The Regular Heptagon by Angle Trisection and Other Constructions

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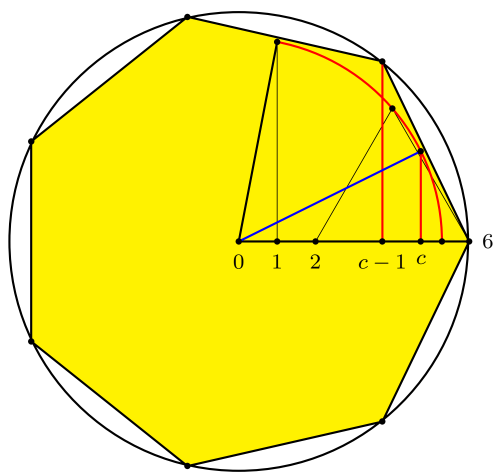
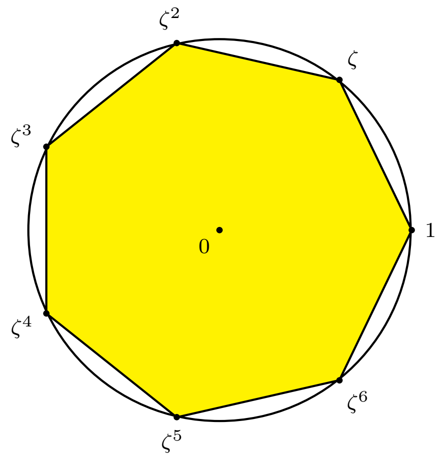
April 24, 2007



Contents

1 Which regular n-gons are constructible?



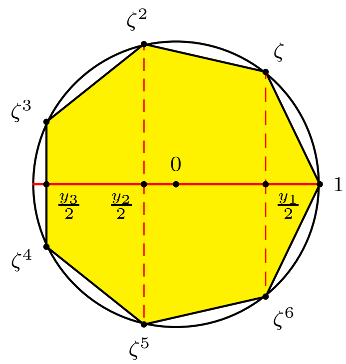
ζ^k, $k = 1, 2, \ldots, 6$, are the roots of

$$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0.$$

These are constructible with ruler and compass if and only if $y = x + \frac{1}{x} = 2\cos k \cdot \frac{2\pi}{7}$ are.

$$y^3 + y^2 - 2y - 1 = 0.$$

Theorem 1. *Let F be a field.*
The roots of a **cubic** *polynomial $p(x) \in F[x]$ are constructible by ruler and compass from F if and only if $p(x)$ has a root in F.*



Corollary 2. *The regular 7-gon is not constructible by ruler and compass.*

Proof. The cubic polynomial $y^3 + y^2 - 2y - 1$ has no rational root, (the only possible candidates being ± 1). □

More examples

1. $\sqrt[3]{2}$ cannot be constructed by ruler and compass.

Proof. $x^3 - 2$ does not have a rational root. $\square$

2. The 60° angle cannot be trisected by ruler and compass.

Proof. Since $\cos 60^\circ = \frac{1}{2}$, the trisection of 60° is equivalent to the solution of $\cos 3\theta = \frac{1}{2}$. With $x = \cos\theta$, this is

$$4x^3 - 3x = \frac{1}{2},$$

or $8x^3 - 6x - 1 = 0$. It is clear that this does not have rational roots. This shows that 60° cannot be trisected with ruler and compass. $\square$

Theorem 3. *A regular n-gon is constructible if and only if $n = 2^a \prod_j p_j$ where*
(i) $a \geq 0$,
(ii) *if $j \geq 1$, each p_j is a Fermat prime of the form $2^{2^k}+1$,*
(iii) *if $j \geq 2$, the Fermat primes p_j are all distinct.*

Proof. (1) If a regular n-gon is constructible, then so is a regular $2n$-gon.

(2) If a regular n-gon is constructible, and $m|n$, then a regular m-gon is constructible.

(3) If p is an odd prime and a regular p-gon is constructible, then $p - 1$ is a power of 2.

(4) If p is an odd prime of the form $2^k + 1$, then a regular p-gon is constructible (Gauss).

(5) If $\gcd(m,n) = 1$ and regular m- and regular n-gons are constructible, then so is a regular mn-gon.

Find integers a and b such that $am + bn = 1$, and note that

$$\frac{2\pi}{mn} = b \cdot \frac{2\pi}{m} + a \cdot \frac{2\pi}{n}.$$

It remains to show why the Fermat primes may not repeat.

(6) Let p be an odd prime divisor of n. If a regular n-gon is constructible, then p^2 cannot divide n.

□

A regular n-gon is constructible if and only if the numbers $\tan \frac{k \cdot 2\pi}{n}$, $k = 1, 2, \ldots, n-1$, are constructible. These are the roots of the polynomial

$$P_n(x) = \sum_{2j \le n-1} (-1)^{j-1} \binom{n}{2j+1} x^{2j}.$$

Proof.

$$\begin{aligned}
\tan n\theta &= \frac{\sum_{2j+1 \le n} (-1)^j \binom{n}{2j+1} \tan^{2j+1} \theta}{\sum_{2j \le n} (-1)^j \binom{n}{2j} \tan^{2j} \theta} \\
&= \frac{\tan \theta \sum_{2j+1 \le n} (-1)^j \binom{n}{2j+1} \tan^{2j} \theta}{\sum_{2j \le n} (-1)^j \binom{n}{2j} \tan^{2j} \theta}.
\end{aligned}$$

□

If n is odd, we may take P_n to be a monic polynomial (with leading coefficient 1).

(6) Let p be an odd prime divisor of n. If a regular n-gon is constructible, then p^2 cannot divide n.

Proof. $P_{p^2}(x) = P_p(x)Q(x)$ for a polynomial $Q(x)$.

(i) $Q(x) \in \mathbb{Z}$, and is monic.

(ii) Each coefficient of $Q(x)$, apart from the leading one, is divisible by p.

(iii) The constant term is not divisible by p^2.

Therefore, $Q(x)$ is irreducible by Eisenstein's criterion.

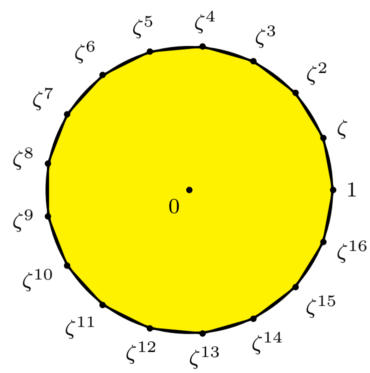
Example: For $p = 3$, $P_3(x) = x^2 - 3$,

$P_9(x) = x^8 - 36x^6 + 126x^4 - 84x^2 + 9$,

$Q(x) = \frac{P_9(x)}{P_3(x)} = x^6 - 33x^4 + 27x - 3$.

Since $\deg Q = p^2 - p = p(p-1)$ is not a power of 2, the roots of $Q(x)$ are not ruler-and-compass constructible. □

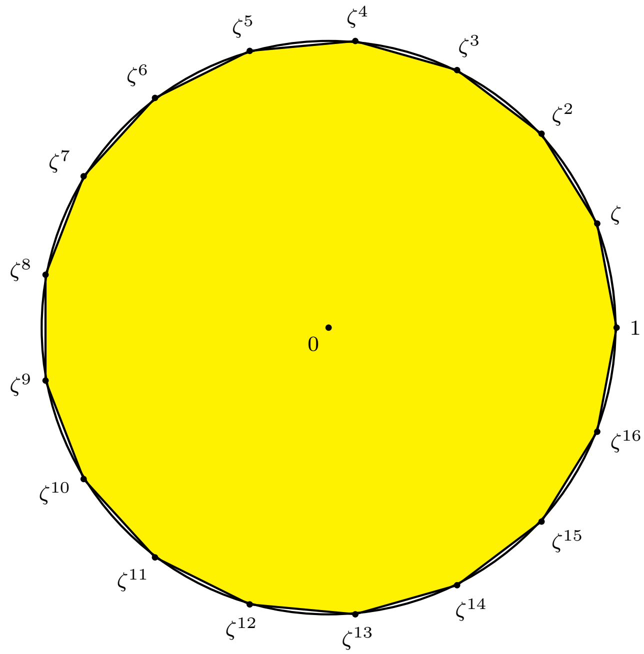
2 Gauss' Construction of the regular 17-gon



First entry of Gauss' diary:

> [1] The principles upon which the division of the circle depend, and geometrical divisibility of the same into seventeen parts, etc.
>
> [1796] March 30 Brunswick

The details appeared in Section VII of his *Disquisitiones Arithmeticae* (1801).

