DIVISION ALGEBRAS AND THE PROBLEM OF SPLITTING PRIMES BY AN AFFINE DOUBLE PLANE

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1. INTRODUCTION

Let \( k \) denote an algebraically closed field of characteristic different from 2. Let \( k[x,y] \) be the affine coordinate ring of \( \mathbb{A}^2 \) and \( K = k(x,y) \) the field of rational functions. Let \( f \) be a non-invertible square-free element of \( k[x,y] \) and \( F = Z(f) \). Let \( Y = \text{Spec} k[x,y][\sqrt{f}] \) and \( \pi: Y \to \mathbb{A}^2 \) the double cover ramified along \( F \). It is an exercise [12, Exercise II.6.4] to show that \( k[x,y][\sqrt{f}] \) is integrally closed, or equivalently, that \( Y \) has only normal singularities. Let \( g \) be an irreducible element of \( k[x,y] \). Denote by \( G = Z(g) \) the irreducible curve on \( \mathbb{A}^2 \) defined by \( g \). Consider the divisor \( \tilde{G} = \pi^{-1}(G) \) on \( Y \).

\[
\begin{array}{c}
G \quad \xrightarrow{\subset} \quad Y \\
\pi \downarrow \quad \quad \quad \pi \\
G \quad \xrightarrow{\subset} \quad \mathbb{A}^2
\end{array}
\]

There are three possibilities for the double cover \( \pi: \tilde{G} \to G \) of curves.

1. \( \tilde{G} \) is irreducible. For this case to occur, it is sufficient (not necessary) that the local intersection multiplicity of \( G \) and \( F \) at some point is odd.

2. \( \tilde{G} = 2G_1 \) is reducible and has one irreducible component. This case occurs if \( g \) divides \( f \).

3. \( \tilde{G} = G_1 + G_2 \) is reducible and has two irreducible components. We say that the double cover \( \pi: Y \to \mathbb{A}^2 \) splits the prime \( G \). If \( G \) is simply connected, then this case occurs if and only if the local intersection multiplicity of \( G \) and \( F \) at each point is even.

In this article we focus on the third case. We study the curves \( F \) and \( G \) such that \( \tilde{G} = G_1 + G_2 \) and \( G_i \) is a non-principal prime divisor on \( Y \). In other words, we look for curves \( F \) and \( G \) such that lying above \( G \) are divisors which are non-trivial in the class group of \( Y \). Our methods rely on the utilization of K-central division algebras and arithmetic in the Brauer group \( B(K) \).

We suggest [17] as a standard reference for all unexplained terminology and notation. All sheaves and all cohomology are for the étale topology. Let \( p \) denote the characteristic of \( k \) (\( p = 0 \) is allowed). Most of our results are concerned only with 2-torsion in the Brauer group. To avoid some complications that arise when \( p > 0 \), we tacitly assume all groups and sequences of groups are 'modulo \( p \)-groups'. For any scheme \( X \) of finite type over \( k \), we denote by \( G_m \) the sheaf of units and we write \( X^* = H^0(X, G_m) \) for the group of global units on \( X \). We identify \( \text{Pic} X \), the Picard group of \( X \), with \( H^1(X, G_m) \). The Brauer group \( B(X) \) embeds into the torsion subgroup of the cohomological Brauer group \( H^2(X, G_m) \) [11,
(2.1), p. 51. For any abelian group $M$ and integer $d$, $dM$ is the subgroup of $M$ annihilated by $d$. The $d$th power map induces

$$1 \to \mu_d \to G_m \overset{d}{\to} G_m \to 1$$

the Kummer sequence [17, p. 66], an exact sequence of sheaves for the étale topology on $X$. By $\mu_d$ we denote the constant sheaf on $X$ of $d$th roots of unity. The long exact sequence of cohomology associated to (2) breaks up into short exact sequences. In degrees one and two we have

$$1 \to X^*/X^{ad} \to H^1(X, \mu_d) \to d\text{Pic}X \to 0$$

$$0 \to \text{Pic}X \otimes \mathbb{Z}/d \to H^2(X, \mu_d) \to dB(X) \to 0$$

The group $H^1(X, \mathbb{Q}/\mathbb{Z})$ classifies the cyclic Galois covers of $X$ and $H^1(X, \mu_d)$ classifies the cyclic Galois covers of degree $d$.

**Theorem 1.1.** Let $k$ be an algebraically closed field of characteristic $p$, $X$ a nonsingular integral surface over $k$, and $K = K(X)$ the function field of $X$. Modulo $p$-groups the sequence

$$0 \to B(X) \to B(K) \overset{a}{\to} \bigoplus_{C \in X_1} H^1(K(C), \mathbb{Q}/\mathbb{Z}) \overset{\mu}{\to} \bigoplus_{p \in X_2} \mu(-1) \overset{S}{\to} H^4(X, \mu) \to 0$$

is a complex which is exact except that in general the image of $a$ is not equal to the kernel of $r$. The first summand is over all irreducible curves $C$ on $X$, the second summand is over all closed points $p$ on $X$. If $H^3(X, \mu) = 0$ (true for example if $X$ is affine, or complete and simply connected), the sequence is exact.

**Proof.** Follows from combining sequences (3.1) and (3.2) of [1, p. 86].

The map $a$ of Theorem 1.1 is called the “ramification map”. Let $\Lambda$ be a central $K$-division algebra which represents a class $[\Lambda]$ in $B(K)$. The curves $C \in X_1$ for which $a([\Lambda])$ is non-zero make up the so-called ramification divisor of $\Lambda$ on $X$.

Let $\alpha, \beta$ be elements of $K$. By $\Lambda = (\alpha, \beta)_d$ we denote the symbol algebra over $K$ of degree $d$ defined by $\alpha$ and $\beta$. Recall that $\Lambda$ is the associative $K$-algebra generated by two elements, $u$ and $v$ subject to the relations $u^d = \alpha, \quad v^d = \beta, \quad uv = \zeta vu$, where $\zeta$ is a fixed primitive $d$th root of unity. Then $\Lambda$ is a $K$-vector space of dimension $d^2$, with basis $u^iv^j, 0 \leq i, j \leq d - 1$. Also $\Lambda$ is a central simple $K$-algebra and we denote by ram. div($\Lambda$) the ramification divisor of $\Lambda$ on $\mathbb{A}^2$. The map $a$ applied to the Brauer class containing a symbol algebra $(\alpha, \beta)_d$ agrees with the so-called tame symbol. Let $C$ be a prime divisor on $X$. Then $\mathcal{O}_{X,C}$ is a discrete valuation ring with valuation denoted by $v_C$. The residue field is $K(C)$, the field of rational functions on $C$. The ramification of $(\alpha, \beta)_d$ along $C$ is the cyclic Galois extension of $K(C)$ defined by adjoining the $d$th root of

$$\alpha^{v_C(\beta)}\beta^{-v_C(\alpha)}.$$  

2. **Theorems**

Throughout this section, let $A = k[x, y]$ and $K = k(x, y)$ the quotient field of $A$. Let $f$ be a non-invertible square-free element of $A$ and $f = f_1f_2\cdots f_s$ the unique factorization of $f$ into irreducibles. Let $T = A[z]/(z^2 - f)$, $R = A[f^{-1}]$ and $S = T[z^{-1}]$. Then $T$ is a ramified quadratic extension of $R$, and $S$ is a quadratic Galois extension of $R$. It follows that Theorem 1.1 applies to the non-singular surface Spec $S$ whose field of rational functions is
The group \( B \) computes that \( B(\sqrt{7}) \) is in \( B(\sqrt{7}) \). Let \( \sigma \) denote both the \( R \)-algebra automorphism of \( S \) and the \( K \)-algebra automorphism of \( K(u) \) defined by \( u \mapsto -u \). Let \( G = \{ 1, \sigma \} \) be the cyclic group generated by \( \sigma \), which we view as a group of automorphisms of both \( S \) and \( K(u) \).

Given a left \( S \)-module \( M \), define another left \( S \)-module action on \( M \) by the rule \( s \cdot x = \sigma(s)x \), for all \( s \in S \) and \( x \in M \). The left \( S \)-module just defined is denoted \( \sigma M \). If \( N \) is a right \( S \)-module, then by \( N_\sigma \) we denote the right \( S \)-module with action \( x \cdot s = x\sigma(s) \). If \( L \) is an invertible \( S \)-module, then \( \sigma L \) is another invertible \( S \)-module and the action \( L \mapsto \sigma L \) induces an action by \( G \) on \( \text{Pic} S \).

The group of units of \( R \) is equal to \( k^* \times \langle f_1 \rangle \times \cdots \langle f_n \rangle \), which is isomorphic to \( k^* \times \mathbb{Z}^n \). Since \( R \) is factorial, \( \text{Pic} R = 0 \). By (3), \( H^1(R, \mu_2) \cong (\mathbb{Z}/2)^{[n]} \). Since \( H^1(R, \mu_2) \) classifies the étale double covers of \( R \), we view \( S \) as a representative of the class \([S]\) in \( H^1(R, \mu_2) \) corresponding to \( f = f_1 \cdots f_n \). Fixing \([S]\) in one factor of the cup product \( \smile: H^1(R, \mu_2) \times H^1(R, \mu_2) \rightarrow H^2(R, \mu_2) \) [17, p. 172] and following with the Kummer theory map \( H^2(R, \mu_2) \rightarrow 2B(R) \), we have a homomorphism \( \smile: H^1(R, \mu_2) \rightarrow 2B(R) \). The image of \( \smile \) is denoted by \( B^-(S/R) \). If we pass to the quotient fields, \( K \rightarrow K(\sqrt{7}) \), every element of the Brauer group \( B(K) \) split by \( K(\sqrt{7}) \) is a cyclic crossed product, hence is in the image of the cup product map. In this sense, the classes of Azumaya algebras in \( B^-(S/R) \) represent the obvious elements in \( B(S/R) \).

**Theorem 2.1.** In the notation established above, there is an exact sequence of abelian groups

\[
0 \rightarrow B^-(S/R) \rightarrow B(S/R) \xrightarrow{\alpha_5} \text{Pic} S \otimes \mathbb{Z}/2 \rightarrow 0.
\]

If \( f \) is irreducible, then \( B^-(S/R) = \{ 0 \} \) and \( \alpha_5 \) is an isomorphism

**Proof.** [7, Theorem 2.1]

In Theorem 2.5 below it is shown that the short exact sequence of Theorem 2.1 is a special case of the so-called seven term exact sequence of Chase, Harrison and Rosenberg [4, Corollary 5.5].

**Example 2.2.** Let \( f = f_1f_2f_3f_4 \in k[x,y] \), where \( f_1, f_2, f_3, f_4 \) are four linear polynomials in general position. Let \( R = k[x,y][f^{-1}] \), \( S = R(\sqrt{7}) \). Using [8, Theorem 4], we see that \( 2B(R) = (\mathbb{Z}/2)^{[6]} \) and a basis consists of the symbol algebras \( \{ (f_i, f_j) : i < j \} \). The group \( B^-(S/R) \) is the subgroup of \( 2B(R) \) generated by \( \{ (f_i, f_j) : 1 \leq i \leq 4 \} \). One computes that \( B^-(R) \) is a group of order \( 2^3 \). Let \( F_i = Z(f_i) \) be the line defined by \( f_i = 0 \). Let \( P_{12} = F_1 \cap F_2 \) and \( P_{34} = F_3 \cap F_4 \). Let \( \ell \) be the linear equation of the line \( L \) through \( P_{12} \) and \( P_{34} \). Let \( \Lambda = (f, \ell) \). As in [8, Theorem 4], one computes

\[
(f, \ell)_2 \sim (f_1f_2, \ell)_2(f_3f_4, \ell)_2
\]

is in \( B(S/R) \) and not in \( B^-(S/R) \). By Theorem 2.1, \( \alpha_5(\Lambda) \) represents a non-trivial element of \( \text{Pic}(S) \otimes \mathbb{Z}/2 \).
Example 2.3. As in Example 2.2, let \( f_1, f_2, f_3, f_4 \) be four linear polynomials in general position. Let \( F_1 = Z(f_1) \) be the line defined by \( f_1 = 0 \). Let \( P_{12} = F_1 \cap F_2, P_{34} = F_3 \cap F_4 \), and let \( \ell \) be the linear equation of the line \( L \) through \( P_{12} \) and \( P_{34} \). Let \( F_0 \) be the line at infinity and let \( P_{05} \) be the point \( F_0 \cap L \). Let \( F_5 \) be a line through \( P_{05} \) which is in general position with respect to \( F_1, F_2, F_3, F_4, L \). Let \( R = k[x,y][f^{-1}] \), and \( S = R[\sqrt{f}] \).

