Zeros of Interval Polynomials

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Abstract

Polynomial equations with perturbed coefficients arise in several areas of engineering sciences, for instance, in automatic control theory, dynamical systems, optimization and in control theory. For such equations, it is necessary to study their roots and to establish a priori estimates to define regions containing such roots. Attending to the fact that these equations can be seen as algebraic interval equations - equations defined by interval polynomials - the computation of the roots can be made, in certain cases, using the interval arithmetic. In this paper, we study the zeros of interval polynomials. We develop a method to compute all zeros of such polynomial with interval coefficients and give the characterization of the roots.

Key words. interval analysis, interval polynomials, interval root, boundary real polynomials.

Classification. 65G40.

1 Introduction

Polynomial equations with perturbed coefficients arise in several areas of engineering sciences, for instance, in automatic control theory [16], dynamical systems [13], optimization [14] and in control theory [15]. For such equations it is necessary to study their roots and to establish a priori estimates to define regions containing such roots [5, 6, 7, 8].

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Attending to the fact that these equations can be seen as algebraic interval equations - equations defined by interval polynomials - the computation of the roots can be made, in certain cases [3, 4], using the interval arithmetic [1, 9, 10]. In [3], several a priori estimates for the zeros of interval polynomials of degree two were established.

A recurring problem in interval computations is finding the real roots of polynomial with interval coefficients. In this paper, we give simple procedure for finding real roots of interval polynomials, which are defined as follows: Let the interval of the polynomial be expressed as \( P(x) = [\underline{p}(x), \bar{p}(x)] \). Given a point \( x \) for which
\[
\underline{p}(x) \leq 0 \leq \bar{p}(x),
\]
let \( \mathcal{I} \) be the largest interval containing values of \( x \) such that every point in \( \mathcal{I} \) satisfies (1). We call \( \mathcal{I} \) an interval root of \( P(x) \).

In the present paper, we give a simple procedure for finding the interval root of quadratic interval polynomials. The characterization of the roots of the interval polynomials is obtained as a function of their interval coefficients adequately defining certain polynomials with real coefficients.

The paper is organized as follows. In Section 2, we give a review of interval arithmetics. In Section 3, we introduce basic definitions of interval polynomials and their roots. In Section 4, we develop an algorithm to compute interval roots of interval polynomials. Finally, we provide numerical results in Section 5 to demonstrate the effectiveness of the method.

## 2 Interval Arithmetic

In this section, a minimum knowledge of interval arithmetic is imported. A through introduction to the whole area of interval arithmetic can be found in [2, 11, 12].

Let \( [a] = [a, \bar{a}], [b] = [b, \bar{b}] \) be real compact intervals and \( \circ \) one of the basic operations ‘addition’, ‘subtraction’, ‘multiplication’ and ‘division’, respectively, for real numbers, that is \( \circ \in \{+,-,\cdot,\div\} \). Then we define the corresponding operations for intervals \([a]\) and \([b]\) by
\[
[a] \circ [b] = \{a \circ b | a \in [a], b \in [b]\},
\]
where we assume \( 0 \notin [b] \) in case of division.

It is easy to prove that the set \( I(\mathbb{R}) \) of real compact intervals is closed with
respect to these operations. What is even more important is the fact that 
\([a] \circ [b]\) can be represented by using only the bounds of \([a]\) and \([b]\). The following roles hold:

\[
\begin{align*}
[a] + [b] &= [a + b, a + b], \\
[a] - [b] &= [a - b, a - b], \\
[a] \cdot [b] &= \left[ \min\{ab, \bar{a}b, \bar{a} \bar{b}, \bar{a} \bar{b}\}, \max\{ab, \bar{a}b, \bar{a} \bar{b}, \bar{a} \bar{b}\} \right], \\
[a] ÷ [b] &= [a, \bar{a}][1/\bar{b}, 1/b], \quad 0 \notin [b].
\end{align*}
\]

These rules show that subtraction and division in \(I\) are not the inverse operations of addition and multiplication, respectively, as in the case of \(\mathbb{R}\). For example, 
\([0, 1] - [0, 1] = [-1, 1], [1, 2]/[1, 2] = [1/2, 2]\). This property is one of the main differences between interval arithmetic and real arithmetic. Another main difference is given by the so called subdistributive law,

\([a](b + [c]) \subseteq [a][b] + [a][c].\)

The simple example \([-1, 1](1 + (-1)) = 0 \subset [-1, 1].1 + [-1, 1].(-1) = [-2, 2]\) illustrates this property. The topological properties, and algebraic structure of interval analysis were developed in [2, 11].