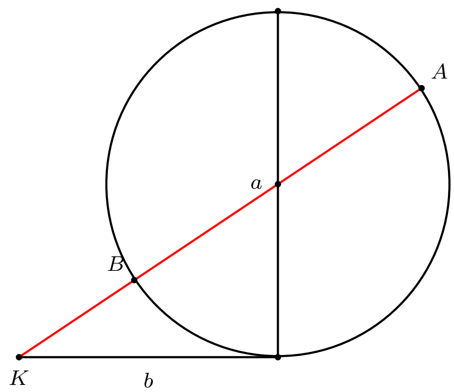


$\zeta, \zeta^2, \ldots, \zeta^{15}, \zeta^{16}$ are the roots of the equation

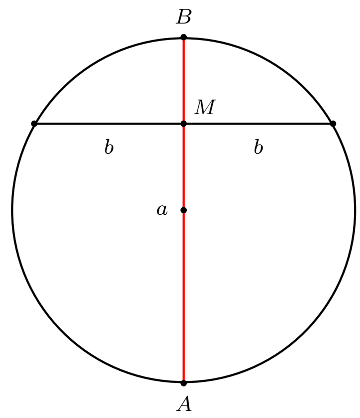
$$\frac{x^{17}-1}{x-1} = x^{16} + x^{15} + \cdots + x + 1 = 0.$$

Gauss' construction depends on two key ideas.

(1) Solution of quadratic equations by ruler and compass:



(a) $x^2 = ax + b^2$: $x_1 = KA,\ x_2 = -KB$



(b) $x^2 + b^2 = ax$: $x_1 = AM,\ x_2 = BM$

(2) Gauss observed that, modulo 17, all the residues are powers of 3 (which is a **primitive root** for the prime 17):

k	0	2	4	6	8	10	12	14
3^k	1	9	13	15	16	8	4	2
k	1	3	5	7	9	11	13	15
3^k	3	10	5	11	14	7	12	6

If we take

$$\begin{aligned} y_1 &= \zeta + \zeta^9 + \zeta^{13} + \zeta^{15} + \zeta^{16} + \zeta^8 + \zeta^4 + \zeta^2, \\ y_2 &= \zeta^3 + \zeta^{10} + \zeta^5 + \zeta^{11} + \zeta^{14} + \zeta^7 + \zeta^{12} + \zeta^6, \end{aligned}$$

then $y_1 + y_2 = -1$. More importantly, the product $y_1 y_2$ does *not* depend on the choice of ζ.

$$\begin{aligned} y_1 &= \zeta + \zeta^9 + \zeta^{13} + \zeta^{15} + \zeta^{16} + \zeta^8 + \zeta^4 + \zeta^2, \\ y_2 &= \zeta^3 + \zeta^{10} + \zeta^5 + \zeta^{11} + \zeta^{14} + \zeta^7 + \zeta^{12} + \zeta^6. \end{aligned}$$

This table gives $\zeta^h \cdot \zeta^k = \zeta^m$, $m \equiv h + k \pmod{17}$.

	1	9	13	15	16	8	4	2
3	4	12	16	1	2	11	7	5
10	11	2	6	8	9	1	14	12
5	6	14	1	3	4	13	9	7
11	12	3	7	9	10	2	15	13
14	15	6	10	12	13	5	1	16
7	8	16	3	5	6	15	11	9
12	13	4	8	10	11	3	16	14
6	7	15	2	4	5	14	10	8

Each of 1, 2, $\ldots$, 16 appears **exactly** 4 times.

$$
\begin{aligned}
y_1 + y_2 &= \zeta + \zeta^2 + \cdots + \zeta^{16} = -1, \\
y_1 y_2 &= 4(\zeta + \zeta^2 + \cdots + \zeta^{16}) = -4.
\end{aligned}
$$

It follows that y_1 and y_2 are the roots of

$$y^2 + y - 4 = 0,$$

and are constructible. We may take

$$y_1 = \frac{-1 + \sqrt{17}}{2}, \qquad y_2 = \frac{-1 - \sqrt{17}}{2}.$$

Recall

$$\begin{aligned} y_1 &= \zeta + \zeta^9 + \zeta^{13} + \zeta^{15} + \zeta^{16} + \zeta^8 + \zeta^4 + \zeta^2, \\ y_2 &= \zeta^3 + \zeta^{10} + \zeta^5 + \zeta^{11} + \zeta^{14} + \zeta^7 + \zeta^{12} + \zeta^6. \end{aligned}$$

Now separate each of y_1, y_2 into two "groups" of four:

$$\begin{aligned} z_1 &= \zeta + \zeta^{13} + \zeta^{16} + \zeta^4, \\ z_2 &= \zeta^9 + \zeta^{15} + \zeta^8 + \zeta^2; \\ z_3 &= \zeta^3 + \zeta^5 + \zeta^{14} + \zeta^{12}, \\ z_4 &= \zeta^{10} + \zeta^{11} + \zeta^7 + \zeta^6. \end{aligned}$$

Then,

$$\begin{aligned} z_1 z_2 &= (\zeta + \zeta^{13} + \zeta^{16} + \zeta^4)(\zeta^9 + \zeta^{15} + \zeta^8 + \zeta^2) \\ &= \zeta + \zeta^2 + \cdots + \zeta^{16} \\ &= -1; \\ z_3 z_4 &= -1. \end{aligned}$$

Numbers	quadratic equation
$z_1,\ z_2$	$z^2 - y_1 z - 1 = 0$
$z_3,\ z_4$	$z^2 - y_2 z - 1 = 0$

Finally, further separating $z_1 = \zeta + \zeta^{13} + \zeta^{16} + \zeta^4$ into

$$t_1 = \zeta + \zeta^{16}, \qquad t_2 = \zeta^{13} + \zeta^4,$$

we obtain

$$\begin{aligned} t_1 + t_2 &= z_1, \\ t_1 t_2 &= (\zeta + \zeta^{16})(\zeta^{13} + \zeta^4) \\ &= \zeta^{14} + \zeta^5 + \zeta^{12} + \zeta^3 \\ &= z_3. \end{aligned}$$

Therefore, $t_1 = \zeta + \zeta^{16}$ and $t_2 = \zeta^{13} + \zeta^4$ are the roots of

$$t^2 - z_1 t + z_3 = 0,$$

and are constructible.

2.1 An explicit construction of a regular 17-gon

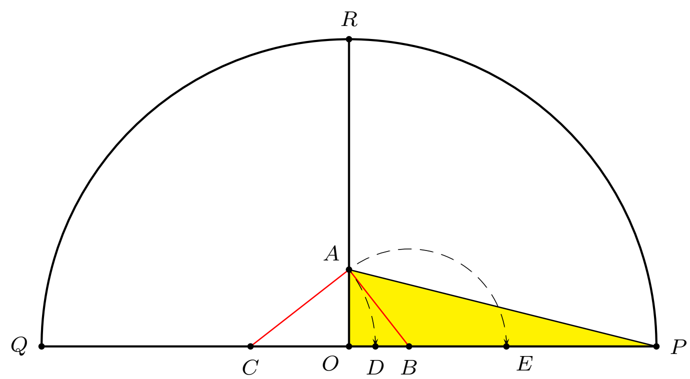
$$\cos\frac{2\pi}{17} = \frac{1}{2}\left(\zeta+\zeta^{16}\right) = \frac{1}{16}\left(\sqrt{17}-1+\sqrt{34-2\sqrt{17}}\right) + \frac{1}{8}\sqrt{17+3\sqrt{17}-\sqrt{34-2\sqrt{17}}-2\sqrt{34+2\sqrt{17}}}.$$

Circle (O) with perpendicular diameters PQ and RS.

(1) A on OR with $OA = \frac{1}{4}OP$.

(2) Construct the bisectors OB and OC of angle OAP.

(3) $D = OP \cap C(A)$ and $E = OP \cap B(A)$.



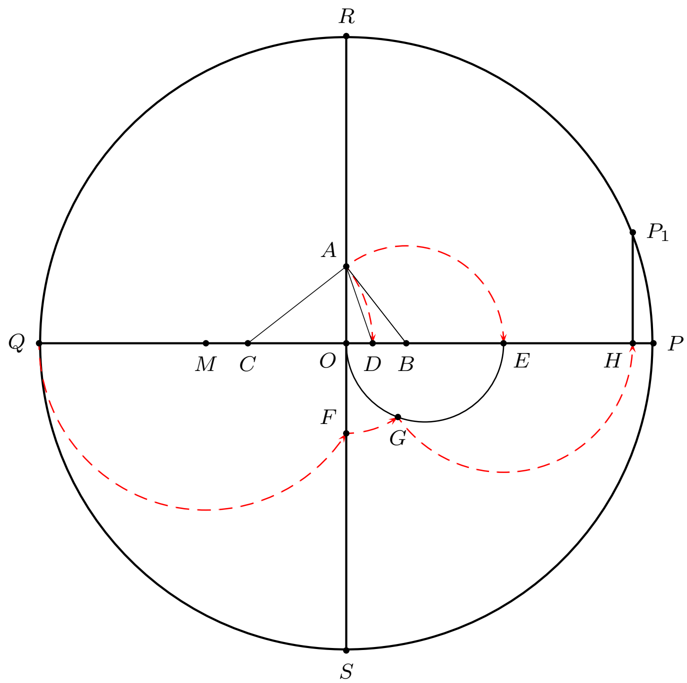
(4) $M =$ midpoint M of QD.

