THE RELATIVE BRAUER GROUP AND GENERALIZED CROSS PRODUCTS FOR A CYCLIC COVERING OF AFFINE SPACE

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ABSTRACT. We study algebras and divisors on a normal affine hypersurface defined by an equation of the form \( z^n = f(x_1, \ldots, x_m) \). The coordinate ring is \( T = k[x_1, \ldots, x_m, z]/(z^n - f) \), and if \( R = k[x_1, \ldots, x_m][f^{-1}] \) and \( S = R[z]/(z^n - f) \), then \( S \) is a cyclic Galois extension of \( R \). If the Galois group is \( G \), we show that the natural map \( H^1(G, Pic(S)) \rightarrow H^1(G, Pic(T)) \) factors through the relative Brauer group \( B(S/R) \) and that all of the maps are onto. Sufficient conditions are given for \( H^1(G, Pic(T)) \) to be isomorphic to \( B(S/R) \).

As an example, all of the groups, maps, divisors and algebras are computed for an affine surface defined by an equation of the form \( z^n = (y - a_1 x) \cdots (y - a_n x)(x - 1) \).

1. Introduction

1.1. Statement of main results. A continuation of [5] and [6], this article is concerned with the study of Azumaya algebras and Weil divisors on a ramified cyclic cover of affine space \( \mathbb{A}^m \). The variety under investigation is a hypersurface in \( \mathbb{A}^{m+1} \) which is defined by an equation of the form \( z^n = f \), where \( f \) is a square-free polynomial in \( k[x_1, \ldots, x_m] \).

Throughout, \( k \) is an algebraically closed field.

Let \( A = k[x_1, \ldots, x_m] \) and let \( K = k(x_1, \ldots, x_m) \) be the quotient field of \( A \). Let \( f \) be a noninvertible square-free element of \( A \). Assume \( n \) is invertible in \( k \). Set \( T = A[z]/(z^n - f) \), \( R = A[f^{-1}] \) and \( S = T[z^{-1}] \). Then \( T \) is a ramified extension of \( A \), and \( S \) is a cyclic Galois extension of \( R \). The ring \( T \) is an integral domain. If \( L \) denotes the quotient field of \( T \), then \( T \) is integrally closed in \( L \) and \( K \rightarrow L \) is a Galois extension of fields. The rings defined so far make up this commutative diagram

\[
\begin{array}{cccc}
T &=& S &=& L = K \\
A &\longrightarrow& R &=& K \\
&\uparrow&\uparrow&\uparrow&\uparrow
\end{array}
\]

where an arrow represents set inclusion. The Hochschild-Serre spectral sequence [11, p. 105] associated to the Galois extension \( S/R \) gives rise to the exact sequence of cohomology groups

\[
1 \rightarrow H^1(G, S^n) \overset{\alpha}{\rightarrow} Pic(R) \overset{\alpha}{\rightarrow} (Pic S)^G \overset{\alpha}{\rightarrow} H^2(G, S^n) \rightarrow B(S/R) \rightarrow H^1(G, Pic S) \rightarrow H^3(G, S^n)
\]

[3, Corollary 5.5]. This article represents an attempt to exhibit explicit generators of the various groups and descriptions of the homomorphisms that appear in (2). All of these
computations are carried out, in Section 3, for those hypersurfaces in \( A^3 \) defined by equations of the form \( z^n = (y - a_1 x) \cdots (y - a_n x)(x - 1) \).

The group \( B(S/R) \) appearing in (2) is the relative Brauer group mentioned in the title. It is the kernel of the natural map on Brauer groups \( B(R) \rightarrow B(S) \). Let \( \zeta \) be a primitive \( n \)th root of unity in \( k \). Let \( \sigma \) denote the \( A \)-algebra automorphism of \( T \), the \( R \)-algebra automorphism of \( S \), as well as the \( K \)-algebra automorphism of \( L \), defined by \( z \mapsto \zeta z \). If \( G = \langle \sigma \rangle \) is the cyclic group generated by \( \sigma \), then \( G \) acts as a group of automorphisms on \( T, S \) and \( L \). We show that there is a commutative diagram

\[
\begin{array}{ccc}
H^1(G, Cl(T)) & \rightarrow & H^1(G, Pic(S)) \\
\downarrow \alpha_5 & & \downarrow \alpha_5 \\
B(S/R) & & B(S/R)
\end{array}
\]

(3)

where the top row is the natural map and \( \alpha_5 \) is from (2). We define \( \Delta \) using generalized crossed product algebras over the ring \( A \). For each rank one reflexive ideal \( I \) in \( T \) representing a divisor class in \( Cl(T) \) we construct in Definition 2.10 an \( A \)-order \( \Delta(I) \) whose localization is an \( R \)-Azumaya algebra which is split by \( S \). Using this construction, we show in Theorem 2.15 that all three maps in (3) are onto. In Theorem 2.16 a sufficient condition for \( \Delta \) being one-to-one is given. Most of Section 2 is devoted to these proofs. No example is known for which \( \Delta \) is not one-to-one.

1.2. **Background material, terminology, and notation.** We suggest [11] as a standard reference for all unexplained terminology and notation. All sheaves and all cohomology are for the \( \acute{e} \)tale topology, except when we use group cohomology. The results in this article are concerned primarily with \( n \)-torsion in the various groups being studied.

We tacitly assume all groups and sequences of groups are ‘modulo the characteristic of \( k \).’ For any variety \( X \) over \( k \), we denote by \( G_m \) the sheaf of units and we write \( X^* = H^0(X, G_m) \) for the group of global units on \( X \). We identify \( Pic_X \), the Picard group of \( X \), with \( H^1(X, G_m) \).

The Brauer group \( B(X) \) embeds into the torsion subgroup of the cohomological Brauer group \( H^2(X, G_m) \) [8, (2.1), p. 51]. If \( X \) is a normal variety, the divisor class group \( Cl(X) \) is the group of Weil divisors \( Div(X) \) modulo the subgroup \( Prin(X) \) of principal Weil divisors [9, Section I.6]. If \( X \) is regular, \( Pic(X) = Cl(X) \). If \( Sing(X) \) is the singular locus of \( X \), we identify \( Cl(X) = Pic(X - Sing(X)) \).