The group \( B^-(S/R) \) is the subgroup of \( 2B(R) \) generated by \( \{(f,f_i) \mid i < j \} \). One computes that \( B^-(S/R) \) is a group of order \( 2^4 \). Let \( A = \langle f, \ell \rangle \). One computes

\[
(f, \ell)_2 \sim (f_1 f_2, \ell)_2 (f_3 f_4, \ell)_2
\]

is in \( B(S/R) \) and not in \( B^-(S/R) \). By Theorem 2.1, \( \alpha_5(A) \) represents a non-trivial element of \( \text{Pic}(S) \otimes \mathbb{Z}/2 \).

Example 2.4. Pick a linear polynomial \( \ell \in k[x,y] \), and let \( L = Z(\ell) \) be the line in \( \mathbb{A}^2 \) defined by \( \ell \). Generalizing Example 2.3, a large class of \( f \) are presented such that \( G = Z(\ell) \) is split by \( \mathbb{R}[\sqrt{f}] \). Let \( m \geq 2 \) and pick distinct points \( P_1, \ldots, P_m \) on \( L \). Let \( F_i \), \( i = 1, \ldots, 2m \), be general lines in \( \mathbb{A}^2 \) satisfying \( P_i \in F_{2i-1} \cap F_{2i} \). Let \( f_j = 0 \) be the linear equation for \( F_j \) and set \( f = f_1 f_2 \cdots f_{2m} \). Let \( R = k[x,y][f^{-1}] \) and \( S = R[\sqrt{f}] \). Then \( 2B(R) = (\mathbb{Z}/2)^{(r)} \) where \( r = 1 + 2 + \cdots + (2m - 1) \) and a basis consists of the symbol algebras \( \{(f,f_i) \mid i < j\} \).

The group \( B^-(S/R) \) is the subgroup of \( 2B(R) \) generated by \( \{(f,f_i) \mid 1 \leq j \leq 2m-1\} \). One computes that \( B^-(S/R) \) is a \( \mathbb{Z}/2 \)-module of rank \( 2m - 1 \). Let \( A = \langle f, \ell \rangle \). One computes

\[
(f, \ell)_2 \sim (f_1 f_2, \ell)_2 (f_3 f_4, \ell)_2 \cdots (f_{2m-1} f_{2m}, \ell)_2
\]

is in \( B(S/R) \) and not in \( B^-(S/R) \). By Theorem 2.1, \( \alpha_5(A) \) represents a non-trivial element of \( \text{Pic}(S) \otimes \mathbb{Z}/2 \).

Theorem 2.5. In the context of Theorem 2.1, the following are true.

(a) For all \( i > 0 \), \( H^{2i-1}(G,P_S^*) = 0 \).

(b) There is an exact sequence of abelian groups

\[
0 \to \text{Pic}(S)^G \xrightarrow{\alpha_5} H^2(G,P_S^*) \xrightarrow{\alpha_5} B(S/R) \xrightarrow{\alpha_5} H^1(G,\text{Pic}S) \to 0.
\]

Proof. Use the Hochschild-Serre spectral sequence [17, p. 105]:

\[
H^p(G,H^q(S,G_m)) \Rightarrow H^{p+q}(R,G_m).
\]

Since \( \text{Pic}R = 0 \), it follows that \( H^1(G,P_S^*) = 0 \). By [21, Theorem 10.35], \( H^{2i-1}(G,P_S^*) = 0 \) for all \( i > 0 \).

Theorem 2.6. In the context of Theorem 2.1, the following are true.

(a) For every invertible \( S \)-module \( L, \alpha L \cong L^* = \text{Hom}_S(L,S) \).

(b) For all \( i \geq 0 \),

\[
H^i(G,\text{Pic}S) = \begin{cases} 
\text{Pic}(S)^G & \text{if } i = 0,2,4,\ldots \\
\text{Pic}(S) \otimes \mathbb{Z}/2 & \text{if } i = 1,3,\ldots 
\end{cases}
\]

Proof. [7, Theorem 2.3]
In the notation of this section, let \( Y = \text{Spec} T \) and \( \pi : Y \to A^2 \) the double plane defined by \( z^2 = f \), where \( f = f_1 f_2 \cdots f_n \). By \( G = \langle \sigma \rangle \) we denote the group of order two, which acts on \( T \) by \( \sigma(z) = -z \). On \( A^2 \), let \( F_i \) denote the divisor \( \mathbb{Z}(f_i) \). The group \( G \) acts as a group of automorphisms of \( \text{Cl}(T) \). This action is induced on the group of divisors \( \text{Div}(T) \) by sending a height one prime \( P \) to its conjugate \( \sigma(P) \).

**Proposition 2.7.** In the notation established above, the following are true.

(a) The divisors on \( Y \) lying above \( F_1, \ldots, F_n \) generate a subgroup of \( 2\text{Cl}(T) \).

(b) For any \( L \in \text{Cl}(T) \), under the action of \( G \) on \( \text{Cl}(T) \), we have \( \sigma(L) = -L \).

(c) For all \( i \geq 0 \),

\[
H^i(G, \text{Cl}(T)) = \begin{cases} (\text{Cl}(T))^G = 2\text{Cl}(T) & \text{if } i = 0, 2, 4, \ldots \\ \text{Cl}(T) \otimes \mathbb{Z}/2 & \text{if } i = 1, 3, \ldots \end{cases}
\]

**Proof.** \cite[Theorem 2.4]{7}

In \cite{13} the exact sequence of Theorem 2.5(b) is constructed without recourse to spectral sequences. Given an invertible \( S \)-module \( L \), we show in Example 2.8 how to construct a Brauer class in \( B(S/R) \) which is in the pre-image of \( L \) under \( \alpha_S \). The construction of a generalized crossed product algebra is based on the Picard group of invertible bimodules. First we review the definitions, which are found in \cite[Chapter II]{2} or \cite[§ 37]{19}.

### 2.1. The Picard group of invertible bimodules.

Throughout this paragraph, \( R \) is any commutative ring and \( A \) is an \( R \)-algebra. A left \( A \otimes_R A^\sigma \)-module \( P \) is also viewed as an \( A \)-bimodule where \( a \cdot x = a \otimes 1 \cdot x \) and \( x \cdot a = 1 \otimes a \cdot x \) for all \( a \in A \) and \( x \in P \). Conversely, an \( A \)-bimodule \( P \) such that \( rx = xr \) for all \( r \in R \) can be viewed as a left \( A \otimes_R A^\sigma \)-module.

Let \( A_P \) denote the left \( A \)-module and \( P_A \) the right \( A \)-module. Given a left \( A \otimes_R A^\sigma \)-module \( P \), there is a natural homomorphism of \( R \)-algebras

\[
A^\sigma \xrightarrow{\rho} \text{Hom}_A(A_P, A_P)
\]

defined by \( a \mapsto \rho_a \), where \( \rho_a(x) = xa \). We say that \( P \) is invertible if \( P \) is a left \( A \otimes_R A^\sigma \)-progenerator and \( \rho \) is an isomorphism. There is another natural homomorphism of \( R \)-algebras

\[
A \xrightarrow{\lambda} \text{Hom}_A(P_A, P_A)
\]

defined by \( a \mapsto \lambda_a \), where \( \lambda_a(x) = ax \). If \( P \) is invertible, then by the Morita Theorems \cite{5}, \( \lambda \) is also an isomorphism. Let \( \text{Pic}_R(A) \) denote the group of isomorphism classes \( [P] \) of invertible left \( A \otimes_R A^\sigma \)-modules \( P \). The group law is \( [P][Q] = [P \otimes_A Q] \) and the inverse of a class is \( [P]^{-1} = [\text{Hom}_A(A_P, A_P)] \). Consider the case where \( A \) is commutative and \( P \) is an invertible \( A \otimes_R A \)-module. Since \( A = A^\sigma \), the isomorphisms \( \rho \) and \( \lambda \) imply that for any \( a \in A \), there exists \( \alpha_P(a) \in A \) such that \( \rho_a = \lambda_{\alpha_P(a)} \). In other words, for all \( a \in A \) there exists \( \alpha_P(a) \in A \) such that \( xa = \alpha_P(a)x \) for all \( x \in P \). This defines an \( R \)-algebra automorphism \( \alpha_P \in \text{Aut}_R(A) \). The assignment \( P \mapsto \alpha_P \) is a homomorphism of groups \( \Psi : \text{Pic}_R(A) \to \text{Aut}_R(A) \). The kernel of \( \Psi \) is \( \text{Pic}(A) \). Given \( \sigma \in \text{Aut}_R(A) \), we can turn \( A \) into a left \( A \otimes_R A \)-module by the rule \( a \otimes b \cdot x = ax \sigma(b) \). As in \cite[Chapter II, § 5]{2} or \cite[(37.12), p. 322]{19}, this \( A \)-bimodule is denoted \( 1_A \). The assignment \( \sigma \mapsto [1_A] \) is a group homomorphism \( \Phi_0 : \text{Aut}_R(A) \to \text{Pic}_R(A) \) such that \( \Psi \Phi_0 = 1_{\text{Aut}_R(A)} \). We have the exact sequence \cite[(5.4), p. 75]{2} or \cite[(37.18), p. 325]{19}

\[
0 \longrightarrow \text{Pic}A \longrightarrow \text{Pic}_R(A) \xrightarrow{\Psi} \text{Aut}_R(A) \longrightarrow 1
\]

where \( \Psi \) is split by \( \Phi_0 \).
2.2. Generalized Crossed Product Algebras. Now return to the context of the introduction to Section 2. Then $S/R$ is a Galois extension of commutative rings with group $\text{Aut}_R(S) = G = \{1, \sigma\}$, $\text{Pic}\ S$ is a normal subgroup of $\text{Pic}\_R(S)$, and there is the coset decomposition $\text{Pic}\_R(S) = \text{Pic}(S) \cup [\sigma S_1] \text{Pic}(S)$. Notice that in this context, $\sigma S_1 \cong 1 S_\sigma$. By sequence (11), $G$ acts on $\text{Pic}\ S$ by conjugation in $\text{Pic}\_R(S)$. That is, for $[P] \in \text{Pic}\ S$, $\sigma[P] = [\sigma S_1 \otimes_R P \otimes_\sigma S_1]$. This action is equal to the action on $\text{Pic}\ S$ induced by $P \mapsto \sigma P$. By Theorem 2.6(a), $P \otimes_\sigma S \cong S$ as left $S$-modules. Consider any $[J]$ in the coset $[\sigma S_1] \text{Pic}(S)$. Then $J = \sigma S_1 \otimes S I$ for some $[I] \in \text{Pic}(S)$ and

$$J \otimes S = \sigma S_1 \otimes S \sigma S_1 \otimes S I = \sigma I \otimes S I \cong S$$

says $[J]$ has order two in $\text{Pic}\_R(S)$. Set $J_\sigma = J$ and $J_1 = S$. As in [13], a factor set for $[J]$ consists of four $S$-bimodule isomorphisms $f_{1,1} : S \otimes S S \cong S$, $f_{\sigma,1} : J \otimes S S \cong J$, $f_{1,\sigma} : S \otimes S J \cong J$, and $f_{\sigma,\sigma} : J \otimes S J \cong S$ such that the diagram

$$\begin{array}{ccc}
J_{\alpha} \otimes S J_{\beta} \otimes S J_{\gamma} & \overset{1 \otimes f_{\beta,\gamma}}{\longrightarrow} & J_{\alpha} \otimes S J_{\beta} \\
\downarrow f_{\alpha,\beta} \otimes 1 & & \downarrow f_{\alpha,\beta} \\
J_{\alpha \beta} \otimes S J_{\gamma} & \longrightarrow & J_{\alpha \beta \gamma}
\end{array}$$

(12)

commutes for every triple $(\alpha, \beta, \gamma) \in G \times G \times G$.

