THE PICARD GROUP OF A GENERAL TORIC VARIETY OF DIMENSION THREE

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ABSTRACT. Let Δ be a complete fan in $\mathbb{R}^3$ such that every three-dimensional cone in Δ is non-simplicial. In any non-empty open neighborhood of Δ there is a fan Δ' such that every Δ' -linear support function is linear and the Picard group of the associated toric variety is zero.

1. INTRODUCTION

Let k be a field. Let $N = \mathbb{Z}^r$ and denote by T_N the k -torus on N . A fan Δ is a finite set of strongly convex polyhedral cones in $N \otimes \mathbb{R} = \mathbb{R}^r$ such that for all σ, τ in Δ , every face of σ is in Δ and $\sigma \cap \tau$ is a face of both σ and τ . If each cone in Δ is rational, then associated to Δ is the toric variety over k denoted $X = T_N \text{emb}(\Delta)$ [1], [6], [7]. To what extent do the combinatorial properties of the fan Δ determine the Picard group of X ? This question was addressed in [3] and [5] where many partial solutions are given.

The relation $\sigma \leq \tau$ if σ is a face of τ defines a partial ordering on the set of cones in Δ . Denote by Δ_{poset} this partially ordered set. Two fans Δ, Σ are of the same *combinatorial type* if there is an isomorphism $\Delta_{\text{poset}} \cong \Sigma_{\text{poset}}$ of partially ordered sets.

For example, if Δ is a simplicial complete rational fan in $\mathbb{R}^r$, the Picard group of X is a free $\mathbb{Z}$ -module of rank $\nu - r$ where ν is the number of one-dimensional cones in Δ . It follows that for complete simplicial fans, the Picard group of X is determined by the combinatorial type of the fan. For the details, see [6, pp. 63–65] or [5, Theorem 3.2].

To see that the Picard group is not determined by the combinatorial type of the fan, consider another example. The reader is referred to [5, Example 4.6] or [6, pp. 25–26] for the details. If Δ is the fan over the faces of the cube with vertices at $(\pm 1, \pm 1, \pm 1)$ in $\mathbb{R}^3$ and X is the associated toric variety, then the Picard group of X is $\mathbb{Z}$. Let Δ' be the fan with cones spanned by the same sets of generators except that the vertex $(1, 1, 1)$ is replaced by $(2, 1, 1)$. If X' is the toric variety associated to Δ' , then the Picard group of X' is zero. The fans Δ and Δ' are of the same combinatorial type but their Picard groups are not isomorphic. Note that the maximal cones of Δ are non-simplicial.

Let Δ be a complete rational fan in $\mathbb{R}^3$ whose maximal cones are non-simplicial. The main result of this article, Theorem 1, proves that there is a rational fan Δ' of the same combinatorial type as Δ which is “close to Δ ” in a way made precise later such that if X' is the toric variety associated to Δ' , then the Picard group of X' is

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zero. Corollary 1 shows that the fan Δ' can be taken as a general rational point of a real manifold parametrizing a large set of fans of the same combinatorial type as Δ . In particular, it follows that the variety X' is non-projective and Δ' is not the fan of cones over the faces of any integral convex polytope containing the origin in its interior. These results were conjectured in [5, §4].

The proof of Theorem 1 uses the fact that the non-maximal cones of Δ are simplicial. For this reason, the proof does not extend to dimensions greater than three. Nevertheless, it seems plausible that a generalization of Theorem 1 should be true in higher dimensions.

There is a topology on the set of cones in Δ where the open sets are the subfans of Δ . Denote by Δ_{top} this topological space. Notice that two fans Δ, Σ are of the same combinatorial type if and only if Δ_{top} is homeomorphic to Σ_{top} . The assignment $\Delta \mapsto \Delta_{top}$ defines a functor $\mathfrak{T}$ from the category of fans on $\mathbb{R}^r$ to the category of finite topological spaces.

Let $\Delta(i) = \{\sigma \in \Delta \mid \dim(\sigma) = i\}$. Set $\Delta(1) = \{r_0, \dots, r_n\}$. The intersection of $\Delta(1)$ with the unit sphere S in $\mathbb{R}^r$ is a finite set of points, say $\{p_0, \dots, p_n\}$. About each p_i we can find an open ball B_i on S such that if p_i is parametrized by B_i , then each choice of $\vec{p} = (p_0, p_1, \dots, p_n)$ in $B_0 \times B_1 \times \dots \times B_n$ defines a fan $\Phi = \Phi(\vec{p})$ such that $\Phi_{top} \cong \Delta_{top}$. The manifold $B = \prod_{i=0}^n B_i$ parametrizes a subset

of fans in the fiber $\mathfrak{T}^{-1}(\Delta_{top})$. Call B an *open neighborhood of Δ* . If $\vec{p} \in B$, then the fan $\Phi = \Phi(\vec{p})$ is not necessarily rational. Sometimes it will be necessary to refer to points in B that give rise to rational fans. In this case let

$$(1) \quad B_{rat} = \{(p_0, \dots, p_n) \mid \text{for each } i,$$

$p_i \text{ is the intersection of a rational 1-dimensional cone } r_i \text{ with } B_i\}.$