## 3 Basic definitions

In this section, we defined quadratic interval polynomials, their roots and their graphs.

**Definition 3.1.** A quadratic interval polynomial is defined by

\[
P(x) = Ax^2 + Bx + C,
\]

where \(A = [a, \bar{a}], B = [b, \bar{b}],\) and \(C = [c, \bar{c}]\) are intervals.

Attending to (2), it is easy to see that \(P(x)\) is a family of polynomials

\[p(x) = ax^2 + bx + c,\]

where \(a \in A, b \in B,\) and \(c \in C\).

Using the definition of the graph of a real function, we introduce the graph of a real interval polynomial.
Definition 3.2. Let $P(x)$ be a quadratic interval polynomial. The graph of $P(x)$ is denoted by $G(P)$ and is given by

$$G(P) = \{(\hat{x}, \hat{y}) \in \mathbb{R}^2 : \exists p(x) \in P(x), \hat{y} = p(\hat{x})\}.$$  

In the next lemma we characterize the graph of a real interval polynomial $P(x)$ using the graph of certain real polynomial called boundary real polynomials. In order to do that, denote

$$l^+(x) = ax^2 + bx + c,$$
$$u^+(x) = \bar{a}x^2 + \bar{b}x + \bar{c},$$
$$l^-(x) = ax^2 + \bar{b}x + c,$$
$$u^-(x) = \bar{a}x^2 + bx + \bar{c}.$$  

Lemma 3.1. Let $P(x)$ be the real interval polynomial given by (2). The graph of $P$ is given by

$$G(P) = \{(x, y) \in \mathbb{R}^2 : l^+(x) \leq y \leq u^+(x) \text{ if } x \geq 0$$
$$\text{and } l^-(x) \leq y \leq u^-(x) \text{ if } x \leq 0\}.$$  

Proof. The proof is straightforward. \qed

Definition 3.3. Let $P(x)$ be a real interval polynomial. The interval root of $P(x)$ is denoted by $\Omega(P)$ and is given by

$$\Omega(P) = \{\bar{x} \in \mathbb{R} : \exists p(x) \in P(x), p(\bar{x}) = 0\}.$$  

In this paper, our aim is to characterize the interval root of $P(x)$. Similar to a non-interval quadratic, there may be no such interval root, or there may be only one (multiple root), or there may be two such disjoint interval roots. The interval case differs than the non-interval case in that there might be three such disjoint interval roots. However, in the latter case, one interval root extends to $-\infty$ and another extends to $+\infty$. Thus we can think of these interval roots as joined at projective infinity to form a single interval.
4 Zeros of Interval Polynomials

In this section our aim is to find zeros intervals for the polynomial $P(x)$. We will assume that $\bar{a} \neq 0$, then we have to study two cases, the first case is $\bar{a} > 0$, and the second is $\bar{a} < 0$. Subsequent work will deal with finding the roots of quadratic polynomial in the interval $[0, \infty)$.

In the first case where we have $\bar{a} > 0$, there are two subcases. We will consider the subcase where $\bar{a}, a > 0$, if both are negative then we need only to change the sign of $P(x)$.

The following theorem consider all the solutions of (2) on the interval $[0, \infty)$. We denote the interval roots by $\Omega^+$. If $l^+$ has two different real roots then we denote them by $x^+_l(1) < x^+_l(2)$, and if $u^+$ has two different real roots then we denote them by $x^+_u(1) < x^+_u(2)$.