Given $[J] = [\sigma S_1] [I]$ as above, together with a factor set $\{f_{a,b}\}$, the generalized crossed product algebra $\Delta(f, S, J, G)$ is defined as follows. As an $S$-bimodule it is $J_1 \oplus J_\sigma$. Multiply homogeneous elements $a \in J_\sigma$, $b \in J_\beta$ according to the rule $a \cdot b = f_{a,b}(a \otimes b)$. As shown in [13], $\Delta\{f_{a,b}\}, S, J, G\}$ is an Azumaya $R$-algebra, contains $S$ as a maximal commutative subring, and represents a class in $B(S/R)$ which is mapped by $\alpha_S$ to the coset represented by $I$.

**Example 2.8.** Let $I$ be a height one prime ideal in $S$. Then $I$ is a projective $S$-module of rank one, and represents a class in $\text{Pic}\ S$. The assignment $x \mapsto \sigma(x)$ is an isomorphism

$$\sigma I \cong \sigma(I)$$

of left $S$-modules. Because $S$ is commutative, we can view $I$ as an $S$-bimodule under the usual left and right multiplication rules. By $I_\sigma$, we denote the $S$-bimodule $I \otimes_S 1 S_\sigma$, where multiplication from the left is $s \cdot x = sx$, and from the right is $x \cdot s = x \sigma(s)$. As we saw above, $[I_\sigma]$ has order two in $\text{Pic}\_R(S)$. By Theorem 2.6(a), $I \sigma(I) = Sg$ is a principal ideal in $S$. There are isomorphisms

$$\begin{array}{ccc}
I_\sigma \otimes S 1_\sigma & \overset{f_{\sigma,\sigma}}{\longrightarrow} & S \\
I_\sigma \otimes S f_{1,1} & \longrightarrow & I_\sigma \\
S \otimes S 1_\sigma & \overset{f_{1,\sigma}}{\longrightarrow} & I_\sigma \\
S \otimes S f_{1,1} & \longrightarrow & S
\end{array}$$

(13)

of $S$-bimodules defined by the assignments $f_{\sigma,\sigma}(a \otimes b) = a \sigma(b) \sigma^{-1}$, $f_{1,\sigma}(a \otimes b) = a \sigma(b)$, $f_{1,\sigma}(a \otimes b) = ab$, $f_{1,1}(a \otimes b) = ab$. Let $J_1 = S$ and $J_\sigma = 1 I_\sigma$. To show that $\{f_{a,b}\}$ is a factor set, it is enough to use $\{f_{a,b}\}$ to define a multiplication rule on the $S$-bimodule $J_1 \oplus J_\sigma$ and prove that the resulting product rule is associative. Let the multiplication rule
be defined on homogeneous \(a \in J_{\alpha}, b \in J_{\beta}\) by the rule \(a \cdot b = f_{\alpha\beta}(a \otimes b)\). If \((a,b), (c,d), (e,f)\) are three typical elements of \(J_{I} \oplus J_{\sigma}\), then the multiplication rule is

\[
(a,b)(c,d) = (ac + b\sigma(d)g^{-1}, b\sigma(c) + ad).
\]

The associative law for multiplication follows from the fact that

\[
((a,b)(c,d))(e,f) = (ae + b\sigma(d)e^{-1} + (b\sigma(c) + ad)\sigma(f)g^{-1},
(b\sigma(c) + ad)\sigma(e) + (ac + b\sigma(d)g^{-1})f)
\]

is equal to

\[
(a,b)((c,d)(e,f)) = (a,b)(ce + d\sigma(f)g^{-1}, d\sigma(e) + cf)
\]

\[
= \left( a(ce + d\sigma(f)g^{-1}) + b(\sigma(d)e + \sigma(c)\sigma(f))g^{-1},
\right.
\]

\[
\left. a(d\sigma(e) + cf) + b(\sigma(c)\sigma(e) + \sigma(d)fg^{-1}) \right).
\]

It follows that \(\{f_{\alpha\beta}\}\) is a factor set. The generalized crossed product \(\Delta(\{f_{\alpha\beta}\}, S, 1_{\sigma}, \sigma)\) is an Azumaya \(R\)-algebra.

2.3. Additional results. Suppose \(\Lambda\) is a central \(K\)-division algebra which is split by \(K(\sqrt{f})\) such that \(\text{ram.div}(\Lambda)\) on \(A^2\) is a subset of \(F = Z(f)\). By Theorem 1.1 applied to \(\text{Spec} \mathcal{R}\), the Brauer class of \(\Lambda\) is the image of the natural map \(B(\mathcal{R}) \to B(K)\). It follows from the Crossed Product Theorem (for example, [22, Corollary 7.11]) that \(\Lambda\) is a symbol \(\Lambda = (f,h)_{2}\) for some \(h \in K^{*}\). Because any algebra of the form \((f,b^2)_{2}\) is split, \((f,h)_{2}\) is Brauer-equivalent to \((f,\sqrt{h^2})_{2}\), so we can assume \(h\) is in \(R\). There exist \(u, v \in \Lambda\) which generate \(\Lambda\) as a \(K\)-algebra such that \(u^2 = f, v^2 = h\) and \(uv = -vu\). Let \(B\) denote the \(R\)-subalgebra of \(\Lambda\) generated by \(u\) and \(v\). Then \(B\) is a free \(R\)-module of rank four with basis \(1, u, v, uv\), and \(B \otimes_{R} K = \Lambda\). So \(B\) is an \(R\)-order in \(\Lambda\) (for example, [22, Chapter 9]). We can identify \(S\) with \(R[u]\), the \(R\)-subalgebra of \(B\) generated by \(u\). It is easy to see that \(B\) is a free right \(S\)-module with basis \(1, v\). By [22, Theorem 9.2] there exists a maximal \(R\)-order \(\mathcal{A}\) such that \(B \subset \mathcal{A} \subset \Lambda\). By [22, Theorem 9.9], we know that \(\mathcal{A}\) is an Azumaya \(R\)-algebra. The diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & B(S) \longrightarrow \longrightarrow \ B(K(\sqrt{f})) \\
& \text{vertical} & \text{vertical} \\
0 & \longrightarrow & B(R) \longrightarrow \longrightarrow \ B(K)
\end{array}
\]

commutes and the rows are exact. Since \(A \otimes_{R} \mathcal{A} = \Lambda\) is split by \(K(\sqrt{f})\), it follows that \(A\) is split by \(S\). Because \(K(\sqrt{f})\) is a maximal subfield of \(\Lambda\) and \(S\) is integrally closed in \(K(\sqrt{f})\), it follows that \(S\) is a maximal commutative subring of \(A\). It follows from [5, Theorem 2.5.5] that \(A\) is an \(S\)-progenerator and \(S \otimes_{\mathcal{R}} A^0 \cong \text{Hom}_{\mathcal{S}}(A, A)\). There is a Morita equivalence [5, Proposition 1.3.3] between the category \(\mathfrak{M}_{S}\) of right \(S\)-modules and the category \(S \otimes_{\mathcal{R}} A^0\)-modules. Given \(L \in \mathfrak{M}_{S}\), the counterpart of \(L\) in \(S \otimes_{\mathcal{R}} A^0\) is \(L \otimes_{S} A\). The set inclusion map \(1 : S \rightarrow A\) is an \(R\)-algebra homomorphism which gives the exact sequence of \(R\)-modules

\[
0 \to S \xrightarrow{1} A \to A/S \to 0.
\]

Notice that (17) is also an exact sequence of left \(S\)-modules.
Theorem 2.9. The element $\alpha_s(\Lambda)$ in $\text{Pic} S \otimes \mathbb{Z}/2$ determined by $\Lambda$ is the coset containing the invertible $S$-module $A/\mathcal{S}$. Of course this is also the coset containing the inverse of $A/\mathcal{S}$, $\sigma(A/\mathcal{S}) = (A/\mathcal{S})^*$ because these two invertible modules differ by an element in $2\text{Pic} S$.

Proof. By [13, Proposition 3], $A$ is isomorphic to a generalized crossed product algebra $\Delta(f, S, [J], G)$. The $S$-module $A/\mathcal{S}$ is an invertible $S$-module, the image of $t$ is an $S$-module direct summand of $A$, and (17) is split-exact as a sequence of left $S$-modules. The rest follows from the construction of $\alpha_s$ in [13].

Proposition 2.10. In the notation established above, let $G = \langle \sigma \rangle$ denote both the Galois group of $S$ over $R$ and of $K(u)$ over $K$.

(a) If $s \in K(u)$ and $x \in \Lambda$, then $sx - x\sigma(s) \in K(u)$. If $s \in S$ and $x \in A$, then $sx - x\sigma(s) \in S$.

(b) There is an isomorphism of left $K(u) \otimes_R \Lambda^S$-modules $\Lambda/K(u) \otimes_{K(u)} \Lambda \cong \sigma\Lambda$.

(c) If $I$ is a prime ideal of $S$ of height one, then $I$ is a free $R$-module of rank two. There exist elements $a, b$ in $I$ such that $I = aS + bS$.

Proof. (a): Turn $\Lambda$ into a left $K(u)$-module by left multiplication. A $K(u)$-basis for $\Lambda$ is $\{1, v\}$. Write $s = \alpha + \beta u$ where $\alpha, \beta \in K$. Write $x = a1 + bv$ where $a, b \in K(u)$. Then

\[sx = (\alpha + \beta u)(a1 + bv) = \alpha a + \alpha bv + \beta au + \beta uv\]

and

\[x\sigma(s) = (a1 + bv)(\alpha - \beta u) = a\alpha - a\beta u + b\alpha v - b\beta vu = \alpha a - a\beta u + b\alpha v + b\beta uv.\]

Therefore $sx - x\sigma(s) = 2a\beta u \in K(u)$.

If $s \in S$ and $x \in A$, then $sx - x\sigma(s) = 2a\beta u \in \Lambda \cap K(u)$ is integral over $R$. But $S$ is the integral closure of $R$ in $K(u)$, so $sx - x\sigma(s) \in S$.

(b): A basis for the left $K(u)$-module $\Lambda$ is $\{1, v\}$ and a basis for the right $K(u)$-module $\Lambda/K(u)$ is $\{v\}$. A typical element of $\Lambda/K(u) \otimes_{K(u)} \Lambda$ can be written in the form $v \otimes (a + bv)$ for $a, b \in K(u)$. Define $\psi : \Lambda/K(u) \otimes_{K(u)} \Lambda \to \sigma\Lambda$ by the rule $v \otimes (a + bv) \mapsto \sigma(a)v + \sigma(b)v^2$. Clearly $\psi$ is an isomorphism of $K$-vector spaces. Since

\[\psi(c(v \otimes (a + bv))) = \psi(v \otimes (c(a + bv))) = \sigma(ca)v + \sigma(cb)v^2 = \sigma(c)(\sigma(a)v + \sigma(b)v^2) = c \cdot \psi(a + bv)\]

for all $c \in K(u)$, it follows that $\psi$ commutes with left multiplication from $K(u)$. That $\psi$ commutes with right multiplication by $\Lambda$ follows from the fact that

\[\psi((v \otimes (a + bv))(c + dv)) = \psi(v \otimes ((a + bv)(c + dv))) = \psi(v \otimes ((ac + b\sigma(d)v) + (ad + b\sigma(c)v)) = \sigma(ac + b\sigma(d)v) + \sigma(ad + b\sigma(c))v^2\]

is equal to

\[\psi(v \otimes (a + bv))(c + dv) = (\sigma(a)v + \sigma(b)v^2)(c + dv) = (\sigma(a)\sigma(c) + \sigma(b)v^2d)v + (\sigma(a)\sigma(d) + \sigma(b)c)v^2\]

for all $c, d \in K(u)$.
(c): Let $I$ be a height one prime ideal in $S$. Then $I$ is a rank one reflexive module and because $S$ is non-singular, $I$ is a rank one projective $S$-module [12, Corollary II.6.16]. Since $S$ is a free $R$-module of rank two, it follows that $I$ is a projective $R$-module of rank two. By [18], the $R$-module $I$ decomposes into a direct sum of two rank one projective modules. Since $\text{Pic} R = 0$, it follows that $I$ is a free $R$-module. 

