Inverse Circular Function and Trigonometric Equation
\((b, a)\) is the reflection of \((a, b)\) across the line \(y = x\).

The graph of \(f^{-1}\) is the reflection of the graph of \(f\) across the line \(y = x\).
Caution

The −1 in $f^{-1}$ is not an exponent.

$$f^{-1} \neq \frac{1}{f(x)}$$
In this section, we discuss the inverse sine function, which is defined as follows:

\[ y = \sin^{-1}x \quad \text{or} \quad y = \arcsin x \]

**Domain:** \([-1, 1]\)  
**Range:** \([-\frac{\pi}{2}, \frac{\pi}{2}]\)

We can represent the inverse sine function by considering some key points:

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-1)</td>
<td>(-\frac{\pi}{2})</td>
</tr>
<tr>
<td>(-\frac{\sqrt{2}}{2})</td>
<td>(-\frac{\pi}{4})</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\frac{\sqrt{2}}{2})</td>
<td>(\frac{\pi}{4})</td>
</tr>
<tr>
<td>1</td>
<td>(\frac{\pi}{2})</td>
</tr>
</tbody>
</table>

**Figure 11**

- The inverse sine function is increasing and continuous on its domain \([-1, 1]\).
- Its \(x\)-intercept is 0, and its \(y\)-intercept is 0.
- Its graph is symmetric with respect to the origin; it is an odd function.
Inverse Cosine Function

**Inverse Cosine Function**

\[ y = \cos^{-1} x \quad \text{or} \quad y = \arccos x \]

**Domain:** \([-1, 1]\) \quad **Range:** \([0, \pi]\)

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>\pi</td>
</tr>
<tr>
<td>(-\frac{\sqrt{2}}{2})</td>
<td>\frac{3\pi}{4}</td>
</tr>
<tr>
<td>0</td>
<td>\frac{\pi}{2}</td>
</tr>
<tr>
<td>\frac{\sqrt{2}}{2}</td>
<td>\frac{\pi}{4}</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

**Figure 14**

- The inverse cosine function is decreasing and continuous on its domain \([-1, 1]\).
- Its \(x\)-intercept is 1, and its \(y\)-intercept is \(\frac{\pi}{2}\).
- Its graph is not symmetric with respect to the \(y\)-axis or the origin.
**Inverse Tangent Function**

### Inverse Tangent Function

\[ y = \tan^{-1} x \quad \text{or} \quad y = \arctan x \]

**Domain:** \((-\infty, \infty)\)  
**Range:** \((-\frac{\pi}{2}, \frac{\pi}{2})\)

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-1)</td>
<td>(-\frac{\pi}{4})</td>
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<td>(-\frac{\sqrt{3}}{3})</td>
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<td>(\frac{\pi}{6})</td>
</tr>
<tr>
<td>1</td>
<td>(\frac{\pi}{4})</td>
</tr>
</tbody>
</table>

**Figure 17**

- The inverse tangent function is increasing and continuous on its domain \((-\infty, \infty)\).
- Its \(x\)-intercept is 0, and its \(y\)-intercept is 0.
- Its graph is symmetric with respect to the origin; it is an odd function.
- The lines \(y = \frac{\pi}{2}\) and \(y = -\frac{\pi}{2}\) are horizontal asymptotes.
# Domain and Range of Inverse Functions

<table>
<thead>
<tr>
<th>Inverse Function</th>
<th>Domain</th>
<th>Interval</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = \sin^{-1}x$</td>
<td>$[-1, 1]$</td>
<td>$[-\frac{\pi}{2}, \frac{\pi}{2}]$</td>
<td>I and IV</td>
</tr>
<tr>
<td>$y = \cos^{-1}x$</td>
<td>$[-1, 1]$</td>
<td>$[0, \pi]$</td>
<td>I and II</td>
</tr>
<tr>
<td>$y = \tan^{-1}x$</td>
<td>$(-\infty, \infty)$</td>
<td>$(-\frac{\pi}{2}, \frac{\pi}{2})$</td>
<td>I and IV</td>
</tr>
<tr>
<td>$y = \cot^{-1}x$</td>
<td>$(-\infty, \infty)$</td>
<td>$(0, \pi)$</td>
<td>I and II</td>
</tr>
<tr>
<td>$y = \sec^{-1}x$</td>
<td>$(-\infty, -1] \cup [1, \infty)$</td>
<td>$[0, \pi], y \neq \frac{\pi}{2}$*</td>
<td>I and II</td>
</tr>
<tr>
<td>$y = \csc^{-1}x$</td>
<td>$(-\infty, -1] \cup [1, \infty)$</td>
<td>$[-\frac{\pi}{2}, \frac{\pi}{2}], y \neq 0$*</td>
<td>I and IV</td>
</tr>
</tbody>
</table>
How to Find Function Values Using Definitions of the Trigonometric Functions?