**Theorem 4.1.** Let 
\[ P(x) = Ax^2 + Bx + C, \]
where $A = [a, \bar{a}]$, $B = [b, \bar{b}]$, and $C = [c, \bar{c}]$ are intervals, and $a > 0$, then we have the following cases:

1. If $b^2 < 4ac$, then $\Omega^+ = \phi$.
2. If $b^2 = 4ac$, then $\Omega^+ = \{-b/2a\}$.
3. If $b^2 > 4ac$, and $\bar{b}^2 \geq 4\bar{a}\bar{c}$, then $\Omega^+ = [x^+_l(1), x^+_l(2)] \cap [0, \infty)$.
4. If $b^2 > 4ac$, and $\bar{b}^2 > 4\bar{a}\bar{c}$, then 
\[ \Omega^+ = \left([x^+_l(1), x^+_u(1)] \cup [x^+_u(2), x^+_l(2)]\right) \cap [0, \infty). \]

**Proof.**

1. If $b^2 < 4ac$, then $l^+(x) > 0$ for all $x \in [0, \infty)$. Therefore $P(x) > 0$ for all $x \in [0, \infty)$.

2. Since $a > 0$, this implies that $c \geq 0$, therefore, if $b > 0$, then by Descartes' Rule, $l^+(x)$ can not have any positive root, thus $b \leq 0$. In this case, $-b/2a \geq 0$ is the only solution.

3. If $b^2 > 4ac$, and $\bar{b}^2 \leq 4\bar{a}\bar{c}$, then the graph of $u^+(x)$ does not cross the $x-$axis, therefore the solution is the nonnegative interval root of $l^+(x)$, i.e., $\Omega^+ = [x^+_l(1), x^+_l(2)] \cap [0, \infty)$. 

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4. If $b^2 > 4ac$, and $\bar{b}^2 > 4\bar{a}\bar{c}$, then the solution is the nonnegative interval root $[x_l^+(1), x_l^+(2)] \setminus (x_u^+(1), x_u^+(2))$, i.e.,

$$
\Omega^+ = ([x_l^+(1), x_u^+(1)] \cup [x_u^+(2), x_l^+(2)]) \cap [0, \infty).
$$

In the previous theorem, we found the interval roots on the interval $[0, \infty)$. Similar rules can be applied to find interval roots on the interval $(-\infty, 0]$, or we can consider the functions $l^+(-x)$ and $u^+(-x)$ and study their roots on the interval $[0, \infty)$.

Now we will consider the second case where $\bar{a} \bar{a} < 0$. The following Theorem shows how to compute interval roots on the interval $[0, \infty)$.

**Theorem 4.2.** Let

$$
P(x) = Ax^2 + Bx + C,
$$

where $A = [\underline{a}, \bar{a}]$, $B = [\underline{b}, \bar{b}]$, and $C = [\underline{c}, \bar{c}]$ are intervals, and $\underline{a} < 0 < \bar{a}$, then we have the following cases:

1. If $\bar{b}^2 > 4\underline{a}\bar{c} > 0$ and $\bar{b} \geq 0$, then $\Omega^+ = [r, x_l^+(2)] \cup [x_l^+(1), \infty)$, where

$$
r = \begin{cases}
0 & \text{if } \bar{c} \geq 0 \\
 x_l^+(2) & \text{if } \bar{c} < 0
\end{cases}
$$

2. If $\bar{c} > 0$ then $\Omega^+ = [x_l^+(2), \infty)$ if $\bar{b}^2 \leq 4\bar{a}\bar{c}$, and

$$
\Omega^+ = [x_l^+(2), \infty) \setminus (x_u^+(1), x_u^+(2)),
$$

otherwise.

3. If $\bar{b}^2 \leq 4\underline{a}\bar{c}$, then $\Omega^+ = [0, \infty)$ if $\bar{b}^2 \leq 4\underline{a}\bar{c}$, and

$$
\Omega^+ = [0, \infty) \setminus (x_u^+(1), x_u^+(2)),
$$

otherwise.