**Question 2.11.** In Proposition 2.10(c), can we prove a similar result for the ring $T$? Are height one primes of $T$ necessarily free over $A$?

**Proposition 2.12.** In the notation of Proposition 2.10, the following are true.

(a) The exact sequence (17) of $S$-modules is split-exact. Equivalently, $S \cdot 1$ is an $S$-module direct summand of $A$. Equivalently, $A/S$ is an invertible $S$-module.

(b) The top Chern class of the $S$-module $A$ is $\wedge^2 (A) = A/S$.

(c) By $\sigma A$ we denote the left $S \otimes_R A^\vee$-module which as a set is $A$ and where multiplication is defined by $s \otimes x \cdot y = \sigma(s)yx$ for all $s \in S$, $x \in A^\vee$, $y \in A$. Under the Morita equivalence $\Omega_S \cong S \otimes_R A^\vee \Omega$, the right $S$-module corresponding to the left $S \otimes R A^\vee$-module $\sigma A$ is $A/S$.

**Proof.** (a): Done already.

(b): The split exact sequence (17) can be used to compute the exterior powers: $\wedge^2 (A) = S \otimes_S (A/S) = A/S$ [3, Proposition 10, p. 515].

(c): As in Theorem 2.9, $A$ is isomorphic to a generalized crossed product algebra $\Delta(f,S,[J],G)$ where $J$ is an invertible $S$-bimodule. By the coset decomposition (11), there is an $S$-bimodule isomorphism $J \cong aS_1 \otimes_S I$, for some $[J] \in \text{Pic} S$. We identify $I$ with $aS_1 \otimes_S J$, the right $S$-module $J$. Multiplication in $A = \Delta(f,S,[J],G)$, $x \otimes y \mapsto x \cdot y$, induces the map

$$I \otimes_S A \longrightarrow \sigma A$$

which is clearly one-to-one and onto. One can check that this is an isomorphism of $S \otimes R A^\vee$-modules.

**Question 2.13.** In the above context, assume the ramification divisor of $\Lambda = (f,g)_2$ is contained in $F = Z(f)$. What are sufficient conditions on $f$ and $g$ so that $G = Z(g)$ is split by the double cover $\sigma : Y \to A^2$ obtained by adjoining $\sqrt{f}$ and the irreducible components of $\tilde{C}$ are non-principal?

### 2.4. Primes of $R$ that are split by $S$.

As in Section 1, $A = k[x,y]$, $f$ is square-free, $T = A[z]/(z^2 - f)$, $R = A[f^{-1}]$ and $S = T[z^{-1}]$. Let $\pi : \text{Spec} T \to \text{Spec} A$ be the corresponding morphism of surfaces. Since $R$ and $S$ are regular surfaces, by Theorem 1.1, $B(S/R) \to B(L/K)$ is one-to-one. An element of $B(S/R)$ is represented by a central $K$-division algebra $\Lambda \in B(L/K)$ and the ramification divisor of $\Lambda$ is contained in $F = Z(f)$. By the crossed product theorem, the division algebra $\Lambda$ is a symbol $(f, h)_2$ for some $h$ in $K^*$ (for instance, see [22, Corollary 7.11]). Since $h$ is unique up to norms from $L^*$, we can assume $h$ is a square-free element of $A$. Factoring $h$ into irreducibles, the Brauer class of $\Lambda$ is a product of classes of the form $(f, g)_2$, where $g$ is an irreducible element of $A$. Denote by $\tilde{C} = Z(g)$ the irreducible curve on Spec$A$ defined by $g$. Consider the divisor $\tilde{C} = \pi^{-1}(C)$ on Spec$T$. The diagrams

\begin{equation}
\begin{array}{c}
\tilde{C} \longrightarrow \text{Spec } T \\
\pi \downarrow \quad \quad \quad \downarrow \pi \\
C \longrightarrow \text{Spec } A \\
\end{array}
\end{equation}

\begin{equation}
\begin{array}{c}
\quad T \longrightarrow \text{Spec } T \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qu
Proposition 2.14. As above, \( \pi : \text{Spec} T \to \text{Spec} A \) is the affine double plane defined by \( z^2 = f \), where \( A = k[x,y] \) and \( K = k(x,y) \). Assume the \( K \)-symbol algebra \( (f,g)_2 \) ramifies only along \( F = Z(f) \). If \( g \) is irreducible and \( C = Z(g) \), then \( \tilde{C} = \pi^{-1}(C) \) is not irreducible. The curve \( \tilde{C} \) is reducible with only one irreducible component if and only if \( g \) divides \( f \). Otherwise \( \tilde{C} = \tilde{C}_1 + \tilde{C}_2 \) is reducible and has two irreducible components.

**Proof.** If \( g \) divides \( f \), then any prime of \( T \) containing \( g \) also contains \( z \). In this case, \( g \) has a unique minimal prime in \( T \), namely \( P = (g,z) \). In the local ring \( T_p \), the element \( g \) has valuation 2. This shows \( \text{Div}(g) = 2P \). So \( \tilde{C} \) is reducible with only one irreducible component. Note that in this case, \( (f,g)_2 \) is in \( B^{-}(S/R) \).

Now assume \( g \) does not divide \( f \). Then \( g \) is irreducible in \( R = A[f^{-1}] \). Let \( Q \) denote the prime ideal \( Rg \) in \( R \). The field \( K(C) = R_Q/QR_Q \) is the function field of \( C \). Because \( S = T \otimes_A R \) is Galois over \( R \), \( S \otimes_R K(C) \) is separable of degree two over \( K(C) \). Either \( S \otimes_R K(C) \) is a field, or a direct sum of two copies of \( K(C) \) [14, Proposition III.4.1]. If \( S \otimes_R K(C) \) is a field, then \( Sg \) is a prime ideal in \( S \), so \( \tilde{C} \) is irreducible. In this case, the ramification of \( (f,g)_2 \) along the divisor \( C \) is the non-zero class of \( S \otimes_R K(C) \) in \( H^1(K(C),\mathbb{Z}_2) \). This case does not arise because we are assuming \( (f,g)_2 \) is unramified along \( C \).

The last possibility is that \( S \otimes_R K(C) \) is a direct sum of two copies of \( K(C) \). In this case there are two minimal primes of \( Sg \). Let \( P \) be one of them. The other is necessarily \( \sigma(P) \), by [16, (5.E), Theorem 5]. Because the residue fields of \( R_Q \) and \( S_P \) are equal, the image of \( QR_Q \) generates the maximal ideal of \( S_P \). This means \( g \) is a local parameter for \( S_P \). The divisor of \( g \) on \( \text{Spec} S \) is \( P + \sigma(P) \).

In Proposition 2.15 we prove a partial converse to Proposition 2.14. If \( C = Z(g) \) splits over \( S \) into \( \tilde{C} = \tilde{C}_1 + \tilde{C}_2 \) where \( \tilde{C}_1 \) and \( \tilde{C}_2 \) are disjoint, then the \( K \)-symbol algebra \( (f,g)_2 \) is shown to represent a Brauer class in the image of \( \mathbb{B}(R) \to \mathbb{B}(K) \).

**Proposition 2.15.** In the context of Proposition 2.14, suppose \( g \in R \) is irreducible and that \( S/(g) \) is isomorphic to the direct sum of two copies of \( R/(g) \).

1. There is an element \( h \) in \( R - (0) \) such that the minimal primes of \( g \) in \( S \) are \( I = (g,z-h) \) and \( \sigma(I) = (g,z+h) \).
2. The symbol algebra \( (f,g)_2 \) over \( K \) represents a class \( \xi \) in \( \mathbb{B}(S/R) \).
3. The coset \( \alpha_S(\xi) \) in \( \text{Pic} S \otimes \mathbb{Z}/2 \) is represented by the ideal \( I \).

**Proof.** We are given that

\[
\frac{S}{(g)} = \frac{(R/(g))[z]}{(z^2 - f)}
\]

is the trivial quadratic extension of \( R/(g) \). This means \( f \) is a non-square in \( R/(g) \). There exist \( u, h \) in \( R - (0) \) such that \( f = u^2 + h^2 \). Look at the ideal \( I = (g,z-h) \) in \( S \).

Since

\[
S/I = \frac{k[x,y,z][f^{-1}]}{(g,z^2 - f, z-h)} = \frac{k[x,y][f^{-1}]}{(g)}
\]

we see that \( I \) is prime of height one. A typical element of \( S \) can be written in the form \( a + b(z-h) \), for \( a, b \in R \). If \( a, b, c, d \) are from \( R \), then a typical element of \( I \) is

\[
(a + b(z-h))g + (c + d(z-h))(z-h) = ag + b(z-h)g + c(z-h) + d(z-h)^2
\]

\[
= ag + b(z-h)g + c(z-h) + d(z^2 - h^2 - 2zh + 2h^2)
\]

\[
= (a + du)g + (bg + c - 2dh)(z-h)
\]
so \( I = Rg + R(z - h) \). By Proposition 2.10, \( g, z - h \) is a free \( R \)-basis for \( I \). Since \( z \) is invertible in \( S \), \( I(\sigma) = (g^2, g(z + h), g(z - h), ug) = Sg \). Let \( \Delta(I) \) be the generalized cross product algebra, as defined in Example 2.8. Then \( \Delta(I) \) is an Azumaya \( R \)-algebra which is split by \( S \). As an \( R \)-module \( \Delta(I) \) is generated by \((1,0), (z,0), (0,g), \) and \((0,z-h)\). Using equation (14), the multiplication table for \( \Delta(I) \) is constructed in Table 1. Upon extending the ring of scalars to \( K \), it is clear that \( \Delta(I) \otimes_R K \) is isomorphic to the symbol algebra \((f,g)_2\). Therefore \((f,g)_2\) is unramified on \( Z(g) \), represents a class \( \xi \) in \( B(S/R) \), and \( \alpha_f(\xi) \) is represented by the divisor class of the ideal \( I = (g,z-h) \).

Suppose \( f \) and \( g \) are as in Proposition 2.14 and \( g \) does not divide \( f \). If \( C = Z(g) \) is rational and simply connected, then \( \tilde{C} = C_1 + C_2 \) is reducible if and only if the local intersection multiplicity of \( C \) and \( F \) at each point is even [12, Corollary IV.2.4].

**Proposition 2.16.** As always, \( A = k[x,y] \) and \( K = k(x,y) \). Suppose \( f \) and \( g \) are in \( A \), \( f \) is square-free, \( g \) is irreducible, \( g \) does not divide \( f \), and the \( K \)-symbol algebra \((f,g)_2\) is unramified along each prime divisor of \( R = A[f^{-1}] \). If \( C = Z(g) \) on \( \text{Spec} R \) is either nonsingular, or has only unibranched singularities, then \( S/(g) \) is isomorphic to a direct sum of two copies of \( R/(g) \).