For a noetherian normal integral domain \( A \) with quotient field \( K \), \( Cl(A) \) is isomorphic to the group of reflexive fractional ideals modulo the subgroup of principal fractional ideals, the group law being \( I \ast J = A : (A : I J) \) [7, §1 - §7], [2, Chapter VII, §1]. The group \( Pic(A) \) parametrizes the isomorphism classes of rank one projective \( A \)-modules, with group law being \( M \ast N = M \otimes A N \). The group \( Cl(A) \) parametrizes the isomorphism classes of rank one reflexive \( A \)-modules, with group law being \( M \ast N = (M \otimes A N)^{**} \). If \( \sigma \) is an automorphism of \( A \), and \( L \) is an \( A \)-module, then \( \sigma L \) is the \( A \)-module with multiplication twisted by \( \sigma \). As a \( \mathbb{Z} \)-module, \( \sigma L \) is identical to \( L \). For \( a \in A \) and \( x \in \sigma L \), the multiplication rule is \( a \cdot x = \sigma(a)x \).

Let \( d > 1 \) an integer that is invertible in \( k \) and \( \zeta_d \) a fixed primitive \( d \)th root of unity in \( k \). If \( K \) is the field of rational functions on \( X \), then for and \( \alpha, \beta \) in \( K^* \), by \( (\alpha, \beta)_d \) we denote the symbol algebra over \( K \) of degree \( d \). Recall that \( (\alpha, \beta)_d \) is the associative \( K \)-algebra generated by two elements, \( u \) and \( v \), subject to the relations \( u^d = \alpha, v^d = \beta, uv = \zeta_d vu \).

**Lemma 1.1.** Let \( G = \{ \sigma_1, \ldots, \sigma_r \} \) be a finite group of automorphisms acting on a noetherian integrally closed integral domain \( B \). Assume \( A = B^G \) is the fixed ring, and that
If $L$ is a rank one reflexive $B$-module, then
\[
\sigma_L \otimes_B \sigma_L \otimes_B \cdots \otimes_B \sigma_L \cong B.
\]

(b) For a reflexive integral ideal $I$ in $B$, 
\[
(B : (B : \sigma(I) \sigma(1) \cdots \sigma_n(I))) = Bg
\]
for some $g \in A$.

**Proof.** It suffices to prove (b). Start with the commutative diagram
\[
\begin{array}{cccc}
0 & \longrightarrow & \text{Prin}(A) & \longrightarrow & \text{Div}(A) & \longrightarrow & \text{Cl}(A) \\
\downarrow & & \downarrow \alpha & & \downarrow & & \downarrow & \\
0 & \longrightarrow & \text{Prin}(B)^G & \longrightarrow & \text{Div}(B)^G & \longrightarrow & \text{Cl}(B)^G &
\end{array}
\]
which is from [7, Theorem 16.1], where it is shown that $\alpha$ is onto. Let $I$ be a height one prime ideal of $B$. The divisor $D = \sigma(I) + \sigma_2(I) + \cdots + \sigma_n(I)$ is fixed by $G$, hence is in the image of $\alpha$. By assumption, $\text{Cl}(A) = 0$, so $D$ is principal. Since the class group of $B$ is generated by height one primes, this proves (b). \hfill \Box

2. **Divisor Classes on $T$ and $A$-Algebras**

In all that follows, $k$ is an algebraically closed field, $n \geq 2$ is invertible in $k$, and $\zeta$ is a fixed primitive $n$th root of unity in $k$. Let $A = k[x_1, \ldots, x_n]$. Let $f$ be a non-invertible square-free element of $A$. Let $f = f_1 \cdots f_v$ be a factorization of $f$ into irreducibles in $A$. Set $T = A[z]/(z^n - f)$. By Eisenstein’s criterion, for instance, $T$ is an integral domain. The singular locus of $\text{Spec} T$ corresponds to the singular locus of the affine hypersurface $F = Z(f)$. Hence $\text{Spec} T$ is regular in codimension one. By the Serre criteria, $T$ is an integral closed integral domain. The assignment $\sigma(z) = \zeta z$ defines an $A$-algebra automorphism of $T$. If $G = \langle \sigma \rangle$, then $T^G = A$. Let $R = A[f^{-1}]$ and $S = T[z^{-1}]$. Then $S$ is a Galois extension of $R$ with cyclic group $G$. Because Spec $S$ is nonsingular, rank one reflexive modules are projective and $\text{Pic}(S) = \text{Cl}(S)$. The group $G$ acts as a group of automorphisms of $\text{Cl}(T)$. This action is induced on the group of divisors $\text{Div}(T)$ by sending a height one prime ideal $I$ to its conjugate $\sigma(I)$. Let $N = 1 + \sigma + \cdots + \sigma^{n-1}$ be the norm operator on $\text{Cl}(T)$. The only height one primes of $T$ that ramify over $A$ are the minimal primes of $z$. By this same action, $G$ acts as a group of automorphisms of $\text{Pic}(S)$. The following elementary proposition is stated for future reference.

**Proposition 2.1.** In the context of the previous paragraph, the following are true.
(a) $T$ is a free $A$-module of rank $n$. A basis is $1, z, \ldots, z^{n-1}$.
(b) $(T^*)^G = k^*$.
(c) $H^2(G, T^*) = \langle 1 \rangle$.

2.1. **Divisor classes on $T$**. The notation established in the first paragraph of Section 2 will be in effect throughout this section.

**Proposition 2.2.** In the context above, the norm $N : \text{Pic} S \to \text{Pic} S$ is the zero map. That is,
(a) If $L$ is a rank one projective $S$-module, then 
\[
L \otimes_S \sigma L \otimes_S \cdots \otimes_S \sigma^{n-1} L \cong S.
\]
(b) For a projective integral ideal $I$ in $S$,

$$I\sigma(I) \cdots \sigma^{n-1}(I) = Sg$$

for some $g \in R = S^G$.

Proof. Parts (a) and (b) are equivalent and follow from Lemma 1.1. \hfill \Box

**Proposition 2.3.** In the context of Section 2.1, let the zero set of $f_i$ on $\mathbb{A}^m$ be $F_i$. The divisors on $X = \text{Spec} T$ lying above $F_1, \ldots, F_v$ generate a subgroup of $\mathfrak{n}\text{Cl}(T)$.

Proof. Factor $f = f_1 \cdots f_v$ into distinct irreducibles in $A$. For $i = 1, \ldots, v$, let $I_i = (z, f_i)$. Then $T/I_i \cong A/\langle f_i \rangle$, so $I_i$ is a height one prime of $T$. Notice that $\sigma(I_i) = I_i$. The only height one primes of $z$ in $T$ are $I_1, \ldots, I_v$. In the local ring $T_i$, we see that $z$ generates the maximal ideal. In the group $\text{Div}(T)$, the divisor of $z$ is $\text{div}(z) = I_1 + \cdots + I_v$. The only height one prime of $f_i$ in $T$ is $I_i$. The valuation of $f_i$ in the local ring $T_i$ is $n_i$. In the group $\text{Div}(T)$, the divisor of $f_i$ is $\text{div}(f_i) = n_i I_i$. The exponent of the subgroup generated by the divisors $I_1, \ldots, I_v$ divides $n$.

