A Partial Order for Extraordinary Subsets

Ralph P. Grimaldi *, John H. Rickert, Rose-Hulman Institute of Technology

For \( n \geq 1 \), let \([n] = \{1, 2, 3, \ldots, n\}\). A subset \( S \) of \([n]\) is called \textit{extraordinary} if the minimal element in \( S \), denoted \( \text{min}_S \), is equal to \( |S| \), the size of \( S \). For example, there are five extraordinary subsets of \([5]\) - namely, \(\{1\}\), \(\{2, 3\}\), \(\{2, 4\}\), \(\{2, 5\}\), \(\{3, 4, 5\}\). In general, for \( n \geq 1 \), there are \( F_n \) extraordinary subsets of \([n]\), where \( F_n \) denotes the \( n \)th Fibonacci number.

For a given \( n \geq 1 \), define the relation \( R \) on the collection of all extraordinary subsets of \([n]\) as follows: If \( A, B \) are extraordinary subsets of \([n]\), we have \( A R B \) when \( A - \{\text{min}_A\} \subseteq B - \{\text{min}_B\} \). This relation provides a partial order for the extraordinary subsets of \([n]\), where one can count results such as (1) the number of edges in the Hasse diagram of the partial order, and (2) the number of maximal elements for the partial order - both results involving the Fibonacci numbers.

Keywords: Fibonacci numbers, extraordinary subsets, partial order, Hasse diagram, maximal elements