of differential equations is a large branch of mathematics that uses a wide variety of techniques. If you are interested in further study of systems of differential equations, you should consult the elementary introduction in Simmons’ book [28] and then look at the treatises by Lefschetz [27], Nemytskii and Stepanov [24], and Hirsch and Smale [13]. This book has drawn material from all these works.

§13 POLYGONAL CHAINS

The goal of this chapter is to prove a famous theorem, first stated by Jordan in 1887, to the effect that every closed path (Jordan curve) divides the plane into two pieces, an inside and an outside. At first glance this may seem trivial; but its simple statement is misleading. The difficulty lies with the possible complexity of Jordan curves. Even the second curve in Figure 13.1 does not

![Figure 13.1](image-url)