\hfill \Box

**Proposition 2.4.** In the context of Section 2.1, the norm $N : \text{Cl}(T) \to \text{Cl}(T)$ is the zero map. That is,

(i) If $L$ is a rank one reflexive $T$-module, then

$$(L \otimes_T \sigma L \otimes_T \cdots \otimes_T \sigma^{n-1} L)^{**} \cong T.$$

(ii) For a reflexive integral ideal $I$ in $T$,

$$(T : (T : I \sigma(I) \cdots \sigma^{n-1}(I))) = Tg$$

for some $g \in A = T^G$.

Proof. As in Proposition 2.3, let $I_i = (z, f_i)$. We can write $\text{Div}(T)$ as the direct sum $\text{Div}(S) \oplus (\mathbb{Z}I_1 \oplus \cdots \oplus \mathbb{Z}I_v)$. Let $H$ denote the subgroup of $\text{Cl}(T)$ generated by $I_1, \ldots, I_v$. Consider the commutative diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z}I_1 & \longrightarrow & (\mathbb{Z}I_1) \oplus \text{Div}(S) & \longrightarrow & \text{Div}(S) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow \theta & & \downarrow & & \\
0 & \longrightarrow & H & \longrightarrow & \text{Cl}(T) & \longrightarrow & \text{Cl}(S) & \longrightarrow & 0
\end{array}
$$

where the vertical maps are onto. In (5) the top row is clearly exact, and the second row is exact by Nagata’s Theorem [7, Theorem 7.1]. The action of $G$ on the top row of (5) induces the action on the second row. A typical element of $\text{Cl}(T)$ can be represented as $\theta(D_1 + D_2)$ where $D_1$ is in $\mathbb{Z}I_1 \oplus \cdots \oplus \mathbb{Z}I_v$ and $D_2$ is in $\text{Div}(S)$. By Proposition 2.3, $\sigma \theta(D_1) = \theta(D_1)$, so $N \theta(D_1) = n \theta(D_1) = 0$. At each prime component of $D_2$, $T$ is unramified over $A$, so by Lemma 1.1, $N \theta(D_2) = 0$. Therefore, $N(\theta(D_1 + D_2)) = N \theta(D_1) + N \theta(D_2) = 0$.

\hfill \Box

**Proposition 2.5.** In the context of Section 2.1, the group $\text{Cl}(T)^G$ is generated by the divisors lying above $F_1, \ldots, F_v$. As a $\mathbb{Z}/n$-module $\text{Cl}(T)^G$ has a generating set of $v-1$ elements.

Proof. As in Proposition 2.3, let $I_i = (z, f_i)$. Let $J_i = I_i \cap A$ be the ideal defining $F_i$. Consider the commutative diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z}J_1 \oplus \cdots \oplus \mathbb{Z}J_v & \longrightarrow & \text{Div}(A) & \longrightarrow & \text{Div}(R) & \longrightarrow & 0 \\
& & \downarrow \phi & & \downarrow \psi & & \downarrow \tau & & \\
0 & \longrightarrow & \mathbb{Z}I_1 \oplus \cdots \oplus \mathbb{Z}I_v & \longrightarrow & \text{Div}(T)^G & \longrightarrow & \text{Div}(S)^G & &
\end{array}
$$

where the top row is the counterpart for $A$ and $R$ of the top row of (5). The bottom row is $(\cdot)^G$ applied to the top row of (5). The map $\phi$ sends $I_i$ to $nI_i$. As shown in [7, Theorem 16.1], the map $\psi$ is one-to-one and $\tau$ is an isomorphism. By the Snake Lemma, the cokernel of $\psi$ is generated by the divisors $I_1, \ldots, I_v$ and is a free $\mathbb{Z}/n$-module of rank $v$. By Proposition 2.1, $H^2(G,T^*) = \langle 1 \rangle$. Given that $\text{Cl}(A) = 0$, the non-zero terms of low degree in Rim’s exact sequence [12] become

(7) 
$$1 \to H^1(G,T^*) \to \text{Div}(T)^G / \text{Div}(A) \to \text{Cl}(T)^G \to 1.$$ 

The claim about the number of generators was proved in Proposition 2.3. 

**Proposition 2.6.** In the context of Section 2.1, if $k^* = T^*$, the following are true.

(a) As a $\mathbb{Z}/n$-module $\text{Cl}(T)^G$ is free of rank $v - 1$.

(b) $H^0(G,S^*) = R^* = k^* \times \langle f_1 \rangle \times \cdots \times \langle f_v \rangle$.

(c) $H^i(G,S^*) = 1$ if $i = 1, 3, 5, \ldots$.

(d) $H^i(G,S^*)$ is a free $\mathbb{Z}/n$-module of rank $v - 1$, if $i = 2, 4, 6, \ldots$.

**Proof.** Part (c) follows from the exact sequence (2) because $\text{Pic}(R) = 0$ and $G$ is cyclic [13, Theorem 10.35]. This is true without the assumption $k^* = T^*$.

Since $G$ acts trivially on $k^* = T^*$, if $i = 1, 3, 5, \ldots$, then

(8) 
$$H^i(G,T^*) = \mu_n.$$ 

In a finite abelian group an element of maximal order generates a direct summand. The sequence (7) reduces to a split short-exact sequence from which (a) follows.

Using the notation from Proposition 2.4, the Nagata sequence

(9) 
$$1 \to T^* \to S^* \to \bigoplus_{i=1}^v \mathbb{Z}I_i \to \text{Cl}(T) \to \text{Cl}(S) \to 0$$

breaks up into three short exact sequences

(10) 
$$1 \to T^* \to S^* \to S^*/T^* \to 1$$

(11) 
$$1 \to S^*/T^* \to \bigoplus_{i=1}^v \mathbb{Z}I_i \to H \to 0$$

(12) 
$$0 \to H \to \text{Cl}(T) \to \text{Cl}(S) \to 0$$

where (12) is the second row of (5). Since $H$ is a finite group, it follows from (11) that $S^*/T^*$ is a finitely generated torsion free abelian group of rank $v$. Each group in (11) is a trivial $G$-module. By part (a), $H$ is a free $\mathbb{Z}/n$-module of rank $v - 1$. Then $H^i(G,H) = H$ for $i = 0, 1, 2, \ldots$. Also, $H^i(G,S^*/T^*)$ is a free $\mathbb{Z}/n$-module of rank $v$ for $i = 2, 4, 6, \ldots$. In fact,

(13) 
$$H^i(G,S^*/T^*) = \begin{cases} S^*/T^* & \text{if } i = 0, \\ 1 & \text{if } i = 1, 3, 5, \ldots, \\ S^*/(S^*)^n & \text{if } i = 2, 4, 6, \ldots. \end{cases}$$

Part (b) follows because $S$ is a Galois extension of $R$. The long exact sequence of cohomology associated to (10) yields the short exact sequence

(14) 
$$1 \to H^0(G,S^*) \to H^2(G,S^*/T^*) \to H^3(G,T^*) \to 1.$$ 

By (8) and (13), the sequence (14) of $\mathbb{Z}/n$-modules splits. This proves (d). 