(5) $F = OS \cap M(Q)$.

(6) G on semicircle on OE, with $OG = OF$.

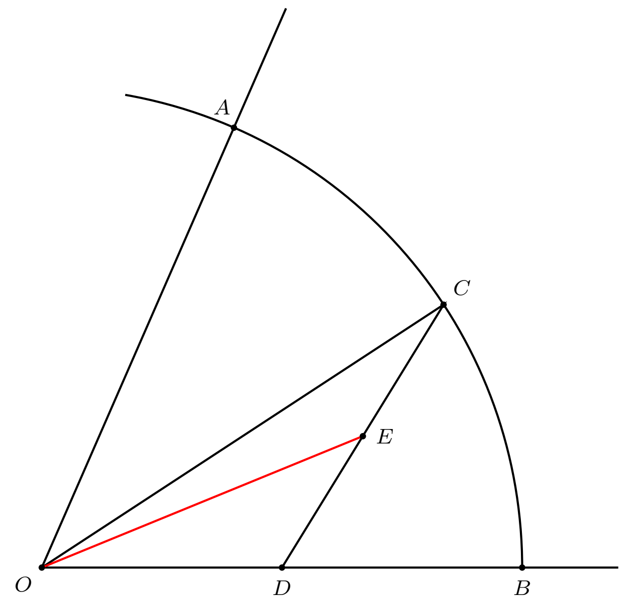
(7) $H = OP \cap E(G)$.

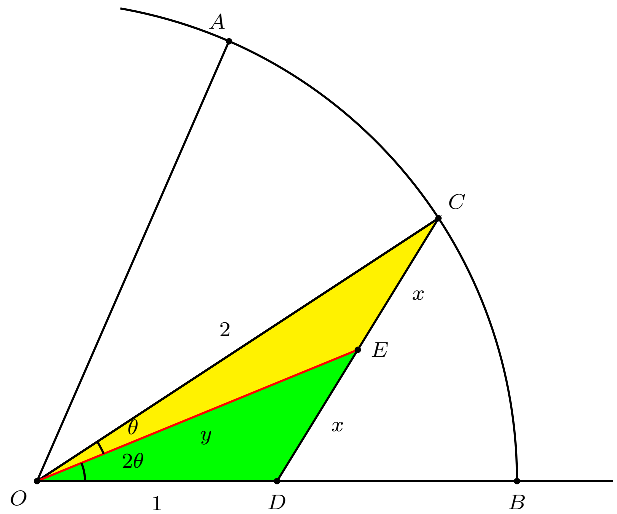
(8) $P_1H \perp OP$.



3 Angle trisection

3.1 A false trisection





Equal areas $\Rightarrow y\sin 2\theta = 2y\sin\theta$

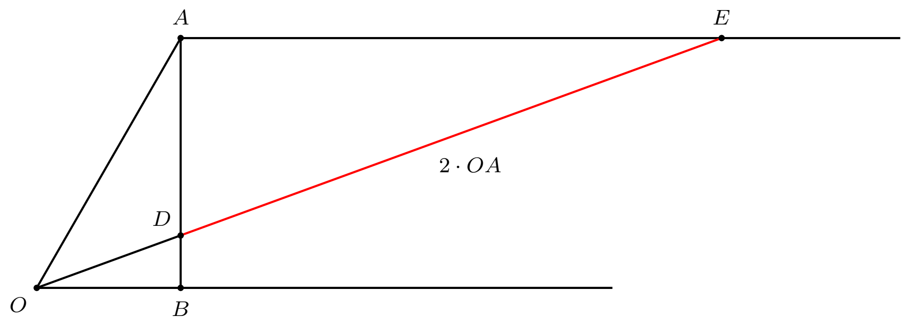
$\Rightarrow \cos\theta = 1 \quad$!!!

In this example, $\cos AOB = \frac{2}{5}$, $\angle AOB \approx 66.422°$,

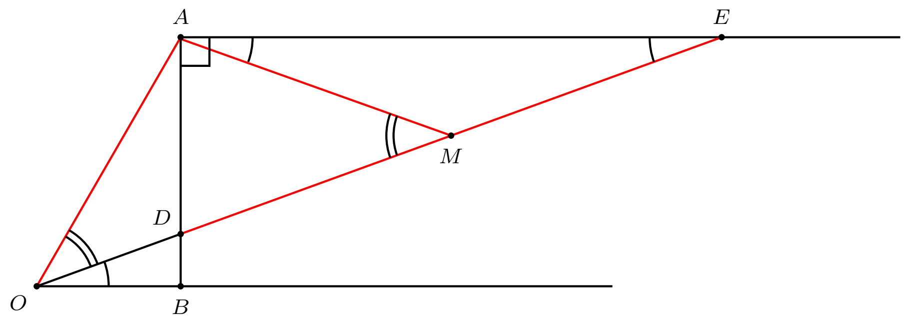
$$2\theta - \frac{1}{3}\angle AOB < 0.142°.$$

3.2 Constructions with a marked ruler

(1) Pass a line through O such that the intercept between the parallel and the perpendicular at A to the line OB is $2 \cdot OA$. Then this line is a trisector of angle AOB.



Proof.

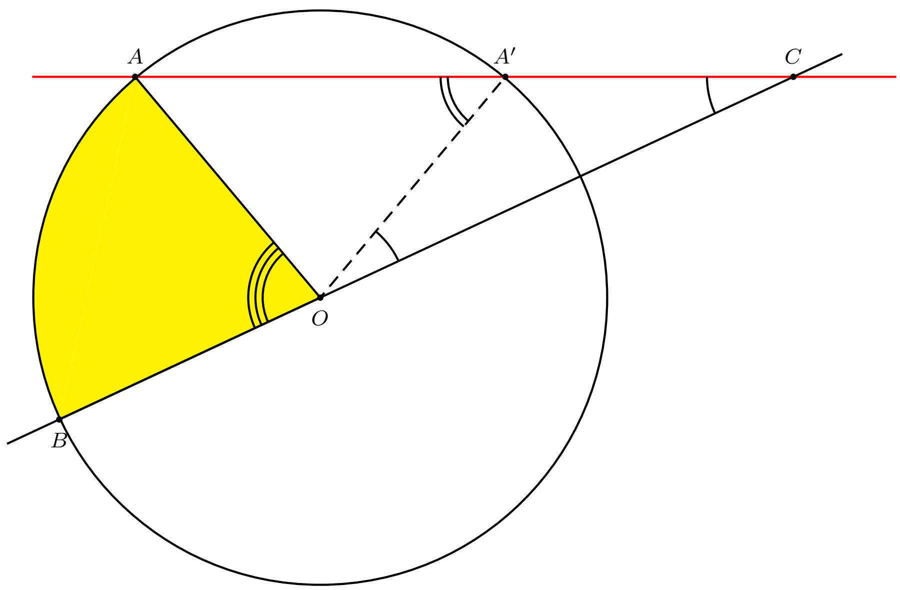


□

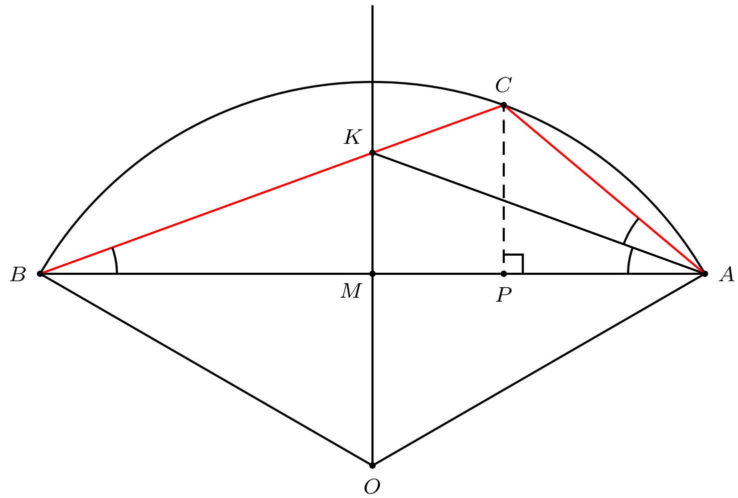
(2) Archimedes [Book of Lemmas, Proposition 8]

Construct a line through A such that the intercept between the circle and the line BO has the same length as the radius of the circle.

Then $\angle A'OC = \frac{1}{3}\angle AOB$.