**Proof.**

1. If $\bar{b}^2 > 4\underline{a}\bar{c} > 0$, and $\bar{b} \geq 0$, then $\bar{c} \leq 0$ and $l^+(x)$ has two positive solutions. Now if $\bar{c} \geq 0$, then $u^+(x)$ does not have any real roots and $\Omega^+ = [0, x_l^+(1)] \cup [x_l^+(2), \infty)$. If $\bar{c} \leq 0$, then $u^+(x)$ has one positive real root and therefore, $\Omega^+ = [x_l^+(2), x_l^+(1)] \cup [x_l^+(2), \infty)$.
2. If $c > 0$, then $b^2 > 0 > 4ac$; therefore, $l^+(x)$ has one positive root, namely, $x_l^+(2)$. Now if $b^2 \leq 4ac$, then the graph of the function $u^+(x)$ does not cross the $x$–axis; therefore, the solution is $\Omega^+ = [x_l^+(2), \infty)$. On the other hand if $b^2 > 4ac$, then $u^+(x)$ has two roots and the solution is $\Omega^+ = [x_u^+(2), \infty) \setminus (x_u^+(1), x_u^+(2))$.

3. If $b^2 \leq 4ac$, then the graph of $l^+(x)$ does not cross the $x$–axis. Now, if $b^2 \leq 4ac$, then the graph of the function $u^+(x)$ does not cross the $x$–axis; therefore, the solution is $\Omega^+ = [0, \infty)$. On the other hand if $b^2 > 4ac$, then $u^+(x)$ has two roots and the solution is $\Omega^+ = [0, \infty) \setminus (x_u^+(1), x_u^+(2))$. 

\[ \Box \]

5 Numerical Examples

Example 5.1. Consider the interval polynomial

$$P(x) = [2, 3]x^2 + [-4, 6]x + [3, 4].$$

Computing the boundary polynomials we can see that the graph of $P(x)$ is

$$G(P) = \{ (x, y) \in \mathbb{R}^2 : l^+(x) \leq y \leq u^+(x), \text{ if } x \geq 0, \text{ and } l^-(x) \leq y \leq u^-(x), \text{ if } x \leq 0 \}$$

where,

$$l^+(x) = 2x^2 - 4x + 3, \quad u^+(x) = 3x^2 + 6x + 4,$$

$$l^-(x) = 2x^2 + 6x + 3, \quad u^-(x) = 3x^2 - 4x + 4.$$

In Figures 1 and 2, we plot the boundary polynomials of $P(x)$ and the function $P(x)$, respectively. In order to find the interval roots of $P(x)$ ($\Omega(P)$), we can see that the function does not have any roots in the interval $[0, \infty)$. On the other hand, $l^-(x)$ has two real zeros $(3 \pm \sqrt{3})/2$ and $u^-(x)$ does not have any zeros; therefore, the interval root on the interval $(-\infty, 0]$ is $[(3 - \sqrt{3})/2, (3 + \sqrt{3})/2] \approx [-2.366, -0.634]$. 

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Example 5.2. Consider the interval polynomial
\[ P(x) = [-2,1]x^2 + [-4,-2]x + [1,2]. \]
Computing the boundary polynomials we can see that the graph of \( P(x) \) is
\[ G(P) = \{(x,y) \in \mathbb{R}^2 : \ l^+(x) \leq y \leq u^+(x), \ if \ x \geq 0, \]
\[ and \ l^-(x) \leq y \leq u^-(x), \ if \ x \leq 0 \} \]
where,
\[ l^+(x) = -2x^2 - 4x + 1, \quad u^+(x) = x^2 - 2x + 2, \]
\( l^-(x) = -2x^2 - 2x + 1, \quad u^-(x) = x^2 - 4x + 2. \)

In Figures 3 and 4, we plot the boundary polynomials of \( P(x) \) and the function \( P(x) \), respectively. In order to find the interval roots of \( P(x) \), we can see that the function \( u^+(x) \) does not have any roots in the interval \([0, \infty)\), and \( l^+(x) \) has only one positive root, namely, \(-1 + \sqrt{3}/2\); therefore, the interval root in the interval \([0, \infty)\) is \([-1 + \sqrt{3}/2, \infty)\). On the other hand, \( u^-(x) \) does not have positive zeros and \( l^-(x) \) has one positive zero \((1 + \sqrt{3})/2\); therefore, the interval root on the interval \((-\infty, 0]\) is \([-\infty, (1 + \sqrt{3})/2]\).

![Figure 3: Boundary polynomials for \( P(x) \) in Example 5.2](image3.png)

![Figure 4: The Graph of \( P(x) \) in Example 5.2](image4.png)
References