**Conjecture 2.7.** In the context of Proposition 2.6, the groups $\mu \text{Cl}(T)$ and $\text{Cl}(T)^G$ are equal.
Proposition 2.8. In the context of Section 2.1, the natural homomorphism $H^1(G,CL(T)) \rightarrow H^1(G,CL(S))$ is onto.

Proof. Because $S$ is regular, $Pic(S) = CL(S)$. Let $D$ denote the homomorphism $\sigma - 1$ on either $CL(T)$ or $CL(S)$. Consider the diagram

$$
\begin{array}{cccc}
Cl(T) & \xrightarrow{D} & Cl(T) & \longrightarrow H^1(G,CL(T)) & \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
Cl(S) & \xrightarrow{D} & Cl(S) & \longrightarrow H^1(G,CL(S)) & \longrightarrow 0
\end{array}
$$

(15)

The vertical arrows in (15) are the natural maps. By Proposition 2.2, Proposition 2.4, and [13, Theorem 10.35], diagram (15) commutes and the rows are exact. The natural map $Cl(T) \rightarrow Cl(S)$ is onto, by Nagata’s Theorem [7, Corollary 7.2]. The proof follows. boxed

2.2. Generalized crossed product algebras over $A$. The notation from the opening paragraph of Section 2 is in effect throughout this section.

We view $T$ as a $T \otimes_A T$-module by $a \otimes b \cdot x = axb$. Given a left $T \otimes_A T$-module $M$, set $T^0(M) = T$, and for $i \geq 1$, set $T^i(M) = M^{\otimes i} = M \otimes T M \otimes T \cdots \otimes T M$ (i factors). The tensor algebra is

$$
T(M) = \bigoplus_{i \geq 0} T^i(M).
$$

(16)

The tensor algebra $T(M)$ is a graded $A$-algebra which contains $T = T^0(M)$ as a commutative $A$-subalgebra. If the left and right $T$-module actions on $M$ agree, then $T^0(M)$ is central. Each $T^i(M)$, and hence $T(M)$ itself, is a left $T \otimes_A T$-module. The product rule on $T(M)$ is induced by

$$
T^i(M) \otimes T^j(M) \overset{\eta_{ij}}{\longrightarrow} T^{i+j}(M)
$$

(17)

which is a $T \otimes_A T$-module isomorphism. As an $A$-algebra, the set $T^0(M) + T^1(M)$ contains a generating set for $T(M)$.

For any left $T$-module $L$, and $\tau \in G$, by $1_L \tau$ we denote the left $T \otimes_A T$-module with action defined by

$$
a \otimes b \cdot x = a \tau(b)x
$$

(18)

where $a, b \in T$ and $x \in L$.

Lemma 2.9. Let $I$ be a rank one reflexive integral ideal of $T$. For any $i \geq 1$, the map

$$
T^i(1_I) \overset{\theta_i}{\longrightarrow} 1 \left( I \sigma(I) \cdots \sigma^{i-1}(I) \right)_{\sigma^i}
$$

$$
x_0 \otimes x_1 \otimes \cdots \otimes x_{i-1} \mapsto x_0 \sigma(x_1) \cdots \sigma^{i-1}(x_{i-1})
$$

is a $T \otimes_A T$-module homomorphism. The map $\theta_i$ is an isomorphism if $I$ is an invertible integral ideal (equivalently, $I$ is projective as a $T$-module).

Proof. First we show that the map

$$
1 \left( I \sigma(I) \cdots \sigma^{i-1}(I) \right)_{\sigma^i} \otimes T \overset{\mu_i}{\longrightarrow} 1 \left( I \sigma(I) \cdots \sigma^{i-1}(I) \right)_{\sigma^{i+1}}
$$

$$
x \otimes y \mapsto x \sigma^i(y)
$$

(19)
is a $T \otimes_A T$-module homomorphism. For any $a \in T$,
\[
\mu_i(x \cdot a \otimes y) = \mu_i(x \sigma^i(a) \otimes y)
\]
(20)
\[
= x \sigma^i(a) \sigma^i(y)
\]
\[
= x \sigma^i(ay)
\]
\[
= \mu_i(x \otimes a \cdot y)
\]
For any $a, b \in T$,
\[
\mu_i(a \otimes b \cdot x \otimes y) = \mu_i(a \cdot x \otimes y \cdot b)
\]
(21)
\[
= \mu_i(ax \otimes y \sigma(b))
\]
\[
= ax \sigma^i(y \sigma(b))
\]
\[
= a \otimes b \cdot x \sigma^i(y)
\]
\[
= a \otimes b \cdot \mu_i(x \otimes y)
\]
Use (20) and (21) to prove that $\mu_i$ is a well-defined $T \otimes_A T$-module homomorphism. Apply (19) inductively to prove that $\theta_i$ is a well-defined $T \otimes_A T$-module homomorphism. □