3. The use of conics



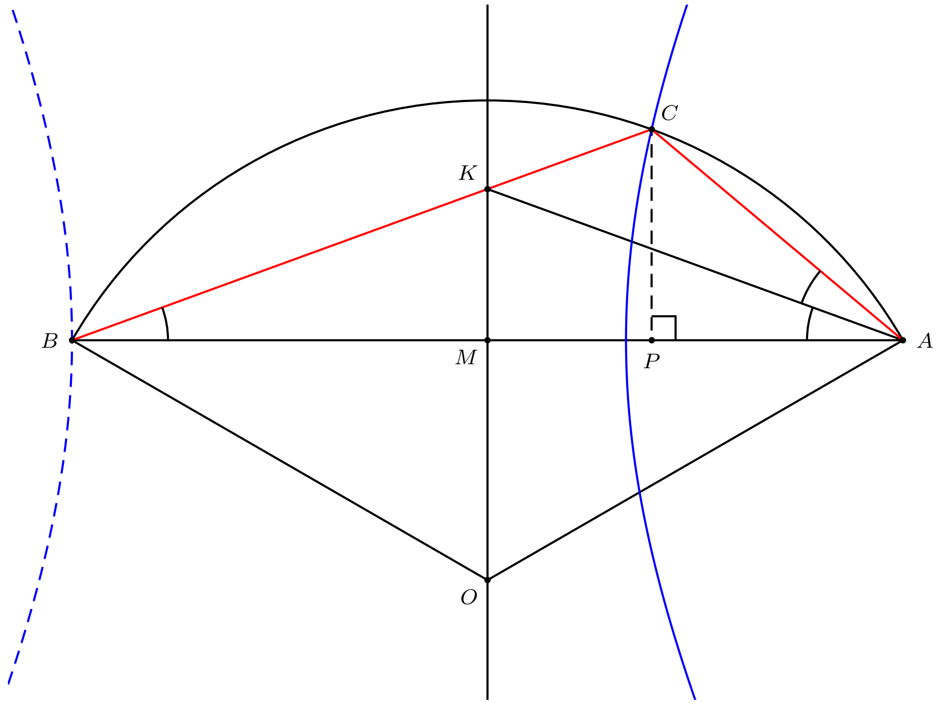
If $\angle AOB = 3\angle AOC$, then

$$AB : AC = BK : CK = BM : PM.$$

Since $AB = 2 \cdot BM$, we have $AC = 2 \cdot PM$.

Therefore, C lies on the **hyperbola** with focus A, directrix OM, and eccentricity 2.

$AC = 2 \cdot PM \Rightarrow C$ lies on the **hyperbola** with focus A, directrix OM, and eccentricity 2.



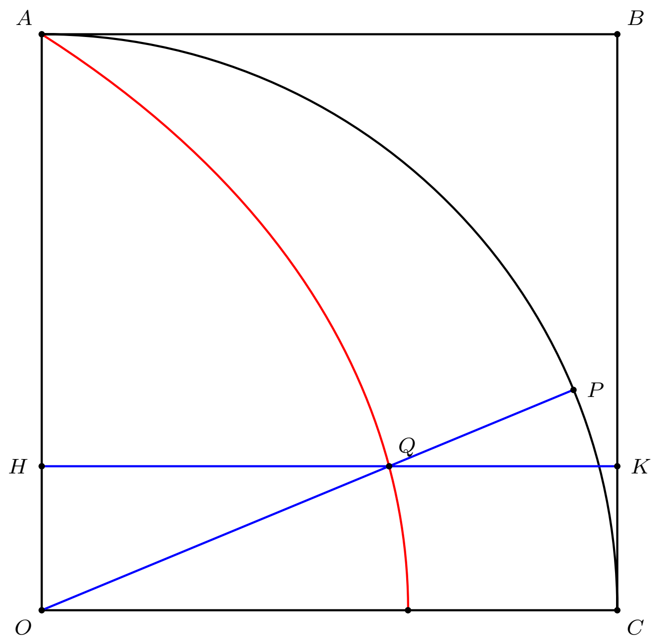
3.3 The quadratrix

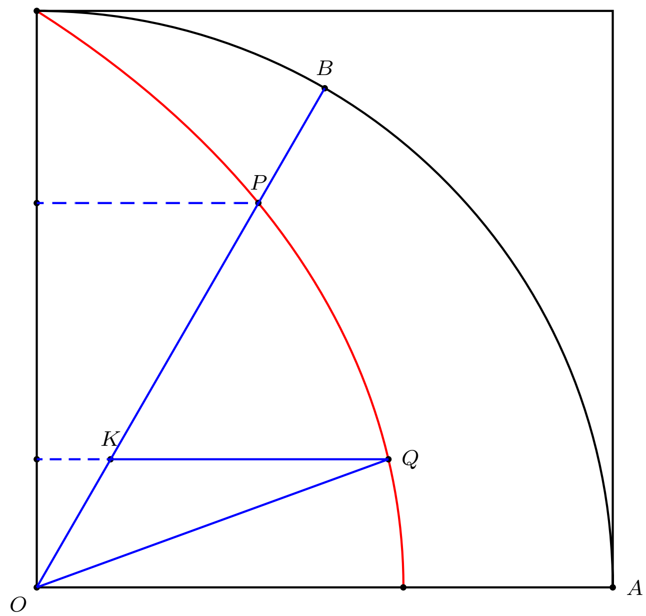
A horizontal line HK (with initial position AB) falls vertically,

and a radius OP (with initial position OA) rotates about O,

both uniformly and arrive at OC at the same time.

The locus of the intersection $Q = HK \cap OP$ is the **quadratrix**.

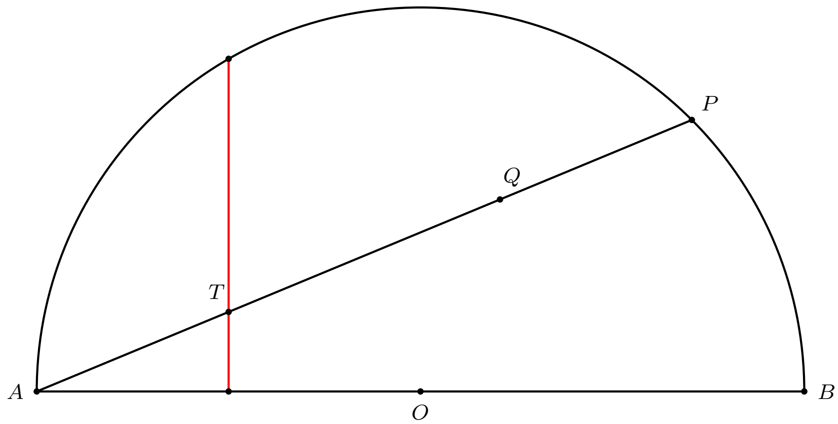




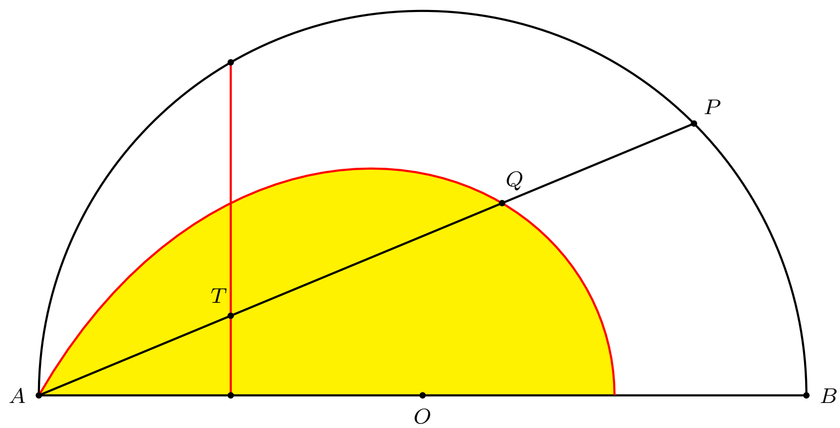
To trisect angle AOB, let OB intersect the quadratrix at P. Trisect the **segment** OP at K. Construct the parallel through K to OA to intersect the quadratrix at Q.

Then OQ is a trisector of angle AOB.

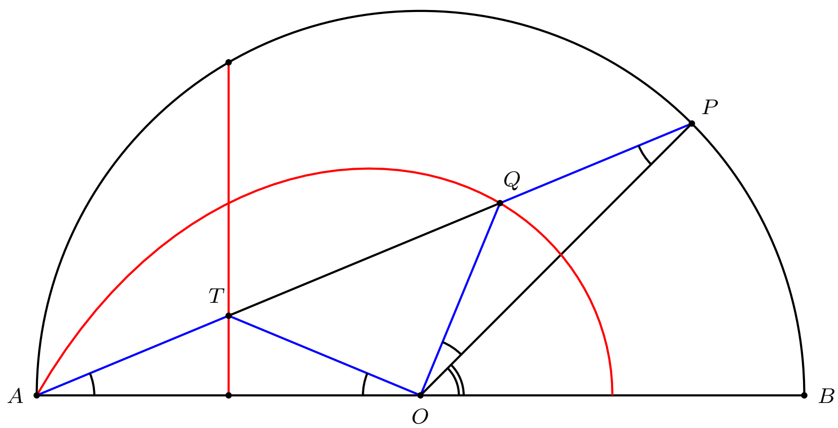
3.4 The trisectrix (MacLaurin)



$T = AP \cap$ perpendicular bisector of OA.
$AT = QP$.
The locus of Q is the **trisectrix**.