If σ is a cone in $N \otimes \mathbb{R}$, define $\mathcal{SF}(\sigma)$ to be $\text{Hom}_{\mathbb{R}}(\mathbb{R}\sigma, \mathbb{R})$. The group of Δ -linear support functions, denoted $\mathcal{SF}(\Delta)$, is by definition the kernel of δ^0 in the Čech complex

$$(2) \quad 0 \rightarrow \bigoplus_i \mathcal{SF}(\sigma_i) \xrightarrow{\delta^0} \bigoplus_{i < j} \mathcal{SF}(\sigma_{ij})$$

where $\{\sigma_0, \dots, \sigma_w\}$ is the set of maximal cones of Δ . Set

$$(3) \quad \kappa = \dim_{\mathbb{R}} \mathcal{SF}(\Delta)$$

Let $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. There is a natural map $M \otimes \mathbb{R} \rightarrow \mathcal{SF}(\Delta)$. A support function in the image of $M \otimes \mathbb{R}$ is said to be *linear*.

2. THE MAIN RESULT

Theorem 1. *Let Δ be a complete fan on $\mathbb{R}^3$ such that every three-dimensional cone in Δ is non-simplicial. Let B be an open neighborhood of Δ . There is a fan Δ' in B such that every Δ' -linear support function is linear. The fan Δ' can be taken to be rational, and if so, the Picard group of the associated toric variety is zero.*

Proof. The proof of Theorem 1 is split into a series of lemmas and occupies the remainder of this section. The statement about the Picard group follows from the first claim, by the next lemma.

FIGURE 1. A cubing of a cone with four edges.

Lemma 2. *For a complete rational fan Δ on $\mathbb{R}^r$, if every Δ -linear support function is linear, then the Picard group of $X = T_N \text{emb}(\Delta)$ is zero.*

Proof. If Δ is a fan of rational cones and $X = T_N \text{emb}(\Delta)$, then it follows from [2, Lemma 8] that there is an exact sequence

$$(4) \quad M \otimes \mathbb{R} \rightarrow \mathcal{SF}(\Delta) \rightarrow \text{Pic } X \otimes \mathbb{R} \rightarrow 0.$$

In this case, set

$$(5) \quad \rho = \dim_{\mathbb{R}} \text{Pic } X \otimes \mathbb{R}$$

If we assume moreover that Δ is a complete fan, then X is compact and $M \otimes \mathbb{R} \rightarrow \mathcal{SF}(\Delta)$ is an injection. The Picard group of X is a finitely generated torsion free abelian group of rank ρ . Combining the above, we have shown $\kappa = \rho + r$ and the lemma follows. $\square$

Let Δ be a fan on $\mathbb{R}^3$ which has at least one three-dimensional cone. Let $\sigma \in \Delta(3)$ and suppose σ has n one-dimensional faces. Define a *cubing* of σ to be any subdivision of σ that inserts n new one-dimensional cones $r_{n+1}, \dots, r_{2n}$, $2n$ new two-dimensional cones and $n + 1$ three-dimensional cones $\sigma_0, \sigma_1, \dots, \sigma_n$. Write $\text{cube}(\sigma)$ for a fan which corresponds to a cubing of the cone σ . This is illustrated in Figure 1 for $n = 4$ and Figure 2 for a cone σ with n edges. There is a morphism of fans $\text{cube}(\sigma) \rightarrow \Delta(\sigma)$. If Σ is the subdivision of Δ corresponding to a cubing of σ , there is a morphism of fans $\Sigma \rightarrow \Delta$. The natural map $\mathcal{SF}(\Delta) \rightarrow \mathcal{SF}(\Sigma)$ is injective. It was shown in [5, Example 3.6] that for a general cubing of a simplicial three-dimensional cone σ , a linear support function on $\text{cube}(\sigma)$ is linear. The same result was shown in [5, Example 4.6] for the cone σ of Figure 1. The next lemma proves this for all n .