**Proof.** We are in the context of the paragraph preceding Proposition 2.14. Let \( \Lambda = (f,g)_2 \).
In Theorem 1.1, the ramification \( \alpha_C(\Lambda) \) along \( C \) is given by the tame symbol (6). But \( R \) is factorial and \( g \) is irreducible. Therefore \( \alpha_C(\Lambda) \) is the quadratic extension \( K(C)[z]/(z^2 - f) \), which by assumption represents the zero class in \( H^1(K(C),Z/2) \). Let \( \tilde{C} \) denote the normalization of \( C \). Because \( C \) has at most unibranched singularities, the natural map \( H^1(C,Q/Z) \to H^1(\tilde{C},Q/Z) \) is an isomorphism. For any closed point \( p \in \tilde{C} \), the natural map

\[
H^1(\tilde{C},Q/Z) \to H^1(\tilde{C} - p, Q/Z)
\]

is one-to-one by cohomological purity [17, Theorem VI.5.1]. By a direct limit argument, the natural map

\[
H^1(\tilde{C},Q/Z) \to H^1(K(C),Q/Z)
\]

is one-to-one. Therefore, the unramified quadratic extension \( S/(g) \) represents the zero class in \( H^1(C,Q/Z) \). So \( S/(g) \) is isomorphic to a direct sum of two copies of \( R/(g) \). \( \square \)

**Example 2.17.** This example shows that if the curve \( R/(g) \) has a nodal singularity, the conclusion of Proposition 2.16 can fail. Let \( f = x + 1 \), \( T = k[x,y,z]/(z^2 - f) \). Let \( g = y^2 - x^2(x + 1) \). In \( T \) the element \( g \) factors into \( (y-xz)(y+xz) \). Each factor is irreducible because the map \( x \mapsto z^2 - 1, y \mapsto xz \) induces \( T/(y-xz) \cong k[z] \). Since

\[
\frac{T}{(y-xz,y+xz)} \cong k[z]/(z^2 - 1)
\]

the elements \( y-xz \) and \( y+xz \) are not relatively prime, even in \( S = T[z^{-1}] \). The conclusion of Proposition 2.16 is not satisfied. Now look at the symbol algebra \( \Lambda = (f,g)_2 \) over
$K = k(x, y)$. Since $1 \sim (x + 1, x)_2$, we have
\[
\Lambda \sim (x + 1, x^{-2})_2(x + 1, y^2 - x^2(x + 1))_2 \\
\sim (x + 1, (y/x)^2 - (x + 1))_2 \\
\sim 1
\]
Therefore, $(f, g)_2$ is split, hence unramified over $R$.

**Remark 2.18.** Suppose our goal is to construct a double plane $\text{Spec } T \to A^2$ with the property that the class group on the unramified set $\text{Spec } S \subseteq \text{Spec } T$ is non-trivial and easy to compute. An approach based on Theorem 2.1 is to find $f$ such that we can compute elements that are in $B(S/R)$ but not in $B^+(S/R)$. The preceding examples provide some insight on how to pick elements $f$ and $g$ in $A$ such that $(f, g)_2$ is in $B(S/R)$ and not in $B^+(S/R)$. Start with a sequence of distinct irreducible polynomials $p_1, \ldots, p_N$ in $A = k[x, y]$, where $N \geq 3$. Put $f = p_1p_2 \cdots p_j + (p_{j+1} \cdots p_N)^2$, for some $j$ such that $2 \leq j < N$. If $f$ is square-free, then $z^2 - f$ is irreducible and $T = A[z]/(z^2 - f)$ is integrally closed. Let $g$ be any one of $p_1, \ldots, p_j$ and $h = p_{j+1} \cdots p_N$. By construction, $g$ does not divide $f$. Let $R = A[f^{-1}]$. The map
\[
\frac{(R/(g))[z]}{(z^2 - h^2)} \xrightarrow{\beta} \frac{R}{(g)} \oplus \frac{R}{(g)}
\]
is an isomorphism, where $\beta$ maps $z \mapsto (h, -h)$. If $S = T[z^{-1}]$, then $S/(g)$ is isomorphic to the ring on the left hand side of (19). By Proposition 2.15, the symbol algebra $\Lambda = (f, g)_2$ ramifies only along the zeros of $f$. Also, the homomorphic image of $[\Lambda]$ under $\alpha_0$ is the divisor class of the ideal $I = (g, z - h)$. Upon restriction to the quotient field $K = k(x, y)$, the symbol algebra $(f, g)_2$ is a division algebra if the ideal $I = (g, z - h)$ represents a non-trivial class in $\text{Pic } S \otimes \mathbb{Z}/2$. The converse of this last statement is false, as shown in Example 2.23.

**Example 2.19.** This example is based on the construction of Remark 2.18. Let $\ell_1, \ell_2, \ell_3$ be three general linear polynomials in $k[x, y]$. Let $f = \ell_1\ell_2 - \ell_3^2$. We can assume $f$ is irreducible. Let $F = Z(f)$, $L_i = Z(\ell_i)$, and $F_0$ the line at infinity. Let $L_1 \cdot L_3 = P_1$ and $L_3 \cdot F_0 = P_{10}$. We see that $F \cdot L_1 = 2P_1$. By a general position argument, we can assume $F_0 \cdot F = P_{10} + P_{02}$. For the symbol algebra $(f, \ell_1)_2$, the weighted path in the graph $G(F + L_1 + F_0)$ is shown in Figure 1. The cycle $F \to P_{01} \to F_0 \to P_{02} \to F$ is non-trivial. This proves that $(f, g)_2$ is a non-trivial element of $B(S/R)$. Since $f$ is irreducible, Theorem 2.1 says $B^+(S/R) = (0)$. Therefore, the ideal $I = (\ell_1, z - \ell_3)$ is a non-trivial element of $\text{Pic } S \otimes \mathbb{Z}/2$.

**Example 2.20.** This example is based on the construction of Remark 2.18. Start with a sequence of distinct irreducible polynomials $p_1, \ldots, p_N$ in $A = k[x, y]$, where $N \geq 3$. Put $f = p_1p_2 \cdots p_j + (p_{j+1} \cdots p_N)^2$, for some $j$ such that $2 \leq j < N$. Let $R = A[f^{-1}]$, and $S = R[z]/(z^2 - f)$. If we assume $f$ is irreducible, by Theorem 2.1, $B(S/R) \cong \text{Pic } (S) \otimes \mathbb{Z}/2$. Let $g$ be any one of $p_1, \ldots, p_j$ and $h = p_{j+1} \cdots p_N$. Let $F = Z(f)$, $F_0$ the line at infinity, $G = Z(g)$, and $H = Z(h)$. At a finite point $P$, the local intersection multiplicity $(F \cdot G)_P$ is divisible by 2. Assume there exists $P_0$ in $F_0 \cap F$ such that $P_0$ is not a point of $G$ and the local intersection multiplicity $(F_0 \cdot F)_P$ is odd. If we assume $\deg G$ is odd, then the weighted path in the graph $G(F + G + F_0)$ of the symbol algebra $(f, g)_2$, has loops of the type $F \to P_0 \to F_0 \to P_{0j} \to F$. Therefore, $(f, g)_2$, is a division algebra and the ideal $I = (g, z - h)$ is a non-trivial element of $\text{Pic } S \otimes \mathbb{Z}/2$. 
DIVISION ALGEBRAS AND THE PROBLEM OF SPLITTING PRIMES

Example 2.21. This example is based on Example 2.20. This example shows that it is not necessary to assume the degree of $p_1$ is odd. Let $\ell_1, \ell_2$ be linear polynomials in $k[x,y]$ and $c$ an irreducible conic such that $f = \ell_1 c + \ell_2^2$ is an irreducible cubic. Assume $\ell_1, \ell_2, c$, and the line at infinity $F_0$ are in general position. In this example we prove that $(f,c)_2$ is a division algebra. Let $C = Z(c)$, $F = Z(F)$, $L_i = Z(\ell_i)$, and $F_0$ the line at infinity. Let
\begin{align*}
C \cdot L_1 &= P_1 + P_2 \\
C \cdot L_2 &= P_3 + P_4 \\
L_1 \cdot L_2 &= P_5 \\
L_1 \cdot F_0 &= P_6 \\
L_2 \cdot F_0 &= P_7 \\
C \cdot F_0 &= P_8 + P_9
\end{align*}
Then
\begin{align*}
F \cdot C &= 2F \cdot L_2 + F \cdot F_0 \\
&= 2L_2 \cdot C + C \cdot F_0 \\
&= 2P_3 + 2P_4 + P_8 + P_9 \\
F \cdot F_0 &= L_1 \cdot F_0 + C \cdot F_0 \\
&= P_6 + P_8 + P_9
\end{align*}
From this we compute the weighted path in the graph $\Gamma(F + C + F_0)$ for the symbol algebra $(f,c)_2$, with coefficients in $\mathbb{Z}/2$. The graph and edge weights are shown in Figure 2. There is one nontrivial loop, $F \rightarrow P_8 \rightarrow F_0 \rightarrow P_9 \rightarrow F$. Therefore $(f,c)_2$ is a division algebra and corresponds to a non-trivial element in $B(S/R)$.

Example 2.22. As in Example 2.2, we consider a double plane ramified over four lines. We consider the case where two of the four lines are parallel. Start with a linear polynomial $\ell \in k[x,y]$ which defines the line $L = Z(\ell)$ in $\mathbb{A}^2$. Pick a point $P$ on $L$. Let $F_1$ and $F_2$ be general lines which are parallel to $L$. Let $F_3$ and $F_4$ be general lines that intersect $L$ at $P$. Let $f_i$ be the equation for $F_i$. Let $f = f_1 f_2 f_3 f_4$, $R = k[x,y][f^{-1}]$, and $S = R[\sqrt{f}]$. Then $\mathbb{Z}/2 B(R)$ is isomorphic to $(\mathbb{Z}/2)^5$. A basis consists of the symbol algebras
\[
\{(f_1,f_3)_2,(f_1,f_4)_2,(f_2,f_3)_2,(f_2,f_4)_2,(f_3,f_4)_2\}.
\]
The group $B^{-}(S/R)$ is the subgroup of $2 \mathbb{B}(R)$ generated by $\{(f,f_1)_2,\ldots,(f,f_4)_2\}$. One computes that $B^{-}(S/R)$ is a $\mathbb{Z}/2$-module of rank 3, with a basis being

$$\{(f_1,f_3)_2,(f_1,f_4)_2,(f_2,f_3)_2,(f_2,f_4)_2,(f_3,f_3)_2,(f_3,f_4)_2\}.$$  

Let $\Lambda = (f,\ell)_2$. One computes $(f,\ell)_2 \sim (f_3,f_4,\ell)_2 \sim (f_3,f_4)_2$ which is in $\mathbb{B}(S/R)$, but not $B^{-}(S/R)$. Theorem 2.1 says $\alpha_5(\Lambda)$ represents a non-trivial element of $\text{Pic}(S) \otimes \mathbb{Z}/2$.

The case where $f_1$ and $f_2$ are parallel, and $f_3$ and $f_4$ are parallel is the subject of Example 2.23, where it is shown that $\alpha_5$ is zero. The double plane ramified over four lines passing through a common point is studied in [6], where it is shown that $\alpha_5$ is zero.