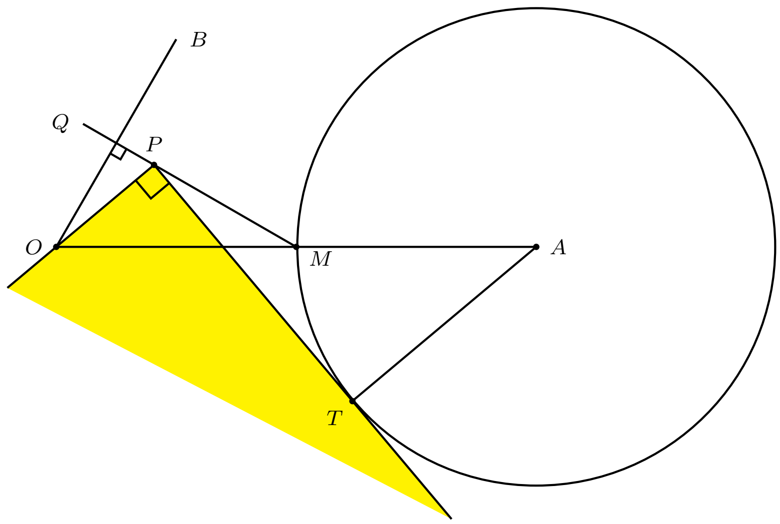
OP is a trisector of angle QOB.


P
Q
T
A
O
B

3.5 A recent trisection

D. A. Brooks, A new method of trisection, *College Math. Journal*, 38 (2007) 78–81.

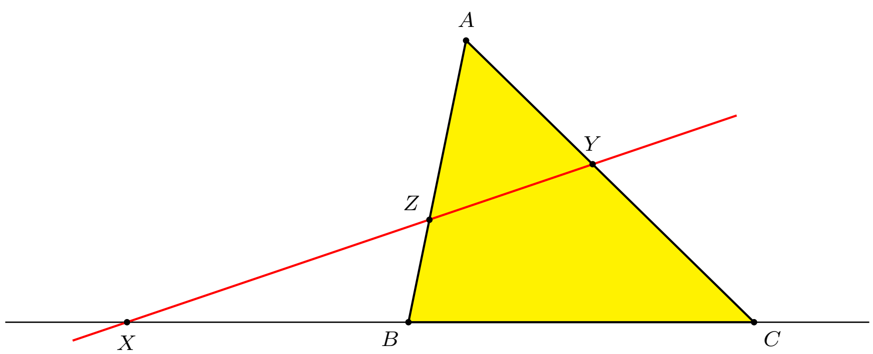
Given an acute angle AOB, let M be the midpoint of OA. Construct (i) the circle, center M, passing through A, (ii) the perpendicular MQ from M to OB. If P is a point on MQ such that the tangent from P to the circle makes a right angle with OP, then OP is a trisector of angle OAB.

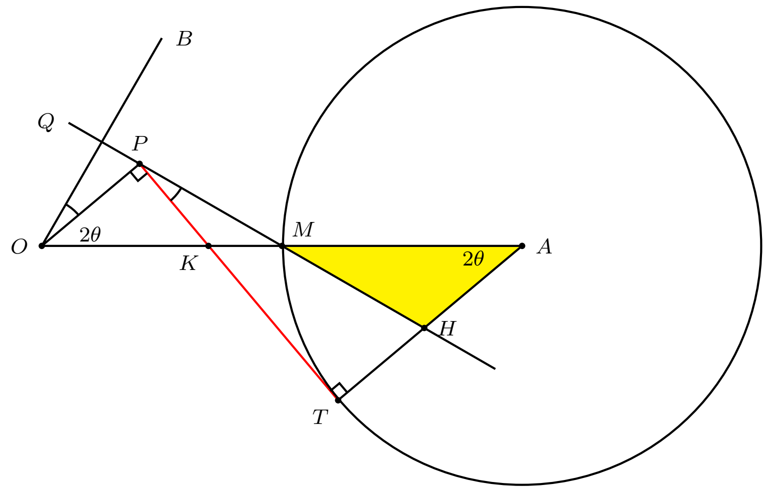


Consider a triangle ABC with points X, Y, Z on the side lines BC, CA, AB respectively.

Theorem 4 (Menelaus)**.** *The points X, Y, Z are collinear if and only if*

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1.$$





Proof. Let $\angle POA = \angle OAT = 2\theta$. We show that $\angle HPT = \theta$. It will follow that $\angle BOP = \theta$, and OP is a trisector of angle AOB.

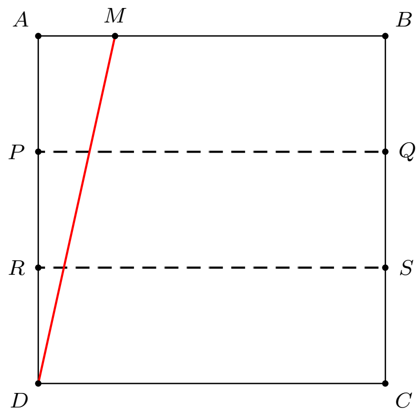
Assume unit radius for the circle. Applying Menelaus' theorem to triangle MHA with transversal TKP, we have $\dfrac{HT}{TA} \cdot \dfrac{AK}{KM} \cdot \dfrac{MP}{PH} = -1$. Therefore,

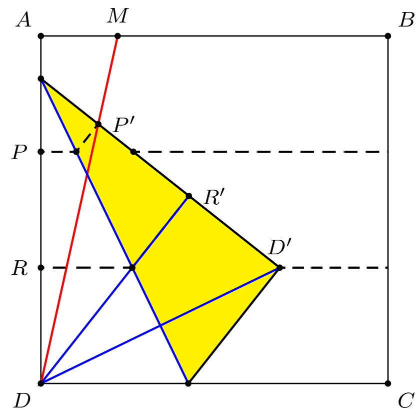
$$\frac{HT}{TA} = -\frac{PH}{MP} \cdot \frac{KM}{AK} = -\frac{OA}{MO} \cdot \frac{AK - AM}{AK} = 2 \cdot \frac{\sec 2\theta - 1}{\sec 2\theta} = 2(1 - \cos 2\theta) = 4\sin^2\theta.$$

Since $TA = 1$, we have $TH = 4\sin^2\theta$. Now, $PT = 2\sin 2\theta = 4\sin\theta\cos\theta$. This gives $\tan HPT = \frac{TH}{PT} = \tan\theta$. □

3.6 Angle trisection by paper folding

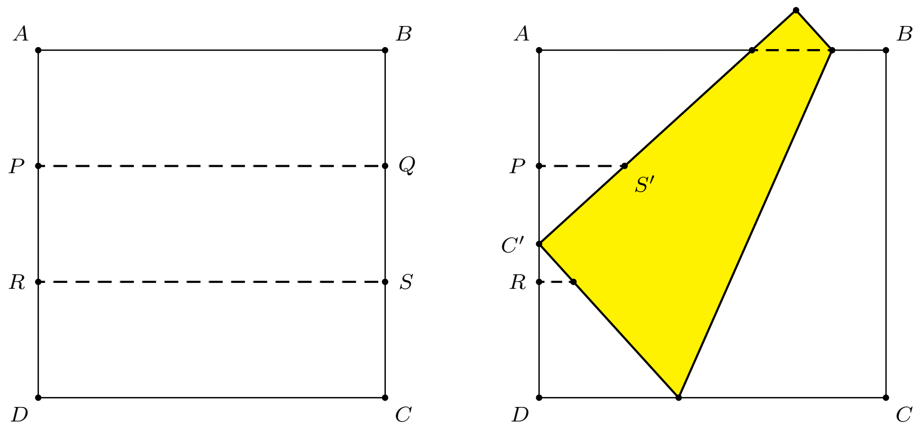
B. Casselman, If Euclid had been Japanese, *Notices of AMS*, 54 (2007) May issue, 626–628.





(2) Cubic root of 2 by paper folding

A paper square $ABCD$ is divided into three strips of equal area by the parallel lines PQ and RS. The square is then folded so that C falls on AD and S falls on PQ (as C' in the second diagram). Then $\frac{AC'}{C'D} = \sqrt[3]{2}$.