Lemma 3. *Let σ be a three dimensional cone in $\mathbb{R}^3$ and suppose σ has n one-dimensional faces. For a general choice of $r_{n+1}, \dots, r_{2n}$ in a cubing of σ , the fan $\Xi = \text{cube}(\sigma)$ has the property that a Ξ -linear support function is linear.*

Proof. A Ξ -linear support function h is determined by its values on the cones of dimension one. The group of Ξ -linear support functions can be embedded in $\prod_{i=1}^{2n} \mathbb{R} \cdot r_i$ as the kernel of a linear transformation defined by a matrix with $2n$

$$S = \begin{bmatrix} a_1 & 0 & 0 & & b_n & x_4 & x_5 & & x_n \\ b_1 & a_2 & 0 & & 0 & y_4 & y_5 & & y_n \\ 0 & b_2 & a_3 & \dots & 0 & z_4 & z_5 & \dots & z_n \\ 0 & 0 & b_3 & & 0 & 1 & 0 & & 0 \\ 0 & 0 & 0 & & 0 & 0 & 1 & & 0 \\ & & & \vdots & & & & \vdots & \\ 0 & 0 & 0 & & a_n & 0 & 0 & & 1 \\ & & & & & & & & \\ 1 & 0 & 0 & \dots & c_n & 0 & 0 & \dots & 0 \\ c_1 & 1 & 0 & & 0 & 0 & 0 & & 0 \\ 0 & c_2 & 1 & & 0 & 0 & 0 & & 0 \\ 0 & 0 & c_3 & & 0 & 0 & 0 & & 0 \\ & & & \vdots & & & & \vdots & \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & & 0 \end{bmatrix}$$

In S , rows 4 through n are independent. The lower left $n \times n$ block of S has determinant $1 + (-1)^{n-1}c_1c_2 \cdots c_n$ which is non-zero for a general cubing. Therefore, we see that S has row rank $2n - 3$ and the kernel of S , which is $\text{SF}(\Xi)$, has dimension three. $\square$

 FIGURE 3. $\text{Star}(r_0)$

Lemma 4. *Let Δ be a fan on $\mathbb{R}^3$ and $\sigma \in \Delta(3)$. Let Σ be the subdivision of Δ corresponding to a cubing of σ . For a sufficiently general choice of $r_{n+1}, \dots, r_{2n}$ in the cubing, the natural map $\text{SF}(\Delta) \rightarrow \text{SF}(\Sigma)$ is an isomorphism.*

Proof. As was shown in Lemma 3, any support function h on a general cube(σ) is linear. So given $h \in \text{SF}(\Sigma)$, h is linear on the subfan $\{0, r_1, \dots, r_{2n}\}$. Thus, h comes from a Δ -linear support function. $\square$

Without loss of generality, fix B and suppose Δ is a general member of B . Let $r_0 \in \Delta(1)$. Define

$$\text{Star}(r_0) = \{\sigma \in \Delta \mid r_0 \text{ is a face of } \sigma \text{ or } \sigma \text{ is a face of } r_0\}.$$

So $\text{Star}(r_0)$ is a subfan of Δ .

Lemma 5. *If $h \in \text{SF}(\Delta)$ is non-linear on Δ , then there exists $r_0 \in \Delta(1)$ such that h is non-linear on $\text{Star}(r_0)$.*

Proof. Pick an arbitrary $\sigma \in \Delta(3)$. Subtract a linear support function from h and assume without loss of generality that h is zero valued on σ . Since h is non-linear, there exists some three-dimensional cone in $\Delta(3)$ on which h is non-zero. Therefore, there are two three-dimensional cones σ_1 and σ_2 such that $\sigma_1 \cap \sigma_2$ has dimension at least one and h is zero on σ_1 and non-zero on σ_2 . Let r_0 be a one-dimensional face of $\sigma_1 \cap \sigma_2$. $\square$

Say $\text{Star}(r_0)(1) = \{r_0, r_1, \dots, r_m\}$, as shown in Figure 3. Check that $r_0, r_1, \dots, r_m$ are distinct. Let $\Sigma \rightarrow \Delta$ be a subdivision of Δ corresponding to cubing every σ in $\Delta(3) - \text{Star}(r_0)(3)$. By Lemma 4, we can pick Σ such that the natural map $\text{SF}(\Delta) \rightarrow \text{SF}(\Sigma)$ is an isomorphism.

