THE PICARD GROUP OF A GENERAL TORIC VARIETY OF DIMENSION THREE

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Abstract. Let $\Delta$ be a complete fan in $\mathbb{R}^3$ such that every three-dimensional cone in $\Delta$ is non-simplicial. In any non-empty open neighborhood of $\Delta$ there is a fan $\Delta'$ such that every $\Delta'$-linear support function is linear and the Picard group of the associated toric variety is zero.

1. Introduction

Let $k$ be a field. Let $N = \mathbb{Z}^r$ and denote by $T_N$ the $k$-torus on $N$. A fan $\Delta$ is a finite set of strongly convex polyhedral cones in $N \otimes \mathbb{R} = \mathbb{R}^r$ such that for all $\sigma, \tau$ in $\Delta$, every face of $\sigma$ is in $\Delta$ and $\sigma \cap \tau$ is a face of both $\sigma$ and $\tau$. If each cone in $\Delta$ is rational, then associated to $\Delta$ is the toric variety over $k$ denoted $X = T_N \operatorname{emb}(\Delta)$ [1], [6], [7]. To what extent do the combinatorial properties of the fan $\Delta$ determine the Picard group of $X$? This question was addressed in [3] and [5] where many partial solutions are given.

The relation $\sigma \leq \tau$ if $\sigma$ is a face of $\tau$ defines a partial ordering on the set of cones in $\Delta$. Denote by $\Delta_{\text{poset}}$ this partially ordered set. Two fans $\Delta, \Sigma$ are of the same combinatorial type if there is an isomorphism $\Delta_{\text{poset}} \cong \Sigma_{\text{poset}}$ of partially ordered sets.

For example, if $\Delta$ is a simplicial complete rational fan in $\mathbb{R}^r$, the Picard group of $X$ is a free $\mathbb{Z}$-module of rank $v - r$ where $v$ is the number of one-dimensional cones in $\Delta$. It follows that for complete simplicial fans, the Picard group of $X$ is determined by the combinatorial type of the fan. For the details, see [6, pp. 63–65] or [5, Theorem 3.2].

To see that the Picard group is not determined by the combinatorial type of the fan, consider another example. The reader is referred to [5, Example 4.6] or [6, pp. 25–26] for the details. If $\Delta$ is the fan over the faces of the cube with vertices at $(\pm 1, \pm 1, \pm 1)$ in $\mathbb{R}^3$ and $X$ is the associated toric variety, then the Picard group of $X$ is $\mathbb{Z}$. Let $\Delta'$ be the fan with cones spanned by the same sets of generators except that the vertex $(1, 1, 1)$ is replaced by $(2, 1, 1)$. If $X'$ is the toric variety associated to $\Delta'$, then the Picard group of $X'$ is zero. The fans $\Delta$ and $\Delta'$ are of the same combinatorial type but their Picard groups are not isomorphic. Note that the maximal cones of $\Delta$ are non-simplicial.

Let $\Delta$ be a complete rational fan in $\mathbb{R}^3$ whose maximal cones are non-simplicial. The main result of this article, Theorem 1, proves that there is a rational fan $\Delta'$ of the same combinatorial type as $\Delta$ which is “close to $\Delta$” in a way made precise later such that if $X'$ is the toric variety associated to $\Delta'$, then the Picard group of $X'$ is zero.

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zero. Corollary 1 shows that the fan $\Delta'$ can be taken as a general rational point of a real manifold parametrizing a large set of fans of the same combinatorial type as $\Delta$. In particular, it follows that the variety $X'$ is non-projective and $\Delta'$ is not the fan of cones over the faces of any integral convex polytope containing the origin in its interior. These results were conjectured in [5, §4].

The proof of Theorem 1 uses the fact that the non-maximal cones of $\Delta$ are simplicial. For this reason, the proof does not extend to dimensions greater than three. Nevertheless, it seems plausible that a generalization of Theorem 1 should be true in higher dimensions.