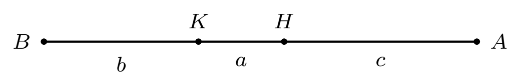


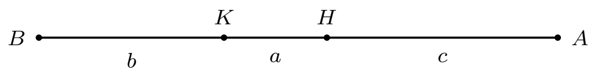
4 Archimedes’ construction of the regular heptagon (Arabic tradition)

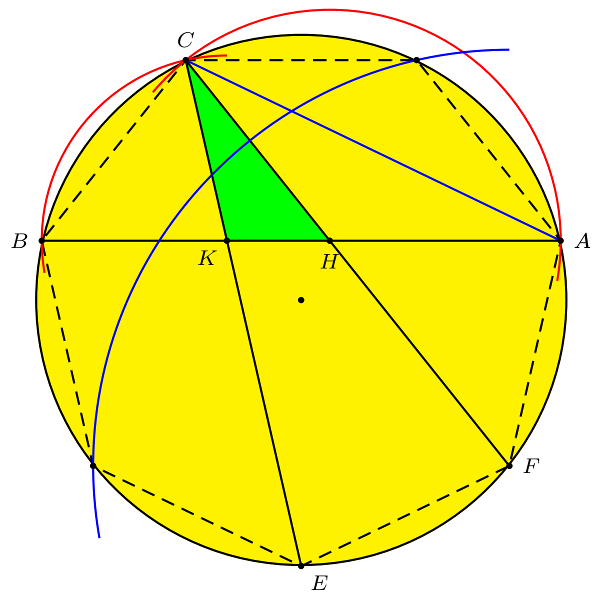
4.1 Archimedes’ construction

Construct a segment AB with division points H and K such that

$$b^2 = a(c + a), \qquad c^2 = (a + b)b.$$

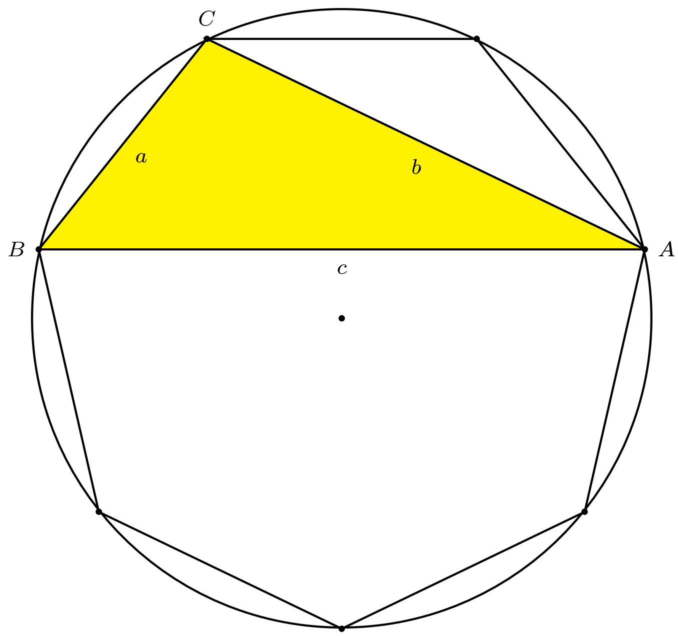


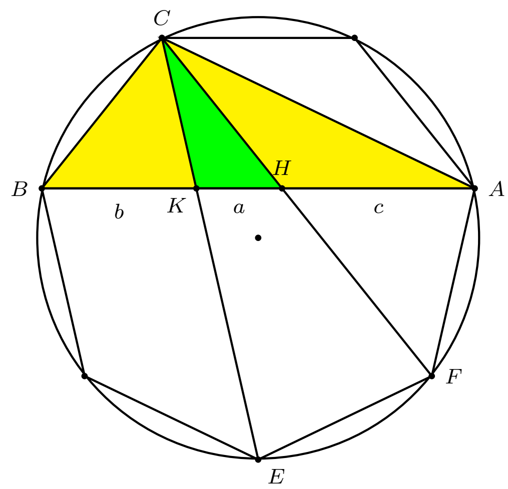

B
K
H
A
b
a
c


C
B
K
H
A
F
E

4.2 The heptagonal triangle

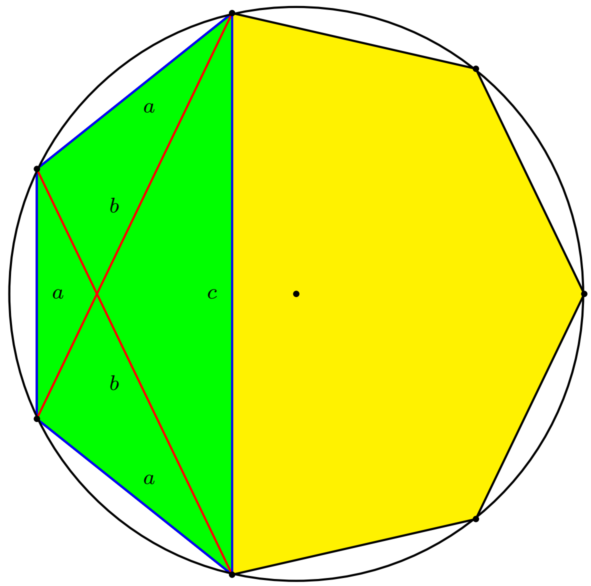
$$A : B : C = 1 : 2 : 4.$$



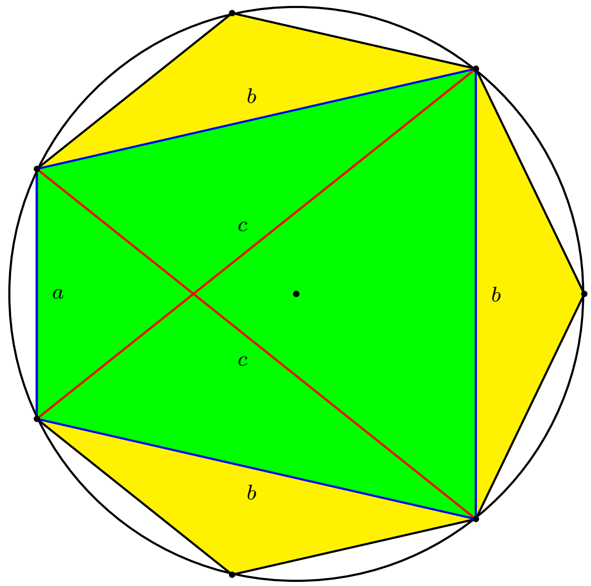


$$b^2 = a(c+a), \qquad c^2 = (a+b)b.$$

Proof. Applications of Ptolemy's theorem. □



$$b^2 = a(c + a).$$



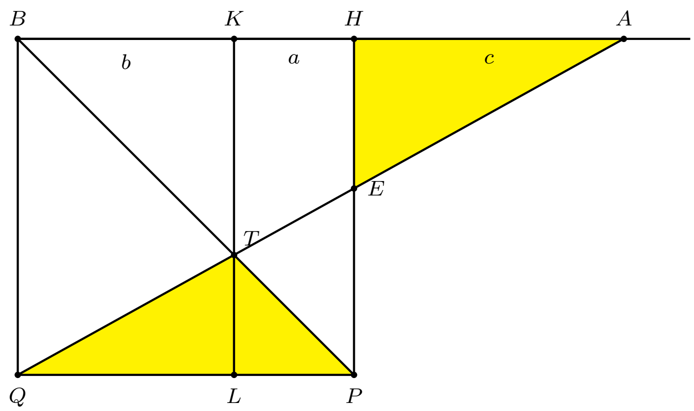
$$c^2 = (a+b)b.$$

4.3 Division of segment

Let $BHPQ$ be a square, with one side BH sufficiently extended.

Draw the diagonal BP. Place a ruler through Q, intersecting the diagonal BP at T, and the side HP at E, and the line BH at A such that the triangles AHE and TPQ have equal areas. Then,

$$b^2 = a(c + a), \qquad c^2 = (a + b)b.$$



GSP

5 Construction of regular 7-gon by angle trisection

$y^3 + py + q = 0$ with **discriminant**

$$D := \frac{p^3}{27} + \frac{q^2}{4}.$$

The cubic equation $y^3 + py + q = 0$ has
(i) three distinct real roots if $D < 0$,
(ii) three real roots with one of multiplicity ≥ 2 if $D = 0$,
(iii) one real root and two imaginary roots if $D > 0$.

The **geometric** solution of a nonsingular cubic equation reduces to one of the two problems:
(1) extraction of a real cube root if $D > 0$;
(2) angle trisection: if $D < 0$, then the equation $y^3 + py + q = 0$ can be converted into

