Fractional Isomorphisms of Cayley Graphs

Robert R. Rubalcaba∗, University of San Diego

Two graphs $G$ and $H$, with adjacency matrices $A$ and $B$ respectively, are isomorphic if and only if there exists a permutation matrix $P$ satisfying $P^{-1}AP = B$ (or equivalently $AP = PB$). Permutation matrices are square matrices with entries from $\{0, 1\}$ such that each row sums to 1 and each column sums to 1. If you relax the condition that entries come from the set $\{0, 1\}$, instead requiring only non-negative entries (and unit row and column sums), then the matrix is doubly stochastic. Two graphs $G$ and $H$, with adjacency matrices $A$ and $B$ respectively, are fractionally isomorphic if and only if there exists a doubly stochastic matrix $S$ satisfying $AS = SB$.

A mixed graph is a graph with directed and undirected edges. The definition of fractional isomorphism does not extend to directed or mixed graphs, since adjacency matrices fail to commute if the graph contains at least one directed edge from $x$ to $y$ (without the reverse directed edge from $y$ to $x$). To get an equivalence relation on mixed and undirected graphs, we modify the definition slightly: Two mixed (or undirected) graphs $G$ and $H$, with adjacency matrices $A$ and $B$ respectively, are fractionally isomorphic if and only if there exists two doubly stochastic matrices $R$ and $S$ satisfying $AS = SB$ and $RA = BR$. We show it suffices to let $R = S^T$.

Cayley graphs are interesting source of mixed graphs. Let $X$ be a group and let $S \subseteq X$ be a subset of elements of $X$ not containing the identity element of $X$. The Cayley graph of $(X, S)$ is defined as associating one vertex with each group element of $X$ and for all $x \in X$ and for all $s \in S$ there is a directed edge from $x$ to $xs$. The Cayley graph of a group depends on the set $S$, however, each choice of $S$ results in a mixed (or undirected) graph.

Keywords: Graph Isomorphism, Fractional Isomorphism, Cayley graphs