Lemma 6. *For every $h \in \text{SF}(\Sigma)$, the values of h on $r_0, r_1, \dots, r_m$ are determined by the values of h on $\Sigma(1) - \{r_0, r_1, \dots, r_m\}$.*

Proof. The subdivision construction employed in the definition of Σ guarantees that for each i in the range $1 \dots m$, there is a three-dimensional cone $\sigma \in \Sigma(3)$, $\sigma \notin \text{Star}(r_0)$, such that r_i is an edge of σ and the other three edges of σ are in $\Sigma(1) - \{r_0, r_1, \dots, r_m\}$. The value of h on r_i is completely determined by the values on the three edges of σ not in $\text{Star}(r_0)$. $\square$

Lemma 7. *Every Σ -linear support function is linear on $\text{Star}(r_0)$.*

Proof. Any Σ -linear support function is determined by the values along the one-dimensional cones of Σ . By this, there is an embedding

$$(6) \quad 0 \rightarrow \text{SF}(\Sigma) \rightarrow \coprod_{r \in \Sigma(1)} \mathbb{R} \cdot r$$

Let h be a Σ -linear support function. The value of h along r_0 is determined by the values along those one-dimensional cones not in $\text{Star}(r_0)$. Therefore (6) yields an embedding

$$(7) \quad 0 \rightarrow \text{SF}(\Sigma) \rightarrow \coprod_{r \in \Sigma(1) - \text{Star}(r_0)} \mathbb{R} \cdot r$$

View r_0 as a variable which parametrizes fans in an open neighborhood of Σ . For each such r_0 , the group of support functions remains constant when viewed as a subspace of $\coprod_{r \in \Sigma(1) - \text{Star}(r_0)} \mathbb{R} \cdot r$. Say σ_1 and σ_2 are adjacent three-dimensional cones in $\text{Star}(r_0)$. Say σ_1 is spanned by $r_0, r_1, r_2, \dots, r_u$ and σ_2 is spanned by $r_0, r_1, s_2, \dots, s_v$. Fix $h \in \text{SF}(\Sigma)$. Subtract a linear support function, and assume h is zero valued on the cone σ_1 . Because Δ is general, we can assume r_0 is general. Moving r_0 to a one-dimensional cone in the interior of σ_2 shows that h is zero valued on a three-dimensional subset of σ_2 . Hence h is zero valued on $r_1, s_2, \dots, s_v$. By iterating this argument, we see that h is zero valued on $r_1, r_2, \dots, r_m$. This proves the lemma. $\square$

Since $\text{SF}(\Delta) \rightarrow \text{SF}(\Sigma)$ is an isomorphism, it follows from Lemma 7 that every Δ -linear support function is linear on $\text{Star}(r_0)$. Theorem 1 follows from Lemma 5. $\square$

3. CONCLUDING REMARKS

We conclude with some corollaries to Theorem 1.

Corollary 1. *Let Δ and B be as in Theorem 1. There is a dense open $U \subseteq B$ such that for all fans Δ' in $U \cap B_{\text{rat}}$,*

- (1) *the Picard group of $T_N \text{emb}(\Delta')$ is zero*
- (2) *$T_N \text{emb}(\Delta')$ is non-projective.*
- (3) *Δ' is not the fan of cones over the faces of any convex integral polytope containing the origin in its interior.*

Proof. As was shown in [5, p. 4044], there is an open subset U of B from which Δ' may be chosen. In fact U is the complement of a Zariski closed. It follows from Theorem 1 that there is a dense open $U \subseteq B$ such that for all rational fans Δ' in U , the Picard group of $T_N \text{emb}(\Delta')$ is zero.

A projective normal variety X will always have a non-principal Cartier divisor corresponding to a hyperplane section. It follows that a projective toric variety has a non-zero Picard group. Therefore, for all rational fans Δ' in U , the toric variety $T_N \text{emb}(\Delta')$ is non-projective.

If Q is a convex integral polytope containing the origin in its interior, and Δ' the fan over the faces of Q , then $T_N \text{emb}(\Delta')$ is projective [7, Proposition 2.19]. $\square$

Remark 2. The proof of [3, Theorem 3.1] shows that for any complete fan of dimension three there exists a rational fan Σ of the same combinatorial type as Δ such that $X = T_N \text{emb}(\Sigma)$ is projective and thus $\text{Pic } X \neq 0$. Can Σ be chosen from within B ?

Theorem 1 also gives information about the rank of the torsion free part of the cohomological Brauer group [2] for a general fan Δ' in B .

Corollary 3. *Let Δ, B, U be as in Corollary 1. For all fans Δ' in $U \cap B_{\text{rat}}$,*

$$\dim H^2(T_N \text{emb}(\Delta'), G_m) \otimes \mathbb{Q} = 3 - (n + 1) + \sum_{i=0}^w q_i$$

where $\Delta(3) = \{\sigma_0, \dots, \sigma_w\}$, $\Delta(1) = \{r_0, \dots, r_n\}$ and $q_i + 3$ is the number of one-dimensional edges in σ_i .

Proof. Follows from Corollary 1 and [5, Theorem 3.5]. □

Remark 4. It follows from Theorem 1 that Conjecture 4.15 of [4] is true.

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