There is a topology on the set of cones in $\Delta$ where the open sets are the subfans of $\Delta$. Denote by $\Delta_{top}$ this topological space. Notice that two fans $\Delta, \Sigma$ are of the same combinatorial type if and only if $\Delta_{top}$ is homeomorphic to $\Sigma_{top}$. The assignment $\Delta \mapsto \Delta_{top}$ defines a functor $\Sigma$ from the category of fans on $\mathbb{R}^r$ to the category of finite topological spaces.

Let $\Delta(i) = \{\sigma \in \Delta | \dim(\sigma) = i\}$. Set $\Delta(1) = \{r_0, \ldots, r_n\}$. The intersection of $\Delta(1)$ with the unit sphere $S$ in $\mathbb{R}^r$ is a finite set of points, say $\{p_0, \ldots, p_n\}$. About each $p_i$ we can find an open ball $B_i$ on $S$ such that if $p_i$ is parametrized by $B_i$, then each choice of $\vec{p} = (p_0, p_1, \ldots, p_n)$ in $B_0 \times B_1 \times \cdots \times B_n$ defines a fan $\Phi = \Phi(\vec{p})$ such that $\Phi_{top} \cong \Delta_{top}$. The manifold $B = \bigcap_{i=0}^n B_i$ parametrizes a subset of fans in the fiber $\Sigma^{-1}(\Delta_{top})$. Call $B$ an open neighborhood of $\Delta$. If $\vec{p} \in B$, then the fan $\Phi = \Phi(\vec{p})$ is not necessarily rational. Sometimes it will be necessary to refer to points in $B$ that give rise to rational fans. In this case let

1. $B_{rat} = \{(p_0, \ldots, p_n) |$ for each $i$, $p_i$ is the intersection of a rational 1-dimensional cone $r_i$ with $B_i\}.$

If $\sigma$ is a cone in $N \otimes \mathbb{R}$, define $SF(\sigma)$ to be $\text{Hom}_\mathbb{R}(\mathbb{R}^n, \mathbb{R})$. The group of $\Delta$-linear support functions, denoted $SF(\Delta)$, is by definition the kernel of $\partial^0$ in the Čech complex

$$0 \rightarrow \bigoplus_i SF(\sigma_i) \rightarrow \bigoplus_{i<j} SF(\sigma_{ij})$$

where $\{\sigma_0, \ldots, \sigma_w\}$ is the set of maximal cones of $\Delta$. Set

$$\kappa = \dim_\mathbb{R} SF(\Delta)$$

Let $M = \text{Hom}_\mathbb{Z}(N, \mathbb{Z})$. There is a natural map $M \otimes \mathbb{R} \rightarrow SF(\Delta)$. A support function in the image of $M \otimes \mathbb{R}$ is said to be linear.

2. The Main Result

**Theorem 1.** Let $\Delta$ be a complete fan on $\mathbb{R}^3$ such that every three-dimensional cone in $\Delta$ is non-simplicial. Let $B$ be an open neighborhood of $\Delta$. There is a fan $\Delta'$ in $B$ such that every $\Delta'$-linear support function is linear. The fan $\Delta'$ can be taken to be rational, and if so, the Picard group of the associated toric variety is zero.

**Proof.** The proof of Theorem 1 is split into a series of lemmas and occupies the remainder of this section. The statement about the Picard group follows from the first claim, by the next lemma.
Lemma 2. For a complete rational fan $\Delta$ on $\mathbb{R}^r$, if every $\Delta$-linear support function is linear, then the Picard group of $X = T_N \text{emb}(\Delta)$ is zero.