**Example 2.23.** Let $f = (x^2 - 1)(y^2 - 1) \in k[x,y]$. Set $R = k[x,y][f^{-1}]$ and $S = R[\sqrt{f}]$. Let $T = k[x,y,z]/(z^2 = f)$. As computed in [15], $\text{Cl}(T) \cong (\mathbb{Z}/2)^3$. By Proposition 2.7, $H^1(G,\text{Cl}(T)) \cong \text{Cl}(T) \otimes \mathbb{Z}/2 \cong (\mathbb{Z}/2)^3$. As shown in [7, Theorem 2.5], $H^1(G,\text{Cl}(T)) \rightarrow B(S/R)$ is onto. Using [8], one can check that $2 \mathbb{B}(R) \cong (\mathbb{Z}/2)^4$ and $B^{-}(S/R) \cong (\mathbb{Z}/2)^3$. This proves $B^{-}(S/R) = B(S/R)$ and $\text{Pic} S \otimes \mathbb{Z}/2 = (0)$. Consider the symbol algebra $(f,y-x)_2$. Check that

$$\begin{align*}
(f,y-x)_2 &\sim ((x-1)(y-1),y-x)_2((x+1)(y+1),y-x)_2 \\
&\sim (x-1,y-1)_2(x+1,y+1)_2 \\
&\sim (f,(x-1)(y+1))_2
\end{align*}$$

(22)

Upon restriction to the field $K = k(x,y)$, $(f,y-x)_2$ is a division algebra. The ideal $S(y-x)$ has two minimal primes, namely $(y-x,z-x^2+1)$ and $(y-x,z+x^2-1)$ and they are comaximal. The ring $S/(y-x)$ is a direct sum of two copies of $R/(y-x)$. The ring in Example 2.2 was a double plane ramified over four lines in general position. The ring in Example 2.22 was a double plane ramified over four lines, three of which are in general position. In both of these examples, it was shown that $\text{Pic}S \otimes \mathbb{Z}/2$ was non-trivial. By comparison, in this example $\text{Pic}S \otimes \mathbb{Z}/2$ is trivial because the four lines are not sufficiently general.
Example 2.24. Now we give an example, where, in the context of Remark 2.18, the symbol algebra \((f, g)\) is split, the ideal \(I = (g, z - h)\) represents a non-trivial class in \(\text{Pic}\, S\) and \(I\) is in \(2\, \text{Pic}\, S\). Start with the special case \(n = 3\) of [7, Example 3.3]. Let \(f_1 = 2xy - 1, \ell_1 = x - 1, \ell_2 = x + 1, f = f_1\ell_1\ell_2, R = k[x,y][f^{-1}],\) and \(S = R[z]/(z^2 - f)\). By [7, Example 3.3] we know that \(\text{Pic}\, S\) is infinite cyclic and is generated by the class of \(I_1 = (z - 1, x)\). The divisor of \(x\) is \(\text{Div}(x) = I_1 + I_2\), where \(I_2 = (z + 1, x)\). Take \(g = x^2 + y^2 - 1\). Check that
\[
\begin{align*}
f_1 &= g - (x - y)^2 \\
 f_2 &= g - y^2
\end{align*}
\]
so the symbol algebras \((f_1, g)\) and \((f_2f_3, g)\) are split. It follows that \((f_1, g)\) \((f_2f_3, g)\) \(\sim (f, g)\) is also split. Multiply on both sides of (23),
\[
f = f_1f_2f_3 = g\left(g - (x - y)^2 - y^2\right) + (x - y)^2 y^2
\]
In the notation of Proposition 2.15, \(u = 2xy - y^2 - 1\) and \(h = (x - y)y\). Let \(I = (g, z - (x - y)y)\). Then \(I\) is a height one prime of \(S\). \(\sigma(I) + I = S\), and \(I\sigma(I) = Sg\). The divisor of \(g\) is \(\text{Div}(g) = I + \sigma(I)\). Let
\[
m = g - (z - xy + y^2)
\]
\[
= x^2 + xy - 1 - z = x(x + y) - (z + 1)
\]
Note that \(m\) is in \(I\) and \(m(z + xy - y^2) = g(1 + z - xy)\). Since \(z + xy - y^2\) and \(1 + z - xy\) are not in \(I\), the valuation of \(m\) at \(I\) is one. Any prime ideal that contains \(m\) must contain \(g\) or \(1 + z - xy\). The ideal \(I\) is generated by \(m\) and \(g\). Since \(m + (1 + z - xy) = x^2\), if a prime contains \(m\) and not \(g\), then it contains \(x\). Any ideal that has both \(m\) and \(x\) also has \(z + 1 = xy + x^2 - m\). Therefore, the only minimal primes of \(m\) are \(I_1\) and \(I_2\). One checks that \((z + 1)^2 = x^2(2xy - 1) - 2(x(x + y) - (z + 1))\), from which it follows \(m \in I_2^2\). Lastly, it is straightforward to check that \((z + 1)^2 - x^2(2xy + 1)\) is not in \(I_1^3\), so the divisor of \(m\) is \(\text{Div}(m) = 2I_2 + I_1\). Therefore, \(I\) is a non-trivial element in \(2\, \text{Pic}\, (S)\).

3. Examples

Example 3.1. Let \(f = ((y - 1)^2 - x^2)y(y + 1) = f_1f_2f_3f_4\) and \(g = x^2 + y^2 - 1\), both in \(k[x,y]\). Let \(F_1 = Z(f_1)\) and \(G = Z(g)\) be the affine curves, which we embed in the projective plane \(\mathbb{P}^2\) in the usual way. The line at infinity is denoted \(H_\infty\). Figure 3 shows the curves and the points of intersection. Figure 4 shows the associated graph \(\Gamma\). Let \(R = k[x,y][f^{-1}]\). The quotient field of \(R\) is \(K = k(x,y)\).

By [8, Theorem 4], \(2\, B(R) \cong H_1(\Gamma(F_1 + F_2 + F_3 + F_4 + H_\infty), \mathbb{Z}/2) \cong (\mathbb{Z}/2)^5\). A basis for \(2\, B(R)\) consists of the Brauer classes of the symbol algebras \((f_1, f_2)_2, (f_1, f_3)_2, (f_1, f_4)_2, (f_2, f_3)_2, (f_2, f_4)_2\). We determine the weighted graph associated to the cyclic \(K\)-algebra \(\Lambda = (f, g)_2\) using the method of [9, Theorem 2.1]. The key observations are \(G, F_1 = P_1 + P_3, G, F_2 = P_2 + P_3, G, F_3 = P_1 + P_2, G, F_4 = 2P_4\). Modulo 2, the element of the cycle space is the cycle shown in Figure 5. The ramification divisor of \(\Lambda\) is \(F_1 + F_2 + F_3\). Since \(A^2 - F = \text{Spec}\, K\), the division algebra \(\Lambda\) is in the image of the natural map \(2\, B(R) \rightarrow 2\, B(K)\). Let \(A\) denote a maximal \(R\)-order in \(\Lambda\). Then \(A \otimes_R K \cong \Lambda\) and \(A\) is an Azumaya \(R\)-algebra [22, Theorem 9.9, p. 65]. Since \(A\) has order two in \(B(K)\), \(\Lambda \cong \Lambda'\). Let \(S = k[x,y][\sqrt{F^{-1}}]\). The quotient field of \(S\) is \(K(\sqrt{F})\). Then \(S/R\) is a quadratic Galois extension which we view as a subring of \(\Lambda\). We can assume \(A\) contains \(S\) as a maximal commutative subring. Therefore, \(A\) is split by \(S\).
Using [8, Theorem 4] we compute $B^{-}(S/R)$. It is the subgroup of $B(R)$ generated by the symbol algebras $(f_1, f)_2$, $(f_2, f)_2$, $(f_3, f)_2$, $(f_4, f)_2$. Using the basis for $2B(R)$ listed above, it is easy to see that this subgroup has $\mathbb{Z}/2$-rank equal to 3 and does not contain the Brauer class of $A$. It follows from Theorem 2.1 that there is associated to $\Lambda$ a non-trivial element of $\text{Pic} S$ which is unique up to an element in $2\text{Pic} S$.

From [7, sequence (9)] we have the exact sequence of $\mathbb{Z}/2$-modules

$$0 \rightarrow H^0(R, \mu_2) \xrightarrow{\partial_0} H^1(R, \mu_2) \xrightarrow{\text{res}} H^1(S, \mu_2) \xrightarrow{\text{cor}} H^1(R, \mu_2) \xrightarrow{\partial_1} B^{-}(S/R) \rightarrow 0.$$  

We know $\dim_{\mathbb{Z}/2} H^0(R, \mu_2) = 1$, $\dim_{\mathbb{Z}/2} H^1(R, \mu_2) = 4$, $\dim_{\mathbb{Z}/2} B^{-}(S/R) = 3$. Hence we calculate $\dim_{\mathbb{Z}/2} H^1(S, \mu_2) = 4$. Consider the commutative diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & R^*/R^{*2} & \xrightarrow{\cong} & H^1(R, \mu_2) & \longrightarrow & 0 \\
\text{res} & & \downarrow \text{res} & & \downarrow & & \\
0 & \longrightarrow & S^*/S^{*2} & \longrightarrow & H^1(S, \mu_2) & \longrightarrow & 2\text{Pic}(S) & \longrightarrow & 0 \\
\text{N}_R^S & & \downarrow \text{cor} & & \downarrow & & \\
1 & \longrightarrow & R^*/R^{*2} & \xrightarrow{\cong} & H^1(R, \mu_2) & \longrightarrow & 0
\end{array}
$$

The rows of (26) are the Kummer sequences (3) for $R$ and $S$, hence are exact since $\text{Pic} R = 0$. The second column of (26) is [7, sequence (9)], hence is exact. The map $N_R^S: S \rightarrow R$ is the norm and if $u$ is the element of $S$ such that $u^2 = f$, then $N_R^S(u) = -f$. Since $f$ is not
a square in \( R^* \), this shows \( N^S_R(u) \) generates the image of \( \text{cor}^1 \). It is routine to check that 
\[ \text{Pic}(S) = 0. \]
It follows that \( S^*/S^2 \cong H^1(S, \mu_2) \), and \( \dim_{\mathbb{Z}/2}(S^*/S^2) = 4. \)

Using [23] it is easy to check that
\[ (x^2 + y^2 - 1)(y^2 - y) = ((y - 1)^2 - x^2)(y^2 + y) + 2x^2y^2 \]
which is equivalent to
\[ g(y^2 - y) = f + 2x^2y^2. \]

The ring \( R = k[x,y][f^{-1}] \) is a unique factorization domain. In \( R \), the element \( g \) is irreducible. In \( S \) there is an element \( u \) such that \( u^2 = f \). In \( S = R[\sqrt{f}] \), the right-hand side of (28) factors \( (u + \sqrt{-2}xy)(u - \sqrt{-2}xy) \). Any prime ideal of \( S \) that contains \( g \) also contains \( t_1 = u + \sqrt{-2}xy \) or \( t_2 = u - \sqrt{-2}xy \). Using [23] one shows that \( I_1 = (g, t_1) \) and
Let \( I_2 = (g,t_2) \) be height one prime ideals in \( S \). Since \( u \) is invertible in \( S \), \( I_1 \) and \( I_2 \) are comaximal. This says that lying above \( G = \mathbb{Z}[g] \) on the double cover \( \text{Spec} \, S \to \text{Spec} \, R \) is a divisor \( \tilde{G} = G_1 + G_2 \) with two irreducible components, each non-principal. In the local ring \( S_{P_3}, t_2 \) is a unit and \( t_1 \) belongs to the principal ideal generated by \( g \). It follows that the divisor of \( g \) on \( \text{Spec} \, S \) is \( G_1 + G_2 \). Therefore, in \( \text{Pic} \, S \) the ideals \( I_1 \) and \( I_2 \) generate the same subgroup.

According to Theorem 2.1, there is an element \( \alpha(A) \) in \( \text{Pic} \, S \otimes \mathbb{Z}/2 \) determined by \( \Lambda \). We prove that \( \alpha(A) \) is the coset represented by the class of the divisor \( I_1 \). Consider the restrictions of \( S \) and \( A \) to the Zariski open set \( \text{Spec} \, R[g^{-1}] \). We know that \( A \otimes_R R[g^{-1}] \) contains the Azumaya \( R[g^{-1}] \)-algebra generated by \( u, v \), where \( u^2 = f, v^2 = g, \) and \( uv = -vu \). Therefore \( A \otimes_R R[g^{-1}] \) is a free \( S[g^{-1}] \)-module of rank two and is generated by \( 1 \) and \( v \). This also means \( A \otimes_R R[g^{-1}] \) is in \( B^{-1}(S[g^{-1}] / R[g^{-1}]) \). Consider the commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & B^{-1}(S/R) & \longrightarrow & B(S/R) & \longrightarrow & \text{Pic} \, S/2 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B^{-1}(S[g^{-1}] / R[g^{-1}]) & \longrightarrow & B(S[g^{-1}] / R[g^{-1}]) & \longrightarrow & \text{Pic} \, S[g^{-1}]/2 \\
\end{array}
\]

where the rows are obtained from Theorem 2.1 applied to \( R \) and \( R[g^{-1}] \), hence are exact. The center vertical map is one-to-one. By Nagata’s Theorem [10, Theorem 7.1], the map \( \gamma \) is onto and the kernel of \( \gamma \) is the subgroup generated by the divisor class of \( I_1 \). It follows that \( \alpha(A) \) is in the kernel of \( \gamma \), hence \( \alpha(A) \) is represented by the divisor \( I_1 \).

