Synchronization and graph endomorphisms
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A finite deterministic automaton consists of a finite set of states and a set of transitions (maps on the set of states); we are allowed to compose transitions. It can be represented by a coloured digraph (with a unique edge of each colour leaving each state) or as a semigroup of mappings on the set of states with a prescribed set of generators. An automaton is synchronizing if there is an element in the semigroup with rank 1 (mapping every state to the same place). Such an element, regarded as a word in the basic transitions, is called a reset word.

The Černý conjecture, still open after more than 40 years, asserts that if an $n$-state automaton is synchronizing, then it has a reset word of length at most $(n - 1)^2$. Though we can check in polynomial time whether an automaton is synchronizing, finding the length of the shortest reset word is NP-hard.

There is a very close connection with graph endomorphisms. It is easy to see that a graph has an endomorphism onto a complete subgraph if and only if its clique number and chromatic number are equal. Now a finite transformation semigroup is non-synchronizing if and only if it is contained in the endomorphism semigroup of a non-null graph with clique number and chromatic number equal.

There has been a lot of interest in automata whose transitions include only one non-permutation, and in the permutation groups the other transitions generate. I will present several conjectures and some recent progress.