- Evaluate a) \( \sin \left( \tan^{-1} \frac{3}{2} \right) \)

- Let \( \theta = \tan^{-1} \frac{3}{2} \), so \( \tan \theta = \frac{3}{2} \)

- The inverse tangent function yields values only in quadrants I and IV, since \( 3/2 \) is positive, \( \theta \) is in quadrant I.
How to Find Function Values Using Definitions of the Trigonometric Functions? (Cont.)

- Sketch and label the triangle.
- The hypotenuse is \( \sqrt{13} \)
- \( \sin\left(\tan^{-1}\frac{3}{2}\right) = \sin \theta = \frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{13} \)
FINDING FUNCTION VALUES USING DEFINITIONS OF THE TRIGONOMETRIC FUNCTIONS (continued)

(b) \( \tan\left(\cos^{-1}\left(-\frac{5}{13}\right)\right) \)

Let \( A = \cos^{-1}\left(-\frac{5}{13}\right) \), so \( \cos A = -\frac{5}{13} \).

Since \( \arccos \) is defined only in quadrants I and II, and \( -\frac{5}{13} \) is negative, \( \theta \) is in quadrant II.

Sketch \( A \) in quadrant II, and label the triangle.
\[
\sqrt{13^2} - \sqrt{(-5)^2} = 12
\]

\[
\tan\left(\cos^{-1}\left(-\frac{5}{13}\right)\right) = \tan A
\]

\[
= \frac{12}{-5} = -\frac{12}{5}
\]

\[
A = \cos^{-1}\left(-\frac{5}{13}\right)
\]
Another Example

- Evaluate the expression \( \tan \left( 2 \arcsin \frac{2}{5} \right) \) without using a calculator.
- Let \( \arcsin \left( \frac{2}{5} \right) = B \)
  \[
  \tan \left( 2 \arcsin \frac{2}{5} \right) = \tan 2B = \frac{2 \tan B}{1 - \tan^2 B}
  \]
- Since \( \arcsin \left( \frac{2}{5} \right) = B, \sin B = \frac{2}{5} \). Sketch a triangle in quadrant I, find the length of the third side, then find \( \tan(B) \).
Another Example (Cont.)

\[ \tan \left(2 \arcsin \frac{2}{5}\right) = \frac{2 \left(\frac{2}{\sqrt{21}}\right)}{1 - \left(\frac{2}{\sqrt{21}}\right)^2} \]

\[ = \frac{\frac{4}{\sqrt{21}}}{1 - \frac{4}{21}} = \frac{4\sqrt{21}}{17} \]
FINDING FUNCTION VALUES USING IDENTITIES

• Evaluate the expression without a calculator. \[ \cos\left(\arctan \sqrt{3} + \arcsin \frac{1}{3}\right) \]

Let \( A = \arctan \sqrt{3} \) and \( B = \arcsin \frac{1}{3} \), so \( \tan A = \sqrt{3} \) and \( \sin B = \frac{1}{3} \).

Use the cosine sum identity:

\[
\cos\left(\arctan \sqrt{3} + \arcsin \frac{1}{3}\right) = \cos(A + B) = \cos A \cos B - \sin A \sin B
\]
• Sketch both A and B in quadrant I. Use the Pythagorean theorem to find the missing side.

\[
\sin A = \frac{\sqrt{3}}{2}, \quad \cos A = \frac{1}{2}
\]

\[
\sin B = \frac{1}{3}, \quad \cos B = \frac{2\sqrt{2}}{3}
\]
\[
\cos \left( \arctan \sqrt{3} + \arcsin \frac{1}{3} \right) = \cos (A + B) \\
= \cos A \cos B - \sin A \sin B \\
= \frac{1}{2} \cdot \frac{2\sqrt{2}}{3} - \frac{\sqrt{3}}{2} \cdot \frac{1}{3} \\
= \frac{2\sqrt{2} - \sqrt{3}}{6}
\]
How to Solve a Trigonometric Equation?