**Question 3.2.** In Example 3.1, if \( g \) is fixed, can \( f \) be extended to an arbitrary configuration of lines in \( \mathbb{A}^2 \)? In other words, can we find a large class of \( f \) such that the results of Example 3.1 hold? In this case, we would have a number of examples of double covers for which \( G \) splits.

**Example 3.3.** Let \( f, R \) and \( S \) be as in Example 3.1. Take \( G_2 = \mathbb{Z}(g_2) \) to be a conic that contains \( P_3, P_{10}, P_{11} \) and is tangential to \( F_3 \). The cycle corresponding to \( \Lambda_2 = (g_2, f_2) \) is shown in Figure 6. One can check that \( [\Lambda] / [\Lambda_2] \) is in the image of \( B^{-1}(S/R) \).

![Figure 6](image-url) **Figure 6.** The cycle corresponding to \( \Lambda = (f, g_2)_2 \) in Example 3.3.

**Example 3.4.** Let \( f, g, R \) and \( S \) be as in Example 3.1. Let \( f_3 = y - 1 \). Consider the localizations \( R[f_3^{-1}], S[f_3^{-1}], A[f_3^{-1}] \). One checks that \( (f_3 f_5, f) \sim (f_3, f_1)(f_3, f_2)(f_5, f_1)(f_5, f_2) \sim (f_3, f_1)(f_3, f_2)(f_1, f_2) \sim \Lambda \). Consider the exact sequence of Theorem 2.1 applied to the Galois extension \( R[f_3^{-1}] \to S[f_3^{-1}] \). We have shown that the Brauer class of \( A[f_3^{-1}] \) is in \( B^{-1}(S[f_3^{-1}] / R[f_3^{-1}]) \). Let the element of \( \text{Pic} \, S \otimes \mathbb{Z}/2 \) corresponding to \( \Lambda \) be denoted...
\[ \alpha(A) = [L], \text{ where } L \text{ is an invertible } S\text{-module By Theorem 2.9, } L \cong \sigma(A/S). \] The diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & B^-(S/R) & \longrightarrow & B(S/R) & \longrightarrow & \frac{\text{Pic}S}{2} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \alpha & & \downarrow & & \gamma \\
0 & \longrightarrow & B^-(S[f_5^{-1}] / R[f_5^{-1}]) & \longrightarrow & B(S[f_5^{-1}] / R[f_5^{-1}]) & \longrightarrow & \frac{\text{Pic}(S[f_5^{-1}])}{2} & \longrightarrow & 0 \\
\end{array}
\]

commutes, the rows are exact, and the center vertical arrow is one-to-one. The kernel of \( \gamma \) is generated by the primes that contain \( f_5 = y - 1 \). In the top row, the class of \( A \) maps onto \([L]\), which is mapped by \( \gamma \) to 0. This says that \( L \) is equivalent to a product of powers of primes that contain \( f_5 = y - 1 \).

### 3.1. More Questions and Examples.

**Example 3.5.** Let \( f = xy, R = k[x,y][f^{-1}], S = R[\sqrt{f}] \). Then \( S \) is rational and factorial. Let \( T = k[x,y,z] / (z^2 = xy) \). Then \( T \) is a ramified quadratic extension of \( A = k[x,y] \) and \( T = A \cdot 1 + A \cdot z \) is a free \( A \)-module. The class group of \( T \) is isomorphic to \( \mathbb{Z}/2 \) and is generated by the prime ideal \( I = Tx + Tz \). A typical element of \( I \) looks like \((a + bc)x + (c + dz)z = (a + dy)x + (e + bx)z\), where \( a, b, c, d \) come from \( A \). As an \( A \)-module, \( I \) is free with generators \( x \) and \( z \). Since \( I^2 \) is generated by \( x^2, xz, \) and \( z^2 = xy \), we see that \( I^2 \subseteq Tx \).

Define an \( A \)-algebra \( \Lambda \) which as a \( T \)-bimodule is \( T \oplus I_\sigma \). As in equation (14), define the multiplication rule on ordered pairs by

\[
(a, b)(c, d) = (ac + b\sigma(d)x^{-1}, b\sigma(c) + ad).
\]

The proof that \( \Lambda \) is an associative \( A \)-algebra follows from an argument similar to that of Example 2.8. As an \( A \)-module, \( \Lambda \) is free of rank four on the elements \((1,0), (z,0), (0,x), (0,z)\). The multiplication rules are given in Table 2. Tensoring with \( K = k(x,y) \), we see immediately that \( \Lambda \otimes_A K \) is equal to the symbol algebra \((xy,x_2)\). The ramification divisor of the algebra \((xy,x_2)\) is \( Z(xy) + H_\sigma \). In the notation of Proposition 2.7, we remark that \( H^1(G, \text{Cl}(T)) \cong \mathbb{Z}/2 \) which is isomorphic to \( 2B(R) = B(S/R) \).

**Question 3.6.** As in Example 2.2, let \( f = f_1 f_2 f_3 f_4 \in k[x,y] \) where \( f_1, f_2, f_3, f_4 \) are four linear polynomials in general position. Set \( R = k[x,y][f^{-1}] \) and \( S = R[\sqrt{f}] \). As was computed in Example 2.2, \( 2B(R) \cong (\mathbb{Z}/2)^6 \) and \( B(S/R) \) has \( \mathbb{Z}/2 \)-rank at least four. Can one compute \( B(S/R) \)?

**Question 3.7.** As in Example 3.1, let \( f = f_1 f_2 f_3 f_4 \in k[x,y] \) where \( f_1, f_2, f_3, f_4 \) are four linear polynomials such that \( f_1, f_2, f_3 \) are in general position and \( f_4 \) is parallel to \( f_3 \). Set \( R = k[x,y][f^{-1}] \) and \( S = R[\sqrt{f}] \). As was computed in Example 3.1, \( 2B(R) \cong (\mathbb{Z}/2)^5 \) and \( B(S/R) \) has \( \mathbb{Z}/2 \)-rank at least four. Can one compute these groups: \( B(S/R), \text{Pic}S \otimes \mathbb{Z}/2 \)?
3.2. The surface \( z^2 = y(y - p(x)) \). The notation in this section agrees with that of Section 2. In this section we study the divisor classes and algebra classes on the surface defined by \( z^2 = y(y - p(x)) \), where \( p(x) \in k[x] \) is a monic polynomial of degree \( d > 1 \).

Let \( f_1 = y, f_2 = y - p(x) \), and \( f = f_1 f_2 \). Let \( A = k[x,y], R = A[ f^{-1}], T = A[z]/(z^2 - f) \) and \( S = T[z^{-1}] \). In \( A \) let \( p(x) = \ell_1^{e_1} \cdots \ell_v^{e_v} \) be the unique factorization into irreducibles. Let \( \alpha_1, \ldots, \alpha_v \) be the distinct roots of \( p(x) \). Let \( F_i = \mathbb{Z}(f_i) \) which we embed into \( \mathbb{P}^2 \) in the usual way. Let \( F_0 \) be the line at infinity. Then \( F_1 \cdot F_2 = e_1 P_1 + \cdots + e_v P_v, F_1 \cdot F_0 = P_{01} \) and \( F_2 \cdot F_0 = dP_{02} \). The graph \( \Gamma \) of the curve \( F = F_0 + F_1 + F_2 \) is seen in Figure 7. If \( i > 1 \), the node \( P_i \) and its edges exist only if \( v \geq i \). This explains why the edges to \( P_2, \ldots, P_v \) are dashed.

We will compute the following, where \( D = \gcd(e_1, \ldots, e_v) \).

\[
\begin{align*}
&\text{Figure 7. The graph in Example 3.2.} \\
&\text{(a) } B(R) \cong (\mathbb{Q}/\mathbb{Z})^{(v)} \\
&\text{(b) } B^{-}(S/R) \cong 0 \text{ if } 2 \mid D, \text{ otherwise } \mathbb{Z}/2 \\
&\text{(c) } B(S/R) = 2B(R) \cong (\mathbb{Z}/2)^{(v)} \\
&\text{(d) } \text{Cl}(T) \cong \mathbb{Z}/(2D) \oplus \mathbb{Z}^{(v-1)} \text{ and } \text{H}^1(G, \text{Cl}(T)) \cong (\mathbb{Z}/2)^{(v)} \\
&\text{(e) } \text{Pic}(S) \cong \mathbb{Z}/D \oplus \mathbb{Z}^{(v-1)} \text{ and } \text{H}^1(G, \text{Cl}(S)) \cong (\mathbb{Z}/2)^{(v)} \text{ if } 2 \mid D, \text{ otherwise } (\mathbb{Z}/2)^{(v-1)} \\
&\text{(f) } B(S) \cong (\mathbb{Q}/\mathbb{Z})^{(v)} 
\end{align*}
\]

Using [8, Theorem 4], \( B(R) \cong (\mathbb{Q}/\mathbb{Z})^{(v)} \), which is (a). The group \( B^{-}(S/R) \) is generated by the symbol \((f, f_1)_2 \sim (f_2, f_1)_2 \sim (f, f_2)_2 \), hence is cyclic. The weighted element of the edge space is computed as in \([9, \S 2]\) and is shown in Figure 8. Therefore we get (b):

\[
B^{-}(S/R) = \begin{cases} 0 & \text{if } 2 \mid e_i \text{ for all } i \\ \mathbb{Z}/2 & \text{otherwise.} \end{cases}
\]

This and [7, sequence (9)] imply

\[
H^1(S, \mu_2) \cong \begin{cases} (\mathbb{Z}/2)^{(3)} & \text{if } 2 \mid e_i \text{ for all } i \\ (\mathbb{Z}/2)^{(2)} & \text{otherwise.} \end{cases}
\]

Let \( L_1 = Z(\ell_1) \). Then \( L_1 \cdot F_0 = P_{02}, L_1 \cdot F_1 = P_1, \) and \( L_1 \cdot F_2 = P_2 + (d-1)P_{02} \). Using the method of [8, Theorem 4], the weighted path associated to the symbol algebra \((f y^{-2}, \ell_i)_m\) is computed to be \( F_2 \to P_{02} \to F_0 \to P_{01} \to F_1 \to P_1 \to F_2 \). For \( i = 1, \ldots, v \), these cycles

\[
\begin{align*}
&\text{(a) } B(R) \cong (\mathbb{Q}/\mathbb{Z})^{(v)} \\
&\text{(b) } B^{-}(S/R) \cong 0 \text{ if } 2 \mid D, \text{ otherwise } \mathbb{Z}/2 \\
&\text{(c) } B(S/R) = 2B(R) \cong (\mathbb{Z}/2)^{(v)} \\
&\text{(d) } \text{Cl}(T) \cong \mathbb{Z}/(2D) \oplus \mathbb{Z}^{(v-1)} \text{ and } \text{H}^1(G, \text{Cl}(T)) \cong (\mathbb{Z}/2)^{(v)} \\
&\text{(e) } \text{Pic}(S) \cong \mathbb{Z}/D \oplus \mathbb{Z}^{(v-1)} \text{ and } \text{H}^1(G, \text{Cl}(S)) \cong (\mathbb{Z}/2)^{(v)} \text{ if } 2 \mid D, \text{ otherwise } (\mathbb{Z}/2)^{(v-1)} \\
&\text{(f) } B(S) \cong (\mathbb{Q}/\mathbb{Z})^{(v)} 
\end{align*}
\]
make up a basis for $H_1(\Gamma, \mathbb{Z}/m)$. Therefore, a basis for $mB(R)$ consists of the classes of the algebras

\[(f y^{-2}, \ell_1)_m, \ldots, (f y^{-2}, \ell_v)_m.\]