Proof. If $\Delta$ is a fan of rational cones and $X = T_N \text{emb}(\Delta)$, then it follows from [2, Lemma 8] that there is an exact sequence
\begin{equation}
M \otimes \mathbb{R} \rightarrow \mathcal{S}\mathcal{F}(\Delta) \rightarrow \text{Pic} X \otimes \mathbb{R} \rightarrow 0.
\end{equation}
In this case, set
\begin{equation}
\rho = \dim_{\mathbb{R}} \text{Pic} X \otimes \mathbb{R}
\end{equation}
If we assume moreover that $\Delta$ is a complete fan, then $X$ is compact and $M \otimes \mathbb{R} \rightarrow \mathcal{S}\mathcal{F}(\Delta)$ is an injection. The Picard group of $X$ is a finitely generated torsion free abelian group of rank $\rho$. Combining the above, we have shown $\kappa = \rho + r$ and the lemma follows. \hfill \Box

Let $\Delta$ be a fan on $\mathbb{R}^3$ which has at least one three-dimensional cone. Let $\sigma \in \Delta(3)$ and suppose $\sigma$ has $n$ one-dimensional faces. Define a cubing of $\sigma$ to be any subdivision of $\sigma$ that inserts $n$ new one-dimensional cones $r_{n+1}, \ldots, r_{2n}$, $2n$ new two-dimensional cones and $n + 1$ three-dimensional cones $\sigma_0, \sigma_1, \ldots, \sigma_n$. Write $\text{cube}(\sigma)$ for a fan which corresponds to a cubing of the cone $\sigma$. This is illustrated in Figure 1 for $n = 4$ and Figure 2 for a cone $\sigma$ with $n$ edges. There is a morphism of fans $\text{cube}(\sigma) \rightarrow \Delta(\sigma)$. If $\Sigma$ is the subdivision of $\Delta$ corresponding to a cubing of $\sigma$, there is a morphism of fans $\Sigma \rightarrow \Delta$. The natural map $\mathcal{S}\mathcal{F}(\Delta) \rightarrow \mathcal{S}\mathcal{F}(\Sigma)$ is injective. It was shown in [5, Example 3.6] that for a general cubing of a simplicial three-dimensional cone $\sigma$, a linear support function on $\text{cube}(\sigma)$ is linear. The same result was shown in [5, Example 4.6] for the cone $\sigma$ of Figure 1. The next lemma proves this for all $n$.

Lemma 3. Let $\sigma$ be a three dimensional cone in $\mathbb{R}^3$ and suppose $\sigma$ has $n$ one-dimensional faces. For a general choice of $r_{n+1}, \ldots, r_{2n}$ in a cubing of $\sigma$, the fan $\Xi = \text{cube}(\sigma)$ has the property that a $\Xi$-linear support function is linear.

Proof. A $\Xi$-linear support function $h$ is determined by its values on the cones of dimension one. The group of $\Xi$-linear support functions can be embedded in $\bigoplus_{i=1}^{2n} \mathbb{R} \cdot r_i$ as the kernel of a linear transformation defined by a matrix with $2n$
rows and \(2n - 3\) columns. Call this matrix \(S\). The first \(n\) columns correspond to the cones \(\sigma_1, \ldots, \sigma_n\). The last \(n - 3\) columns correspond to the cone \(\sigma_0\). Each column has exactly four non-zero entries corresponding to the linear dependence relation among four vectors in \(\mathbb{R}^3\). Upon normalization, one of the four numbers in each column can be chosen to be 1. The matrix \(S\) looks like

\[
S = \begin{bmatrix}
    a_1 & 0 & 0 & b_4 & x_4 & x_5 & x_n \\
    b_1 & a_2 & 0 & 0 & y_4 & y_5 & y_n \\
    0 & b_2 & a_3 & 0 & z_4 & z_5 & \cdots & z_n \\
    0 & 0 & b_3 & 0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\
    1 & 0 & 0 & \cdots & c_n & 0 & 0 & \cdots & 0 \\
    c_1 & 1 & 0 & 0 & 0 & 0 & 0 \\
    0 & c_2 & 1 & 0 & 0 & 0 & 0 \\
    0 & 0 & c_3 & 0 & 0 & 0 & 0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

In \(S\), rows 4 through \(n\) are independent. The lower left \(n \times n\) block of \(S\) has determinant \(1 + (-1)^{n-1}c_1c_2 \cdots c_n\) which is non-zero for a general cubing. Therefore, we see that \(S\) has row rank \(2n - 3\) and the kernel of \(S\), which is \(SF(\Xi)\), has dimension three.
Lemma 4. Let $\Delta$ be a fan on $\mathbb{R}^3$ and $\sigma \in \Delta(3)$. Let $\Sigma$ be the subdivision of $\Delta$ corresponding to a cubing of $\sigma$. For a sufficiently general choice of $r_{n+1}, \ldots, r_{2n}$ in the cubing, the natural map $\text{SF}(\Delta) \rightarrow \text{SF}(\Sigma)$ is an isomorphism.