**Definition 2.10.** Define an $A$-algebra $\Delta(I)$. Let $I$ be a rank one reflexive ideal in $T$. By Proposition 2.4, there exists an element $g$ in $A = T^G$ such that
\[
\theta_e(T^n(I_\sigma)) = I \sigma(I) \sigma^2(I) \cdots \sigma^{n-1}(I) \subseteq T g.
\]
Following $\theta_e$ by “multiplication by $g^{-1}$”, defines a $T \otimes_A T$-module homomorphism
\[
T^n(I_\sigma) \overset{\lambda_0}{\longrightarrow} T^0(I_\sigma) = T
\]
(22)
\[
x \mapsto \theta_e(x) g^{-1}
\]
which is an isomorphism if and only if $I$ is projective. Define $\Delta(I)$ to be the $A$-algebra quotient $T(I_\sigma)/\Gamma$, where $\Gamma$ is the two-sided ideal of $T(I_\sigma)$ generated by the set
\[
\{ x - \lambda_0(x) \mid x \in T^n(I_\sigma) \}.
\]
(23)
For $0 \leq i < n$, the image of $T^i(I_\sigma)$ in $\Delta(I)$ is denoted $\Delta_{\sigma^i}(I)$. One can verify that $\Delta_{\sigma^i}(I)$ and $T^i(I_\sigma)$ are naturally isomorphic as $T \otimes_A T$-modules. The $A$-algebra $\Delta(I)$ is $G$-graded:
\[
\Delta(I) = \bigoplus_{i=0}^{n-1} \Delta_{\sigma^i}(I).
\]
(24)
For each $i = 0, \ldots, n-1$, using (17) we identify $T^{n+i}(I_\sigma) = T^n(I_\sigma) \otimes_T T^i(I_\sigma)$. The $T \otimes_A T$-module homomorphism $\lambda_i$ is defined by the diagram
\[
T^{n+i}(I_\sigma) \overset{\lambda_i}{\longrightarrow} T^i(I_\sigma)
\]
(25)
\[
\eta_{i,0} \mid \cong \eta_{0,i} \mid \cong
\]
\[
T^n(I_\sigma) \otimes_T T^i(I_\sigma) \overset{\lambda_{0 \otimes 1}}{\longrightarrow} T^0(I_\sigma) \otimes_T T^i(I_\sigma)
\]
Then $\lambda_i$ is an isomorphism if and only if $I$ is projective. The multiplication rule on $\Delta(I)$ is defined on homogeneous components,
\[
\Delta_{\sigma^i}(I) \otimes_T \Delta_{\sigma^j}(I) \overset{f_{i,j}}{\longrightarrow} \Delta_{\sigma^{i+j}}(I).
\]
(26)
If \( 0 \leq i + j < n \), then \( f_{i,j} \) is simply the natural map \( \eta_{i,j} \) of (17). If \( i + j \geq n \), \( f_{i,j} = \chi_{n+j-n} \circ \eta_{i,j} \).

In the terminology of [10], \( \{ f_{i,j} \} \) is a factor set, and the \( A \)-algebra \( \Delta(I) \) of Definition 2.10 is a generalized crossed product. The \( R \)-algebra \( \Delta(I) \otimes_A R \) is an Azumaya \( R \)-algebra which is split by \( S \), hence represents a class in \( B(S/R) \).

**Proposition 2.11.** Let \( I \) be a rank one reflexive ideal in \( T \). Let \( g \) be an element of \( A \) that generates the norm of \( I \), \( (T : (T : I\sigma(I) \cdots \sigma^{n-1}(I))) = Tg \). Upon restriction of scalars to \( K \), \( \Delta(I) \otimes_A K \) is isomorphic to the symbol algebra \( (f, g^{n-1})_n \), hence is Brauer equivalent to \( (g, f)_n \).

**Proof.** The element \( g \) exists by Proposition 2.4. Restrict the ring of scalars to \( R \), so that we can assume \( I \) is a projective integral ideal in \( S \). Then \( I\sigma(I) \cdots \sigma^{n-1}(I) = Sg \). Therefore, \( g \) is an element of \( I \). We work in the tensor algebra \( T_1 \). Let \( u \) denote the element of \( S = T^0(I\sigma) \) represented by \( z \). Let \( v \) denote the element of \( I\sigma = T^1(I\sigma) \) represented by \( g \). Then \( v^n \) is the element of \( S = T^0(I\sigma) \) represented by \( z^n \), which is equal to \( f \). The element \( v^n \) in \( T(I\sigma) \) is represented by \( g \otimes \cdots \otimes g \). Therefore, \( \lambda_0(v^n) = g^n g^{-1} = g^{n-1} \), which we view as an element of \( S = T^0(I\sigma) \). Also, \( uv \) is the element of \( T^1(I\sigma) \) represented by \( zg \), and \( vu \) is the element of \( T^1(I\sigma) \) represented by \( \sigma(z)g \). This proves that \( \Delta(I) \otimes_A K \) generated by \( K \) is a two-sided ideal in \( A \) whose relations are \( u^n = f, \ n \in \mathbb{Z} \), and \( vu = \xi uv \).

**Proposition 2.12.** Let \( I_1, I_2 \) be rank one reflexive ideals in \( T \). Upon restriction of scalars to \( K \), \( \Delta(I_1) \otimes_A \Delta(I_2) \otimes_A K \) is Brauer equivalent to \( \Delta((T : (T : I_1I_2))) \otimes_A K \) in \( B(K) \).

**Proof.** For an ideal \( I \) in \( T \), let \( N(I) \) denote \( I\sigma(I) \cdots \sigma^{n-1}(I) \). Use Proposition 2.4 to find elements \( g_i \) in \( T \) such that \( (T : (T : N(I_i))) = Tg_i \), for \( i = 1, 2 \). Then \( N(I_1)N(I_2) = N(I_1I_2) \) and \( (T : (T : N(I_1I_2))) = Tg_1g_2 \). If \( \sim \) stands for Brauer equivalence over \( K \), then by Proposition 2.11,

\[
\Delta(I_1) \otimes_A \Delta(I_2) \otimes_A K \sim (g_1, f)_n \otimes_K (g_2, f)_n
\]

\[
\sim (g_1g_2, f)_n
\]

\[
\sim \Delta((T : (T : I_1I_2))) \otimes_A K
\]

**Proposition 2.13.** The assignment which sends a height one prime ideal \( I \) in \( T \) to \( \Delta(I) \) defines a homomorphism of abelian groups \( \Delta : \text{Div}(T) \to B(S/R) \).

**Proof.** The ring \( R \) is regular, so \( B(R) \to B(K) \) is one-to-one [1, Theorem 7.2]. Over \( K \), \( (g_1, f)_n \otimes (g_2, f)_n \) is Brauer equivalent to \( (g_1g_2, f)_n \). This follows from Propositions 2.11 and 2.12.

**Proposition 2.14.** The homomorphism \( \Delta : \text{Div}(T) \to B(S/R) \) induces an epimorphism of abelian groups \( \Delta : \text{Cl}(T) \to B(S/R) \).

**Proof.** Let \( a \in T \) and consider the principal ideal \( I = Ta \). Then \( N(I) = TN(a) \). By Proposition 2.11, \( \Delta(I) \otimes K \) is Brauer equivalent to \( (N(a), f)_n \), which is split. Therefore, \( \Delta : \text{Div}(T) \to B(S/R) \) induces \( \Delta : \text{Cl}(T) \to B(S/R) \). As in Proposition 2.3, let \( I_v = (z, f_v) \).