This shows that a basis of $B(R)$ consists of $(f, \ell_1)_2, \ldots, (f, \ell_v)_2$, all of which are in $B(S/R)$, which proves (c). By [7, Theorem 2.1], this shows

\[\text{Pic } S \otimes \mathbb{Z}/2 \cong \begin{cases} (\mathbb{Z}/2)^{(v)} & \text{if } 2 | e_i \text{ for all } i \\ (\mathbb{Z}/2)^{(v-1)} & \text{otherwise.} \end{cases}\]

Consider the homomorphism of $k$-algebras

\[T = \frac{k[x, y, z]}{(z^2 - y(y - p(x)))} \xrightarrow{\beta} U = k[x, w][(1 - w^2)^{-1}] \]

defined by $x \mapsto x$, $y \mapsto p(x)/(1 - w^2)$, $z \mapsto wp(x)/(1 - w^2)$. One checks that $\beta$ is well defined and becomes an isomorphism upon adjoining $1/p(x), 1/z$ to $T$ and $1/p(x), 1/w$ to $U$. Both rings in (35) are integral domains of Krull dimension two, hence $\beta$ is one-to-one. Since $U$ is rational, so is $T$. We have

\[k[x, w][p(x)^{-1}, w^{-1}, (1 - w^2)^{-1}]^* = k^v \times \langle w \rangle \times \langle 1 - w \rangle \times \langle 1 + w \rangle \times \prod_{i=1}^{v} \langle \ell_i \rangle.\]

Using $\beta$ we find $zy^{-1} \mapsto w$, $(z - y)y^{-1} \mapsto w - 1$, $(z + y)y^{-1} \mapsto w + 1$, hence

\[T[p(x)^{-1}, z^{-1}]^* = k^v \times \langle zy^{-1} \rangle \times \langle (z - y)y^{-1} \rangle \times \langle (z + y)y^{-1} \rangle \times \prod_{i=1}^{v} \langle \ell_i \rangle.\]
Since $U$ is factorial, Nagata’s Theorem says the class group of $T$ is generated by the minimal primes of $z, z^2 - y^2, y,$ and $\ell_1, \ldots, \ell_v$. It is routine to verify that

$$
\begin{align*}
\text{Div}(z) &= (z,y) + (z,y - p(x)) \\
\text{Div}(z-y) &= (z,y) + e_1(z-y, \ell_1) + \cdots + e_v(z-y, \ell_v) \\
\text{Div}(z+y) &= (z,y) + e_1(z+y, \ell_1) + \cdots + e_v(z+y, \ell_v) \\
\text{Div}(y) &= 2(z,y) \\
\text{Div}(\ell_i) &= (z-y, \ell_i) + (z+y, \ell_i) \\
\text{Div}(y - p(x)) &= 2(z,y - p(x)).
\end{align*}
$$

(38)

The class group $\text{Cl}(T)$ is generated by the $2v + 2$ prime divisors

$$
(z,y), (z,y - p(x)), (z-y, \ell_1), \ldots, (z-y, \ell_v), (z+y, \ell_1), \ldots, (z+y, \ell_v).
$$

(39)

Using the principal divisors $\text{Div}(\ell_i) = (z-y, \ell_i) + (z+y, \ell_i) \sim 0$ and $\text{Div}(z) = (z,y) + (z,y - p(x)) \sim 0$, we can eliminate half of the generators and all but two of the relations. The group $\text{Cl}(T)$ is generated by the $v + 1$ divisors $(z,y), (z-y, \ell_1), \ldots, (z-y, \ell_v)$ modulo the two principal divisors $2(z,y), (z,y) + e_1(z-y, \ell_1) + \cdots + e_v(z-y, \ell_v)$. If $D = \gcd(e_1, \ldots, e_v)$, then

$$
\text{Cl}(T) \cong \mathbb{Z}/(2D) \oplus \mathbb{Z}^{(v-1)}
$$

(40)

$$
\text{H}^1(G, \text{Cl}(T)) \cong (\mathbb{Z}/2)^{(v)}
$$

which is (d). Note that (40) together with part (c) agree with the conclusion of [7, Theorem 2.6]. The kernel of $\text{Cl}(T) \to \text{Cl}(S)$ is generated by the divisor $(z,y)$. So $\text{Cl}(S)$ is generated by the $v$ divisors $(z-y, \ell_1), \ldots, (z-y, \ell_v)$ modulo the principal divisor $e_1(z-y, \ell_1) + \cdots + e_v(z-y, \ell_v)$. If $D = \gcd(e_1, \ldots, e_v)$, then

$$
\text{Cl}(S) \cong \mathbb{Z}/D \oplus \mathbb{Z}^{(v-1)}
$$

(41)

$$
\text{H}^1(G, \text{Cl}(S)) \cong \begin{cases} 
(\mathbb{Z}/2)^{(v)} & \text{if } 2 \mid e_i \text{ for all } i \\
(\mathbb{Z}/2)^{(v-1)} & \text{otherwise},
\end{cases}
$$

proving (d). Note that (41) agrees with (34). Using (37), we see that $z, y, z-y, \ell_1, \ldots, \ell_v$ make up a basis for $S[p(x)^{-1}]^* / k^*$. The elements $z$ and $y$ are units of $S$. The minimal primes of $z-y, \ell_1, \ldots, \ell_v$ in $S$ are $(z-y, \ell_1), \ldots, (z-y, \ell_v), (z+y, \ell_1), \ldots, (z+y, \ell_v)$. In the Nagata sequence

$$
1 \to S^* \to S[p(x)^{-1}]^* \xrightarrow{\text{Div}} \bigoplus_{i=1}^v (\mathbb{Z}(z-y, \ell_i) \oplus \mathbb{Z}(z+y, \ell_i))
$$

(42)

the elements $\text{Div}(z-y), \text{Div}(\ell_1), \ldots, \text{Div}(\ell_v)$ generate a free $\mathbb{Z}$-module of rank $v + 1$. This proves $S^* = k^* \times (z) \times \langle y \rangle$. Using Theorem 2.5 and [21, Theorem 10.35],

$$
\text{H}^i(G, S^*) = \begin{cases} 
R = k^* \times \langle y \rangle \times \langle y - p(x) \rangle & \text{if } i = 0 \\
\langle 1 \rangle & \text{if } i = 1, 3, \ldots \\
\langle y \rangle / \langle y^2 \rangle & \text{if } i = 2, 4, \ldots
\end{cases}
$$

From the Nagata sequence

$$
1 \to T^* \to S^* \xrightarrow{\text{Div}} \mathbb{Z}F_1 \oplus \mathbb{Z}F_2
$$


we know that \( S^*/T^* \) is free of rank two. It follows that \( T^* = k^* \). Consider the isomorphism
\[
B(S[p(x)^{-1}]) \xrightarrow{\beta} B(k[x,w][p(x)^{-1}, w^{-1}, (1 - w^2)^{-1}])
\]
induced by the map \( \beta \) of (35). Using [8, Theorem 4], compute the Brauer group on the right hand side of (43). It is isomorphic to \((\mathbb{Q}/\mathbb{Z})^{3v}\), and a basis for the subgroup annihilated by \( m \) is made up of the \( 3v \) symbol algebras \((w, \ell_m), (w - 1, \ell_m), (w + 1, \ell_m)\) for \( i = 1, \ldots, v \). Using \( \beta \), it follows that the symbol algebras
\[
(z^y, \ell_m), ((z - y)y^{-1}, \ell_m), ((z + y)z^{-1}, \ell_m)
\]
for \( i = 1, \ldots, v \), make up a basis for the subgroup \( mB(S[p(x)^{-1}]) \). There is an exact sequence [9, Corollary 1.4]
\[
0 \to B(S) \to B(S[p(x)^{-1}]) \to H^1(S/(p(x)), \mu) \to 0
\]
The ring \( S/(p(x)) \) is the disjoint union of \( 2v \) copies of the algebraic torus \( k[z, z^{-1}] \). Therefore, \( H^1(S/(p(x)), \mu) \cong (\mathbb{Q}/\mathbb{Z})^{2v} \). Look at the component of \( S/(p(x)) \) corresponding to the minimal prime \( I_i = (z - y, \ell_i) \). The residue field at \( I_i \) is isomorphic to the quotient field of \( S/I_i \cong k[y, y^{-1}] \) which we identify with \( k(y) \). In the local ring \( S/I_i \), the valuations are \( v(z) = 0, v(y) = 0, v(\ell_i) = 1, v(z - y) = e_i \). For the algebras in (44), the ramification map \( a \) in (5) agrees with the tame symbol (6). For \( ((z - y)y^{-1}, \ell_i) \), the tame symbol is \( y^{-1} \), which gives rise to an element of order \( m \) in \( H^1(k(y), \mathbb{Z}/m) \). This is the only algebra in the list (44) which is ramified at \( I_i \). Similarly, the only algebra in the list (44) which ramifies at the prime \( (z + y, \ell_i) \) is \( ((z + y)y^{-1}, \ell_i) \). It follows that in sequence (45), the group \( mB(S[p(x)^{-1}]) \) maps onto \( H^1((S/(p(x)), I_m) \) and a basis for \( mB(S) \) consists of \( (z^y, \ell_i) \) for \( i = 1, \ldots, v \). This shows \( B(S) \cong (\mathbb{Q}/\mathbb{Z})^{10v} \), which is (f). By (33)
\[
(z^y, \ell_i)_m \sim (z^y, \ell_i)_{2m} \sim (z^y, \ell_i)_{2m} \sim (zy^{-2}, \ell_i)_{2m}
\]
is in the image of \( B(R) \). Therefore, the sequence
\[
0 \to B(R) \to B(R) \to B(S) \to 0
\]
is exact. As a homomorphism of abstract groups, the natural map \( B(R) \to B(S) \) is “multiplication by 2”.

3.3. Interpretation of Rim’s Sequence. In [20], Rim constructed an exact sequence involving some of the groups which we are studying in this article. In our context \( R \) is factorial so \( \text{Cl}(R) = 0 \) and Rim’s exact sequence is
\[
0 \xrightarrow{\beta} H^1(G, T^*) \xrightarrow{\gamma} \text{Div}(T)^G / \text{Div}(R) \xrightarrow{\gamma} \text{Cl}(T)^G \xrightarrow{\delta} H^2(G, T^*) \xrightarrow{\simeq} \bigcap_{P \in X_t(R)} \text{Im}(H^2(G, T \otimes_R P^*) \to H^2(G, L^*)) \xrightarrow{\delta} H^1(G, \text{Cl}(T)) \xrightarrow{\delta} H^3(G, T^*)
\]
where the intersection is over \( X_t(R) \), the set of height one primes in \( \text{Spec}R \). Denote by \( M \) the mysterious term defined by this intersection. It would be interesting to have an interpretation this sequence, at least in the context of the affine double plane with coordinate ring \( T \). If we assume the group of units of \( T \) is equal to \( k^* \), then \( H^2(G, T^*) = 0 \), so (47) breaks into two exact sequences
\[
0 \to H^1(G, T^*) \xrightarrow{\beta} \text{Div}(T)^G / \text{Div}(R) \xrightarrow{\beta} \text{Cl}(T)^G \to 0
\]
\[
0 \to M \xrightarrow{\beta} B(S/R) \xrightarrow{\delta} H^1(G, T^*)
\]
If we assume moreover that $H^1(G, \text{Cl}(S)) = 0$, then $M = H^1(G, \text{Cl}(T)) \cong B(S/R)$, by diagram (13).

**REFERENCES**