Proof. As was shown in Lemma 3, any support function $h$ on a general cube($\sigma$) is linear. So given $h \in \text{SF}(\Sigma)$, $h$ is linear on the subfan $\{0, r_1, \ldots, r_{2n}\}$. Thus, $h$ comes from a $\Delta$-linear support function. □

Without loss of generality, fix $B$ and suppose $\Delta$ is a general member of $B$. Let $r_0 \in \Delta(1)$. Define

$$\text{Star}(r_0) = \{ \sigma \in \Delta | r_0 \text{ is a face of } \sigma \text{ or } \sigma \text{ is a face of } r_0 \}.$$ 

So Star($r_0$) is a subfan of $\Delta$.

Lemma 5. If $h \in \text{SF}(\Delta)$ is non-linear on $\Delta$, then there exists $r_0 \in \Delta(1)$ such that $h$ is non-linear on Star($r_0$).

Proof. Pick an arbitrary $\sigma \in \Delta(3)$. Subtract a linear support function from $h$ and assume without loss of generality that $h$ is zero valued on $\sigma$. Since $h$ is non-linear, there exists some three-dimensional cone in $\Delta(3)$ on which $h$ is non-zero. Therefore, there are two three-dimensional cones $\sigma_1$ and $\sigma_2$ such that $\sigma_1 \cap \sigma_2$ has dimension at least one and $h$ is zero on $\sigma_1$ and non-zero on $\sigma_2$. Let $r_0$ be a one-dimensional face of $\sigma_1 \cap \sigma_2$. □

Say Star($r_0$) = $\{r_0, r_1, \ldots, r_m\}$, as shown in Figure 3. Check that $r_0, r_1, \ldots, r_m$ are distinct. Let $\Sigma \rightarrow \Delta$ be a subdivision of $\Delta$ corresponding to cubing every $\sigma$ in $\Delta(3) - \text{Star}(r_0)(3)$. By Lemma 4, we can pick $\Sigma$ such that the natural map $\text{SF}(\Delta) \rightarrow \text{SF}(\Sigma)$ is an isomorphism.

Lemma 6. For every $h \in \text{SF}(\Sigma)$, the values of $h$ on $r_0, r_1, \ldots, r_m$ are determined by the values of $h$ on $\Sigma(1) - \{r_0, r_1, \ldots, r_m\}$.

Proof. The subdivision construction employed in the definition of $\Sigma$ guarantees that for each $i$ in the range $1 \ldots m$, there is a three-dimensional cone $\sigma \in \Sigma(3)$, $\sigma \notin \text{Star}(r_0)$, such that $r_i$ is an edge of $\sigma$ and the other three edges of $\sigma$ are in $\Sigma(1) - \{r_0, r_1, \ldots, r_m\}$. The value of $h$ on $r_i$ is completely determined by the values on the three edges of $\sigma$ not in Star($r_0$). □
Lemma 7. Every $\Sigma$-linear support function is linear on $\text{Star}(r_0)$.

Proof. Any $\Sigma$-linear support function is determined by the values along the one-dimensional cones of $\Sigma$. By this, there is an embedding

$$0 \to \text{SF}(\Sigma) \to \bigoplus_{r \in \Sigma(1)} \mathbb{R} \cdot r$$

(6)

Let $h$ be a $\Sigma$-linear support function. The value of $h$ along $r_0$ is determined by the values along those one-dimensional cones not in $\text{Star}(r_0)$. Therefore (6) yields an embedding

$$0 \to \text{SF}(\Sigma) \to \bigoplus_{r \in \Sigma(1)-\text{Star}(r_0)} \mathbb{R} \cdot r$$

(7)