By Proposition 2.11, \( \Delta(I_v) \otimes_A R \) is isomorphic to the symbol algebra \( (f, f_v^{n-1})_n \). Let \( H \) denote the subgroup of \( \text{Cl}(T) \) spanned by \( I_1, \ldots, I_v \). In (2) the image of \( \alpha_0 \) is called \( B^{-}(S/R) \).
Then $\Delta$ maps $H$ onto $B^\sim(S/R)$. The diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & H & \longrightarrow & \Cl(T) & \longrightarrow & \Cl(S) & \longrightarrow & 0 \\
\downarrow \Delta_0 & & \downarrow \Delta & & \downarrow & & \downarrow \\
0 & \longrightarrow & B^\sim(S/R) & \longrightarrow & B(S/R) & \longrightarrow & H^1(G,\Cl(S)) & \longrightarrow & 0
\end{array}
$$

(27)

commutes. By Proposition 2.2, the rightmost vertical arrow of (27) is onto.

\[\qed\]

**Theorem 2.15.** The homomorphism $\Delta : \Div(T) \to B(S/R)$ induces a homomorphism of abelian groups $\Delta : H^1(G,\Cl(T)) \to B(S/R)$. The diagram (3) commutes and all of the maps are onto.

**Proof.** As in diagram (15), let $D$ denote the homomorphism $\sigma - 1$ on $\Cl(T)$. The cokernel of $D$ is isomorphic to $H^1(G,\Cl(T))$. Let $I$ be a height one prime ideal in $T$. The norm of $I$ is equal to the norm of $\sigma(I)$. By Proposition 2.11, $\Delta(I) \otimes K$ and $\Delta(\sigma(I)) \otimes K$ are Brauer equivalent over $K$. This implies the image of $D : \Cl(T) \to \Cl(T)$ is in the kernel of $\Delta : \Cl(T) \to B(S/R)$. Diagram (3) commutes by the construction of the map $\theta_3$ in [10, §3]. By Proposition 2.14, $\Delta$ is onto. By Proposition 2.6 (c), the map $\alpha_5$ is onto. \[\qed\]

**Theorem 2.16.** If $k^* = T^*$, then $\Delta : H^1(G,\Cl(T)) \to B(S/R)$ is an isomorphism.

**Proof.** The diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \Cl(S)^G & \longrightarrow & H^1(G,H) & \longrightarrow & H^1(G,\Cl(T)) & \longrightarrow & H^1(G,\Cl(S)) & \longrightarrow & 0 \\
\downarrow = & & \downarrow \Delta_0 & & \downarrow \rho_1 & & \downarrow \Delta & & \downarrow = & & \downarrow \\
0 & \longrightarrow & \Cl(S)^G & \longrightarrow & H^2(G,S^*) & \longrightarrow & B(S/R) & \longrightarrow & H^1(G,\Cl(S)) & \longrightarrow & 0
\end{array}
$$

(28)

commutes. The bottom row is the exact sequence (2). By Proposition 2.6, $H^1(G,H)$ and $H^2(G,S^*)$ are both free $\mathbb{Z}/n$-modules of the same rank. The image of $\rho_1$ is a finite group with the same order as the image of $\alpha_5$. By Proposition 2.14, $\Delta$ maps the image of $\rho_1$ onto the image of $\alpha_4$. It follows that all of the vertical maps of (28) are one-to-one. \[\qed\]

### 3. AN EXAMPLE

To illustrate the theorems from Section 2, computations are carried out for a nontrivial example. The example we consider is a hypersurface in $\mathbb{A}^3$. Let $n \geq 3$ be an integer that is invertible in $k$. In this section the affine coordinates for $\mathbb{A}^2$ are written $x$ and $y$. As in Section 2, $\zeta$ will denote a primitive $n$th root of unity in $k$. We assume moreover that $\zeta = \omega^2$, for a fixed primitive $2n$th root of unity $\omega$ in $k$. In the polynomial ring $A = k[x,y]$, let

$$f(x,y) = (y-a_1x)\cdots(y-a_nx)(x-1)$$

(29)

where the $a_i$ are distinct elements of $k$. The equation $x^n - f$ defines an affine hypersurface $X$ in $\mathbb{A}^3$. The rings $R$, $K$, $T$, $S$, and $L$ are defined as in (1). The extension $S/R$ is Galois of degree $n$, with cyclic group $G = \langle \sigma \rangle$, and $\sigma(\zeta) = \zeta^2$.

In Section 3.1, we explicitly construct a generalized crossed product algebra $\Delta(I)$ as defined in Definition 2.10, where the ideal $I$ is a height one prime in $T$ containing $x$. We find an expression for the $K$-algebra $\Delta(I) \otimes_K K$ as a symbol algebra. In Section 3.2 we compute the relative Brauer group $B(S/R)$. It is shown that $\Delta(I) \otimes_K R$ represents a nontrivial element of $B(S/R)$ which is not in the image of the crossed product map $\alpha_4$ of (3).

**Proposition 3.1.** If \( I \) denotes the integral ideal of \( T \) generated by \( x \) and \( y - \omega z \), then \( I \) is a height one prime ideal of \( T \) and \( \text{div}(x) = I + \sigma(I) + \cdots + \sigma^{n-1}(I) \).

**Proof.** There is a \( k \)-algebra isomorphism

\[
\frac{T}{(x, y - \omega z)} \cong k[z]
\]

induced by the assignments \( x \mapsto 0, y \mapsto \omega z \). Therefore \( I = (x, y - \omega z) \) is a height one prime ideal of \( T \). Since \( \sigma \) is an automorphism, each of the ideals \( \sigma^i(I) \) is a height one prime as well. Suppose \( P \) is a height one prime ideal in \( T \) which contains \( x \). From (29) we see that \( f + y^n \) is divisible by \( x \). For some \( \alpha \in A \) we can write \( f + y^n = x\alpha \). In \( T \), the equation

\[
z^n = f = (y - a_1x) \cdots (y - a_nx)(x - 1)
\]

can be rearranged to

\[
y^n + z^n = f + y^n = x\alpha
\]

hence \( P \) contains

\[
y^n + z^n = (y - \omega z)(y - \omega^2 z) \cdots (y - \omega^{2n-1} z).
\]

Being a prime ideal, \( P \) must contain at least one of the factors \( y - \omega^{2i+1} z \), for some \( i \). Since \( \sigma^i(I) = (x, y - \omega^{2i+1} z) \), we have \( \sigma^i(I) \subseteq P \), for some \( i \). Consequently, \( \sigma^i(I) = P \), for some \( i \). Equation (32) shows that \( x \) generates the maximal ideal of each of the local rings \( T_{\sigma^i(I)} \), and the proof follows. \( \square \)