- **Step 1:** Decide whether the equation is linear or quadratic in form, so you can determine the solution method.
- **Step 2:** If only one trigonometric function is present, first solve the equation for that function.
- **Step 3:** If more than one trigonometric function is present, rearrange the equation so that one side equals 0. Then try to factor and set each factor equal to 0 to solve.
How to Solve a Trigonometric Equation? (Cont.)

- **Step 4:** If the equation is quadratic in form, but not factorable, use the quadratic formula. Check that solutions are in the desired interval.

- **Step 5:** Try using identities to change the form of the equation. It may be helpful to square both sides of the equation first. If this is done, check for extraneous solutions.
Example of Solving Trigonometric Equation Using the Linear Method

- Solve $2 \cos^2 x - 1 = 0$
- **Solution:** First, solve for $\cos x$ on the unit circle.

$$2 \cos^2 x - 1 = 0$$
$$2 \cos^2 x = 1$$
$$\cos^2 x = \frac{1}{2}$$
$$\cos x = \pm \sqrt{\frac{1}{2}}$$
$$\cos x = \pm \frac{\sqrt{2}}{2}$$

$x = 45^\circ, 135^\circ, 225^\circ, 315^\circ$

*or* $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
Example of Solving Trigonometric Equation by Factoring

- Solve $2 \cos x + \sec x = 0$

- Solution:

  
  
  \[
  2 \cos x + \frac{1}{\cos x} = 0
  \]

  \[
  \frac{1}{\cos x} (2 \cos^2 x + 1) = 0
  \]

  \[
  \frac{1}{\cos x} = 0
  \]

  \[
  2 \cos^2 x + 1 = 0
  \]

  \[
  2 \cos^2 x = -1
  \]

  \[
  \cos^2 x = \frac{-1}{2}
  \]

  \[
  \cos x = \pm \sqrt{-\frac{1}{2}}
  \]

  Since neither factor of the equation can equal zero, the equation has no solution.
Example of Solving Trigonometric Equation by Squaring

- Solve \( \cos x + 1 = \sin x \) \([0, 2 \pi]\)

\[
\begin{align*}
\cos x + 1 &= \sin x \\
\cos^2 x + 2 \cos x + 1 &= \sin^2 x \\
\cos^2 x + 2 \cos x + 1 &= 1 - \cos^2 x \\
2 \cos^2 x + 2 \cos x &= 0 \\
2 \cos x (\cos x + 1) &= 0 \\
2 \cos x &= 0 \quad \text{and} \quad \cos x + 1 = 0 \\
\cos x &= 0 \quad \text{and} \quad \cos x = -1 \\
x &= \frac{\pi}{2}, \frac{3\pi}{2} \\
x &= \pi
\end{align*}
\]

Check the solutions in the original equation. The only solutions are \( \pi/2 \) and \( \pi \).
Example of Solving a Trigonometric Equation Using a Half-Angle Identity

- Solve \( 2 \sin \frac{x}{2} = 1 \)

- a) over the interval \([0, 2\pi)\), and

- b) give all solutions

**Solution:**
- Write the interval as the inequality \(0 \leq x \leq 2\pi\).
Example of Solving a Trigonometric Equation Using a Half-Angle Identity (Cont.)

- The corresponding interval for $x/2$ is $0 \leq \frac{x}{2} < \pi$.
- Solve $2 \sin \frac{x}{2} = 1$

$$\sin \frac{x}{2} = \frac{1}{2}$$

- Sine values that corresponds to $1/2$ are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

\begin{align*}
\frac{x}{2} &= \frac{\pi}{6} & \text{or} & \quad \frac{x}{2} &= \frac{5\pi}{6} \\
&\quad \text{or} & \quad x &= \frac{\pi}{3} & \text{or} & \quad x &= \frac{5\pi}{3}
\end{align*}
b) Sine function with a period of $4\pi$, all solutions are given by the expressions

$$\frac{\pi}{3} + 4n\pi \text{ and } \frac{5\pi}{3} + 4n\pi$$

where $n$ is any integer.
Example of Solving a Trigonometric Equation Using a Double-Angle Identity

- Solve $\cos(2x) = \cos(x)$ over the interval $[0, 2\pi)$.