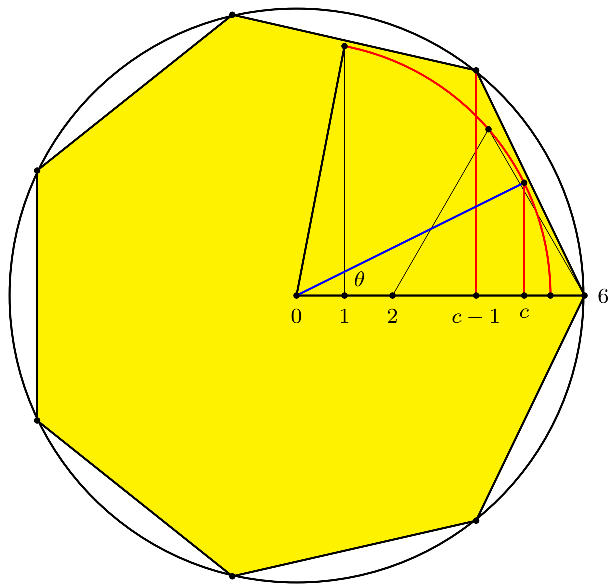
$$\cos 3\theta = f(p, q).$$

Construction of regular 7-gon by angle trisection

$$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0.$$

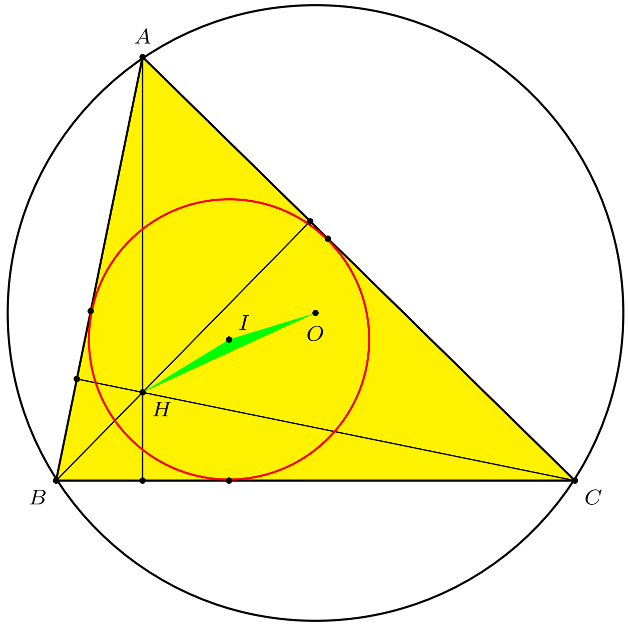
transformation	equation
$y = x + \frac{1}{x}$	$y^3 + y^2 - 2y - 1 = 0$
$y = \frac{t-1}{3}$	$t^3 - 21t - 7 = 0$
$t = r\cos\theta$	$r^3\cos^3\theta - 21r\cos\theta = 7$
$t = 2\sqrt{7}\cos\theta$	$\cos 3\theta = \frac{1}{2\sqrt{7}}$

$$\cos\frac{2\pi}{7} = \frac{2\sqrt{7}\cos\theta - 1}{6}, \qquad \cos 3\theta = \frac{1}{2\sqrt{7}}.$$



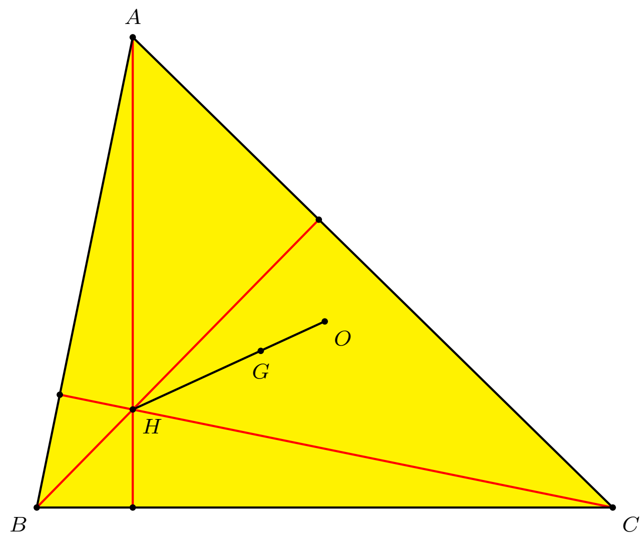
6 Euler's problem of construction of ABC from OHI

6.1 Euler's fundamental results



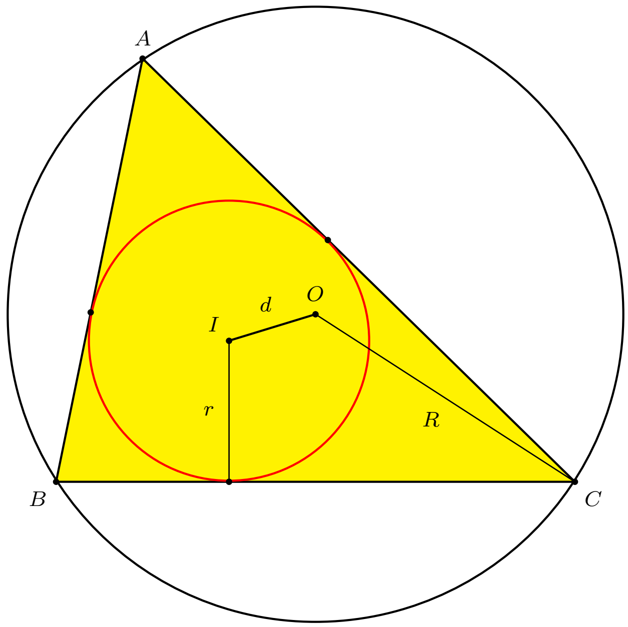
Euler line

O, H, and G are collinear, and $HG : GO = 2 : 1$.



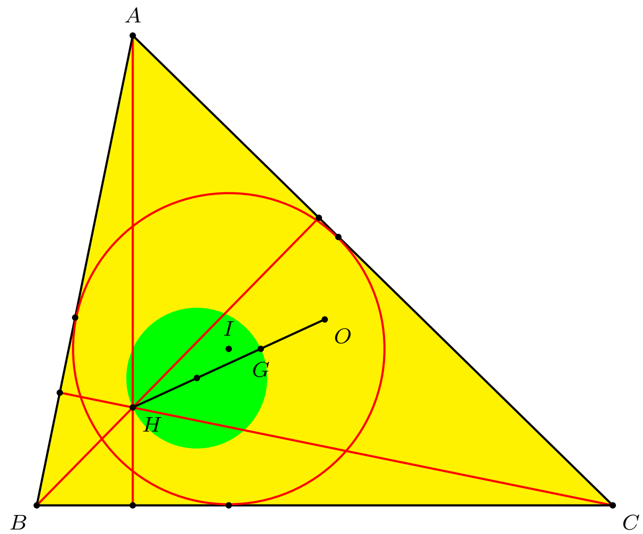
Euler's formula

$$OI^2 = R(R - 2r).$$



Condition for existence

There is a triangle ABC with given O, H, I if and only if I lies in the interior of the **orthocentroidal disk** (except the midpoint of OH).



6.2 Euler's examples

Given O, H, and I, Euler found a cubic equation whose roots are the lengths of the sides of triangle ABC.

(1) $HI^2 = 3$, $IO^2 = 2$, $HO^2 = 9$.

The sides of the triangle are the roots of

$$z^3 - \sqrt{71}z^2 + 22z - 2\sqrt{71} = 0.$$

Euler noted that the roots of this equation is equivalent to the trisection of the angle

$$\alpha = \arccos\sqrt{\frac{71}{125}} \approx 41^\circ 5' 30''.$$

The side lengths are

$$\frac{\sqrt{71}}{3} - \frac{2\sqrt{5}}{3} \cdot \cos\frac{1}{3}\alpha \quad \text{and} \quad \frac{\sqrt{71}}{3} + \frac{2\sqrt{5}}{3} \cdot \cos(60^\circ \pm \frac{1}{3}\alpha).$$

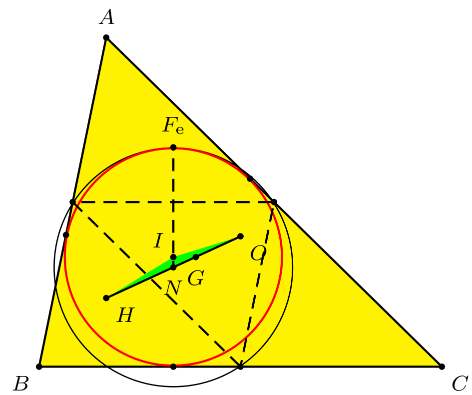
(2) Euler also showed that if $OI = IH$, the cubic polynomial factors. This means that the triangle can be constructed by ruler and compass.

6.3 Conic solution of Euler's problem

Given O, H, I, the circumcircle and the incircle of the required triangle can be constructed, making use of

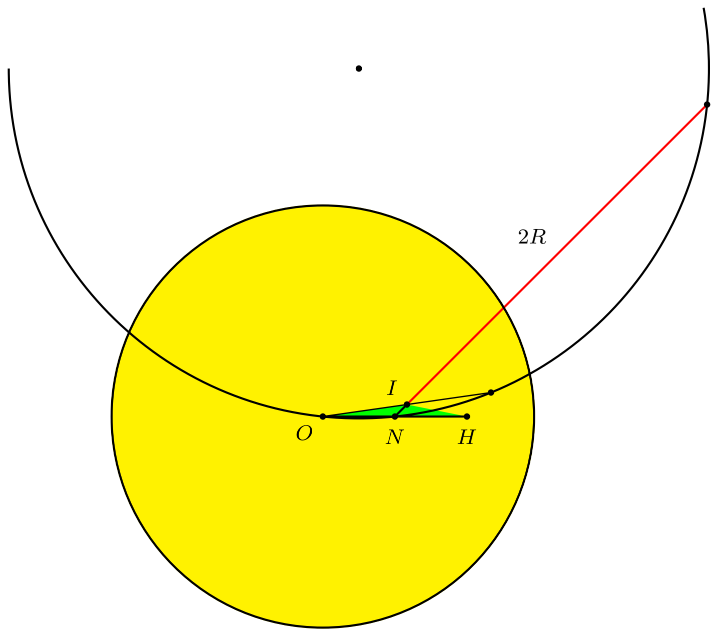
(1) Euler's formula $OI^2 = R(R - 2r)$,

(2) **Feuerbach theorem**: The nine-point circle (through the midpoints of the sides and with center the midpoint of OH) is tangent internally to the incircle.