For the rest of this section, \( I \) is the ideal \( (x, y - \omega z) \) of Proposition 3.1. We explicitly construct the generalized crossed product algebra \( \Delta(I) \) of Definition 2.10. Being a height one prime in \( T \), \( I \) is a rank one reflexive ideal. Viewing the group \( \text{Cl}(T) \) as the group of reflexive fractional ideals, the formula \( \text{div}(x) = I + \sigma(I) + \cdots + \sigma^{n-1}(I) \) from Proposition 3.1 becomes

\[
Tx = T : (T : I\sigma(I) \cdots \sigma^{n-1}(I)).
\]

The role of the element \( g \) discussed in Definition 2.10 will be played by \( x \). A generating set for the tensor algebra \( T(1_{\sigma}) \) as an \( A \)-algebra is contained in the sum of \( T^0(1_{\sigma}) = T \) and \( T^1(1_{\sigma}) = 1_{\sigma} \). The ring \( T \) is generated as an \( A \)-algebra by 1 and \( z \), while \( I \) is generated as a \( T \)-module by \( x \) and \( y - \omega z \). If \( t \in T^r(1_{\sigma}) \), denote by \( (t) \), the homogeneous element of degree \( i \) in \( T(1_{\sigma}) \) represented by \( t \). Similarly, given an element \( \delta \in \Delta_{\sigma^i}(I) \), \( (\delta)_{\sigma^i} \) will denote the homogeneous component of degree \( \sigma^i \) in the crossed product algebra \( \Delta(I) \). With this notation, \( \Delta(I) \) is generated as an \( A \)-algebra by the four elements \( (1), (z), (x), \) and \( (y - \omega z) \), where \( \sigma^0 \) is written as 1.

The multiplication rule on \( \Delta(I) \) is defined on homogeneous components using the factor set \( \{ f_{i, j} \} \) as in (26). If \( 0 \leq i + j < n \), and \( j > 0 \), the product \( (\delta)_{\sigma^i} \cdot (\epsilon)_{\sigma^j} \) is defined to be \( (\delta \otimes \epsilon)_{\sigma^{i+j}} \), which simplifies to \( (\delta \cdot \epsilon)_{\sigma^i} \), if \( i = 0 \). Let \( t \) be an element of \( T \cong \Delta_1(I) \). Multiplication by \( t \) from the left is

\[
(t)_{1} \cdot (\delta)_{\sigma^i} = (t \cdot \delta)_{\sigma^i},
\]

and from the right is twisted by a power of \( \sigma \):

\[
(\delta)_{\sigma^i} \cdot (t)_{1} = (\sigma^i(t) \cdot \delta)_{\sigma^i}.
\]
If $i + j \geq n$, then we can assume $i > 0$, $j > 0$, and

\begin{equation}
(d_1 \otimes \cdots \otimes d_i)_{\sigma^i} \cdot (e_1 \otimes \cdots \otimes e_j)_{\sigma^j} = (x^{-1}d_1 \sigma(d_2) \cdots \sigma^{i-1}(d_i) \sigma^i(e_1) \cdots \sigma^{n-1}(e_{n-i}) \cdot e_{n-i+1} \otimes \cdots \otimes e_j)_{\sigma^{i+j-n}}.
\end{equation}

Table 1 contains a multiplication table for the four generators of $\Delta(I)$. For simplicity, let $u = (z)_1$ and $v = (x)_1$. Note that

\begin{equation}
vu = (x)_1(z)_1 = (\xi x)_1 = (\xi)_1(x)_1 = (\xi)_1uv.
\end{equation}

Taking $n$th powers of $u$ and $v$ gives

\begin{equation}
u^n = (\xi^n)_1 = (f)_1
\end{equation}

and

\begin{equation}
u^n = (x \otimes \cdots \otimes x)_{\sigma^{n-1}} : (x)_1 = (x^{-1}x \sigma(x) \cdots \sigma^{n-1}(x))_1 = (x^{n-1})_1,
\end{equation}

where in (40) we have used the fact that $x$ is fixed by $\sigma$. There is a $K$-algebra homomorphism $\varphi : (f,x^{n-1})_n \to \Delta(I) \otimes_A K$, which maps the symbol algebra $(f,x^{n-1})_n$ onto the subalgebra generated by $u$ and $v$. Since the symbol algebra $(f,x^{n-1})_n$ is simple, $\varphi$ is one-to-one. On the other hand, $\Delta(I) \otimes_A K$ has dimension $n^2$ over $K$ because $\Delta(I) \otimes_A R$ is a projective $R$-module of rank $n^2$. Thus $\varphi$ is an isomorphism. Given the Brauer equivalences

\begin{equation}(f,x^{n-1})_n \sim (f,x^{n-1})_n \sim (f,x)_n(f,x)_{n-1} \sim (x,f)_n,
\end{equation}

we have shown

**Theorem 3.2.** With notation as above, $\Delta(I) \otimes_A K \cong (f,x^{n-1})_n$, which is Brauer equivalent to $(x,f)_n$.

### 3.2. The Relative Brauer Group $B(S/R)$

We show that the Brauer class of the Azumaya $R$-algebra $\Lambda = \Delta(I) \otimes_A R$ is not in the image of the crossed product map $\alpha_S$ of (2). We will show that for the surface $X$, (2) reduces to a split short exact sequence of $\mathbb{Z}/n\mathbb{Z}$-modules. Explicit generators for $B(S/R)$ are exhibited. We begin with

**Proposition 3.3.** The following are true.

(a) $T^* = k^*$

(b) $\text{Pic}(S) \cong \mathbb{Z}^{(n-1)}$

(c) $H^1(G, \text{Pic}S) \cong \mathbb{Z}/n\mathbb{Z}$

(d) $\text{Pic}(S)^G = 0$

**Proof.** With $f$ defined by (29), let $g(x,y) = (y - a_1x) \cdots (y - a_nx)$. We obtain (a) by examining the ring $T[x^{-1}, g(x,y)^{-1}]$. Consider the homomorphism of $k$-algebras

\begin{equation}
T[x^{-1}, g(x,y)^{-1}] \xrightarrow{\alpha} k[u,v,g(1,u)^{-1},(v^n + g(1,u))^{-1}]
\end{equation}
where $\alpha$ is defined by

$$y \mapsto xu, \quad z \mapsto xv, \quad x \mapsto \frac{v^a + g(1, u)}{g(1, u)}.$$ 