View $r_0$ as a variable which parametrizes fans in an open neighborhood of $\Sigma$. For each such $r_0$, the group of support functions remains constant when viewed as a subspace of $\bigoplus_{r \in \Sigma(1)-\text{Star}(r_0)} \mathbb{R} \cdot r$. Say $\sigma_1$ and $\sigma_2$ are adjacent three-dimensional cones in $\text{Star}(r_0)$. Say $\sigma_1$ is spanned by $r_0, r_1, r_2, \ldots, r_u$ and $\sigma_2$ is spanned by $r_0, r_1, s_2, \ldots, s_v$. Fix $h \in \text{SF}(\Sigma)$. Subtract a linear support function, and assume $h$ is zero valued on the cone $\sigma_1$. Because $\Delta$ is general, we can assume $r_0$ is general. Moving $r_0$ to a one-dimensional cone in the interior of $\sigma_2$ shows that $h$ is zero valued on a three-dimensional subset of $\sigma_2$. Hence $h$ is zero valued on $r_1, s_2, \ldots, s_v$. By iterating this argument, we see that $h$ is zero valued on $r_1, r_2, \ldots, r_m$. This proves the lemma. $\square$

Since $\text{SF}(\Delta) \to \text{SF}(\Sigma)$ is an isomorphism, it follows from Lemma 7 that every $\Delta$-linear support function is linear on $\text{Star}(r_0)$. Theorem 1 follows from Lemma 5. $\square$

3. Concluding Remarks

We conclude with some corollaries to Theorem 1.

Corollary 1. Let $\Delta$ and $B$ be as in Theorem 1. There is a dense open $U \subseteq B$ such that for all fans $\Delta'$ in $U \cap B_{\text{rat}},$

1. the Picard group of $T_N \text{emb}(\Delta')$ is zero
2. $T_N \text{emb}(\Delta')$ is non-projective.
3. $\Delta'$ is not the fan of cones over the faces of any convex integral polytope containing the origin in its interior.

Proof. As was shown in [5, p. 4044], there is an open subset $U$ of $B$ from which $\Delta'$ may be chosen. In fact $U$ is the complement of a Zariski closed. It follows from Theorem 1 that there is a dense open $U \subseteq B$ such that for all rational fans $\Delta'$ in $U$, the Picard group of $T_N \text{emb}(\Delta')$ is zero.

A projective normal variety $X$ will always have a non-principal Cartier divisor corresponding to a hyperplane section. It follows that a projective toric variety has a non-zero Picard group. Therefore, for all rational fans $\Delta'$ in $U$, the toric variety $T_N \text{emb}(\Delta')$ is non-projective.

If $Q$ is a convex integral polytope containing the origin in its interior, and $\Delta'$ the fan over the faces of $Q$, then $T_N \text{emb}(\Delta')$ is projective [7, Proposition 2.19]. $\square$
Remark 2. The proof of [3, Theorem 3.1] shows that for any complete fan of dimension three there exists a rational fan $\Sigma$ of the same combinatorial type as $\Delta$ such that $X = T_N \text{emb}(\Sigma)$ is projective and thus $\text{Pic} \, X \neq 0$. Can $\Sigma$ be chosen from within $B$?

Theorem 1 also gives information about the rank of the torsion free part of the cohomological Brauer group [2] for a general fan $\Delta'$ in $B$.

Corollary 3. Let $\Delta, B, U$ be as in Corollary 1. For all fans $\Delta'$ in $U \cap B_{\text{rat}}$,

$$\dim H^2(T_N \text{emb}(\Delta'), \mathbb{G}_m) \otimes \mathbb{Q} = 3 - (n + 1) + \sum_{i=0}^{w} q_i$$

where $\Delta(3) = \{v_0, \ldots, v_w\}$, $\Delta(1) = \{r_0, \ldots, r_n\}$ and $q_i + 3$ is the number of one-dimensional edges in $v_i$.

Proof. Follows from Corollary 1 and [5, Theorem 3.5].

Remark 4. It follows from Theorem 1 that Conjecture 4.15 of [4] is true.

References


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