- First, change $\cos(2x)$ to a trigonometric function of $x$. Use the identity $\cos 2x = 2\cos^2 x - 1$.

\[
\begin{align*}
\cos 2x &= \cos x \\
2\cos^2 x - 1 &= \cos x
\end{align*}
\]
Example of Solving a Trigonometric Equation Using a Double-Angle Identity (Cont.)

\[ \cos 2x = \cos x \]
\[ 2 \cos^2 x - 1 = \cos x \]
\[ 2 \cos^2 x - \cos x - 1 = 0 \]
\[ (2 \cos x + 1)(\cos x - 1) = 0 \]
\[ 2 \cos x + 1 = 0 \quad \text{or} \quad \cos x - 1 = 0 \]
\[ \cos x = -\frac{1}{2} \quad \text{or} \quad \cos x = 1 \]

- Over the interval \( x = \frac{2\pi}{3} \) or \( x = \frac{4\pi}{3} \) or \( x = 0 \).
Example of Using the Multiple-Angle Identity

- Solve $4 \sin \theta \cos \theta = \sqrt{3}$ over the interval $[0, 360^\circ)$.

  
  \[2 \sin \theta \cos \theta = \sin 2\theta\]
  
  \[4 \sin \theta \cos \theta = \sqrt{3}\]
  
  \[2(2 \sin \theta \cos \theta) = \sqrt{3}\]
  
  \[2 \sin 2\theta = \sqrt{3}\]
  
  \[\sin 2\theta = \frac{\sqrt{3}}{2}\]
Example of Using the Multiple-Angle Identity (Cont.)

- List all solutions in the interval.

\[ 2\theta = 60^\circ, 120^\circ, 420^\circ, 480^\circ \]

or \[ \theta = 30^\circ, 60^\circ, 210^\circ, 240^\circ \]

- The final two solutions were found by adding 360° to 60° and 120°, respectively, giving the solution set \( \{30^\circ, 60^\circ, 210^\circ, 240^\circ \} \).
Another Example of Using the Multiple-Angle Identity

- Solve \( \tan 3x + \sec 3x = 2 \) over the interval \([0, 2\pi)\).
- Tangent and secant are related so use the identity

\[
1 + \tan^2 \theta = \sec^2 \theta.
\]

\[
\tan 3x + \sec 3x = 2
\]

\[
\tan 3x = 2 - \sec 3x
\]

\[
\tan^2 3x = 4 - 4\sec 3x + \sec^2 3x
\]

\[
\sec^2 3x - 1 = 4 - 4\sec 3x + \sec^2 3x
\]
Another Example of Using the Multiple-Angle Identity (Cont.)

\[
\sec^2 3x - 1 = 4 - 4 \sec 3x + \sec^2 3x
\]

\[
4 \sec 3x = 5
\]

\[
\sec 3x = \frac{5}{4}
\]

\[
\frac{1}{\cos 3x} = \frac{5}{4}
\]

\[
\cos 3x = \frac{4}{5}
\]
How to Solve an Equation Involving an Inverse Trigonometric Function?

• **Example**: Solve \(2 \arcsin x = \pi\).

• **Solution**: First solve for \(\arcsin x\), and then for \(x\).

\[
2 \arcsin x = \pi \\
\arcsin x = \frac{\pi}{2}
\]

\[
x = \sin \frac{\pi}{2} = 1
\]

• The solution set is \(\{1\}\).
Another Example

• **Example:** Solve \( \cos^{-1} x = \sin^{-1} \frac{1}{2} \).

• **Solution:** Let \( \sin^{-1} \frac{1}{2} = u \). Then \( \sin u = \frac{1}{2} \) and for \( u \) in quadrant I, the equation becomes

\[
\cos^{-1} x = u
\]

\[
\cos u = x.
\]
Example of Simplifying Expression Using the Half-Angle Identities

- Sketch a triangle and label it using the facts that $u$ is in quadrant I and $\sin u = \frac{1}{2}$.

- Since $x = \cos u$, $x = \frac{\sqrt{3}}{2}$,