It is routine to check that $\alpha$ is well defined. Notice that $\alpha(yx^{-1}) = u$, $\alpha(zx^{-1}) = v$, and $\alpha(g(x, y)x^{-n}) = g(1, u)$, so $\alpha$ is onto. Both rings in (41) are integral domains of Krull dimension two. Consequently, $\alpha$ is an isomorphism. The ring on the right hand side of (41) is a unique factorization domain and the group of units decomposes into the internal direct product

$$(42) \quad k[u, v, g(1, u)^{-1}, (v^n + g(1, u))^{-1}]^s = k^s \times \langle u - a_1 \rangle \times \langle u - a_2 \rangle \times \cdots \times \langle u - a_n \rangle \times (v^n + g(1, u)).$$

The isomorphism $\alpha$ of (41) induces an isomorphism on the groups of units. One can check that the subgroup

$$(43) \quad k^s \times \langle x \rangle \times \langle y - a_1 x \rangle \times \langle y - a_2 x \rangle \times \cdots \times \langle y - a_n x \rangle$$

of $T[x^{-1}, g(x, y)^{-1}]^s$ maps onto the group described in (42). Therefore the group in (43) is equal to $T[x^{-1}, g(x, y)^{-1}]^s$, and the product in (43) is direct. Any element of the group (43) is fixed by $G$. Since the group of units of $T$ is a subgroup of (43), we have $(T^s)G = T^s$. By Proposition 2.1(b), $(T^s)G = k^s$, proving (a).

For (b), note that Spec($S$) is nonsingular, so Pic($S$) = Cl($S$). The class group is computed by applying Nagata’s sequence [7, Theorem 7.1] to the rings $S$ and $S[x^{-1}]$. By Proposition 3.1, div($x$) = IS + $\sigma(I)S + \cdots + \sigma^n(I)S$. The Nagata sequence becomes

$$(44) \quad 1 \rightarrow S^* \rightarrow S[x^{-1}]^s \xrightarrow{\text{div}} \bigoplus_{i=0}^{n-1} \mathbb{Z} \cdot \sigma^i(I)S \rightarrow \text{Cl}(S) \rightarrow \text{Cl}(S[x^{-1}]) \rightarrow 0.$$ 

Extend the isomorphism $\alpha$ of (41) by adjoining $(x - 1)^{-1}$ to each ring. Use the fact that $\alpha(x - 1) = v^a g(1, u)^{-1}$ to show that

$$(45) \quad S[x^{-1}] = T[x^{-1}, f(x, y)^{-1}] \xrightarrow{\alpha} k[u, v, v^{-1}, g(1, u)^{-1}, (v^n + g(1, u))^{-1}]$$

is an isomorphism. Since $S[x^{-1}]$ is factorial, Cl($S[x^{-1}]$) = 0 in (44). The isomorphism (45) can be used to compute the group of units

$$(46) \quad S[x^{-1}]^s = k^s \times \langle x \rangle \times \langle y - a_1 x \rangle \times \cdots \times \langle y - a_n x \rangle \times \langle z \rangle$$

as an internal direct product. The generators $y - a_1 x, \ldots, y - a_n x, z$ appearing in (46) are already units in $S$ and will be mapped to zero under the map div in (44). The image of div is generated by div($x$) which is the diagonal element of $\bigoplus_{i=0}^{n-1} \mathbb{Z} \cdot \sigma^i(I)S$. Therefore, Cl($S$) $\cong \mathbb{Z}^{(n-1)}$, proving (b).

A basis for Pic($S$) consists of the ideals IS, IS, $\sigma^{-1}(I)S$. We see that the action of $\sigma$ on Pic($S$) has matrix equal to the companion matrix $M$ of the polynomial $x^n + x^{n-2} + \cdots + x + 1$. Since 1 is not an eigenvalue of $M$, Pic($S$)$^G = 0$. This proves (d). To prove (c), apply [13, Theorem 10.35]. \hfill \Box

By Proposition 2.6(c), $H^3(G, S^*)$ is trivial. By Proposition 2.6(d), $H^2(G, S^*)$ is a free $\mathbb{Z}/n$-module of rank $n$. Together with Proposition 3.3, this implies that the final five terms of the exact sequence (2) become

$$(47) \quad 0 \rightarrow (\mathbb{Z}/n)^{(n)} \xrightarrow{\alpha_5} B(S/R) \xrightarrow{\alpha_5} \mathbb{Z}/n \rightarrow 0.$$
Because $B(S/R)$ embeds into $B(L/K)$, the crossed product theorem implies (47) is a sequence of $\mathbb{Z}/n$-modules. Sequence (47) is split exact.

**Theorem 3.4.** The relative Brauer group $B(S/R)$ is isomorphic to $(\mathbb{Z}/n)^{(n+1)}$.

With this, we may state our claim about the Azumaya $R$-algebra $\Lambda$.

**Theorem 3.5.** A basis for the $\mathbb{Z}/n$-module $B(S/R)$ consists of the Brauer classes of the algebras $(y - a_1 x, f)_n, \ldots, (y - a_n x, f)_n$, together with $\Lambda = \Delta(I) \otimes_A R$. In sequence (2), $(y - a_1 x, f)_n, \ldots, (y - a_n x, f)_n$ generate the image of the crossed product map $\alpha_4$. The Brauer class of $\Lambda$ maps under $\alpha_4$ to a generator of $H^1(G, \text{Pic}(S))$.

**Proof.** We already know the rank of $B(S/R)$ is equal to $n + 1$. Proposition 2.14 shows that $\Lambda$ maps to a generator of $H^1(G, \text{Pic}(S))$. We apply [4, Theorem 4] to compute the image of the crossed product map $\alpha_4$. The subgroup of $B(R)$ annihilated by $n$ is $\mathbb{Z}/n$. It is a free $\mathbb{Z}/n$-module of rank $2n - 1$. A basis for $\mathbb{Z}/n$ consists of the Brauer classes of the symbol algebras

$$(48) \quad (y - a_1 x, y - a_2 x)_n, \ldots, (y - a_1 x, y - a_n x)_n, (y - a_1 x, x - 1)_n, \ldots, (y - a_n x, x - 1)_n.$$ 

The image of the crossed product map $\alpha_4$ is generated by all of the symbols $(u, f)_n$, where $u \in R^*$. A basis for $R^*/k^*$ is $y - a_1 x, \ldots, y - a_n x, x - 1$. To complete the proof, express each of the symbols $(y - a_i x, f)_n$ in terms of the basis (48).

**REFERENCES**