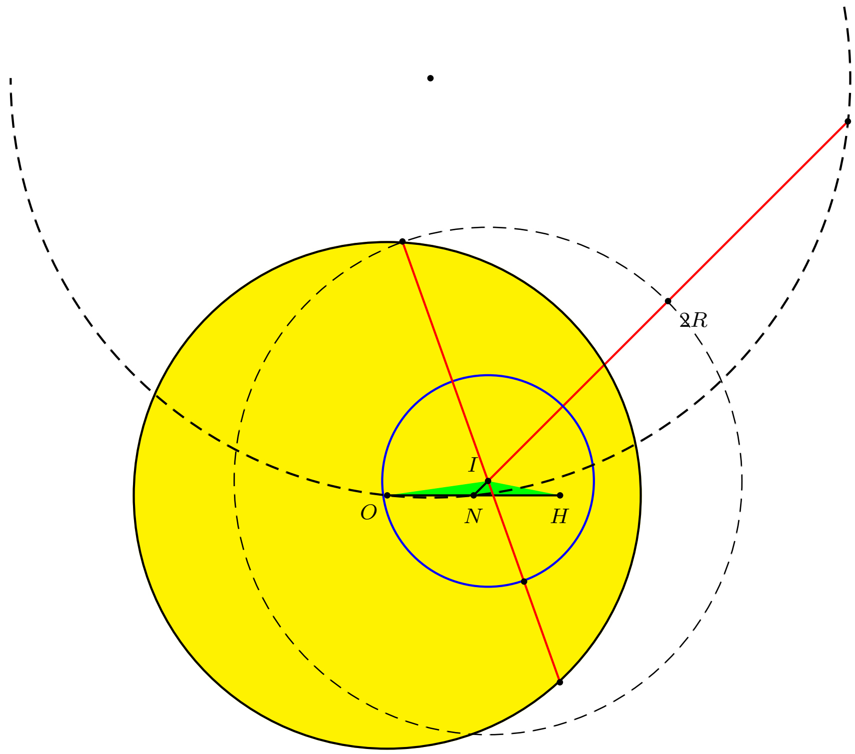


Circumradius

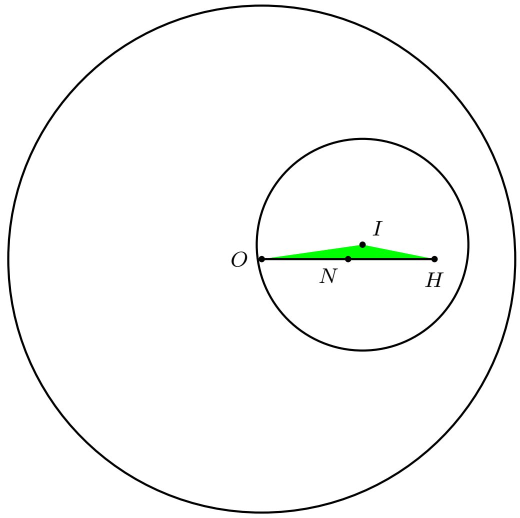
$$OI^2 = R(R-2r) \text{ and } IN = \frac{1}{2}(R-2r)$$
$$\Rightarrow OI^2 = 2R \cdot IN.$$



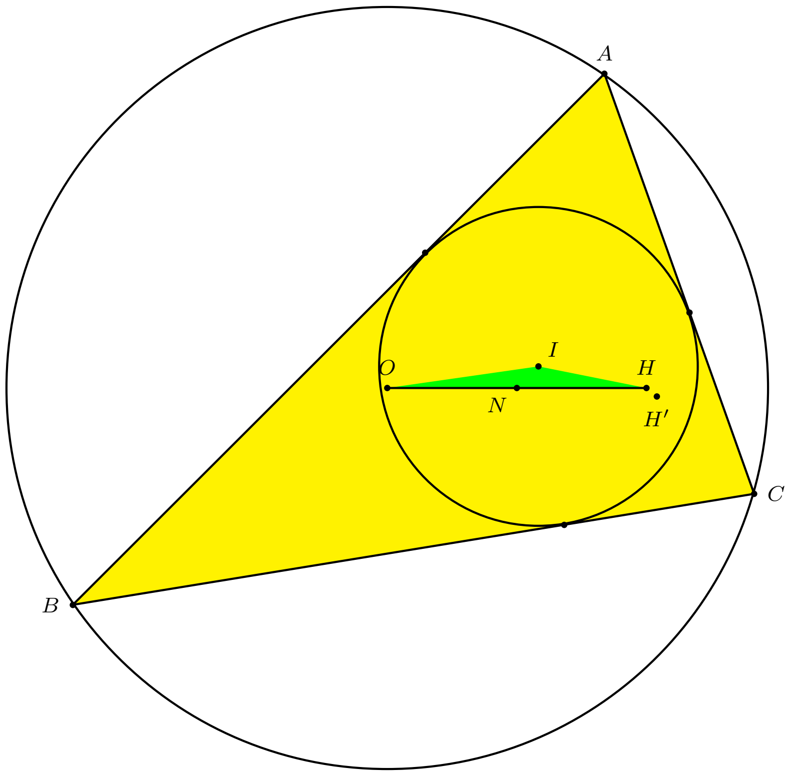
Circumcircle and incircle



Thus, given O, H, I, we can have constructed the circumcircle and the incircle.



In fact, starting with an arbitrary point A on (O), by drawing tangents, we can complete a triangle ABC with incircle (I) and circumcircle (O).



The orthocenter, however, is not always correct!

GSP

Note that $\angle HAI = \angle OAI$. If we put

$$O = (0,0), \quad H = (k,0), \quad I = (p,q),$$

the locus of point P for which $\angle HPI = \angle OPI$ is the curve

$$\begin{aligned} F := & 2qx^3 - (2p-k)x^2y + 2qxy^2 - (2p-k)y^3 \\ & - 2(p+k)qx^2 + 2(p^2-q^2)xy + 2(p-k)qy^2 \\ & + 2kpqx - k(p^2-q^2)y \\ = & 0. \end{aligned}$$

Note that $F(k,0) = 0$, *i.e.*, F contains the point H.
The circumcircle has equation

$$G := x^2 + y^2 - \frac{(p^2+q^2)^2}{(2p-k)^2 + 4q^2}.$$

We find a linear function L such that $F - L \cdot G$ has no third degree terms. With $L = 2qx - (2p - k)y - 2qk$, we have

$$F - L \cdot G = E,$$

where

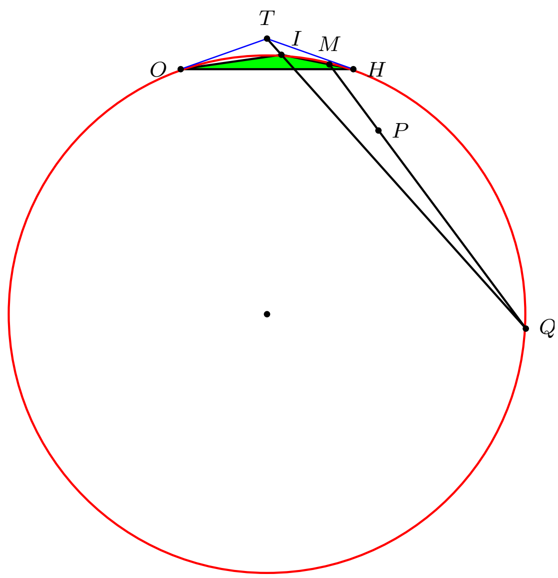
$$\begin{aligned} E := & -2pqx^2 + 2(p^2 - q^2)xy + 2pqy^2 \\ & - \frac{k^2(k - 4p)(p^2 - q^2) + k(3p^2 - 5q^2)(p^2 + q^2) + 2p(p^2 + q^2)^2}{(2p - k)^2 + 4q^2} \cdot x \\ & + \frac{2q(kp((2p - k)^2 + 4q^2) + (p^2 + q^2)^2)}{(2p - k)^2 + 4q^2} \cdot y \\ & - \frac{2k(p^2 + q^2)^2 q}{(2p - k)^2 + 4q^2}. \end{aligned}$$

The quadratic part factors as $-2(px + qy)(qx - py)$. This means that E represents a rectangular hyperbola with asympotes with slopes $\frac{q}{p}$ and $-\frac{p}{q}$.

Note that L represents a line through $H = (k, 0)$.

Therefore, the conic $E = 0$ also contains H.

The center of the rectangular hyperbola is a point which can be described geometrically as follows.

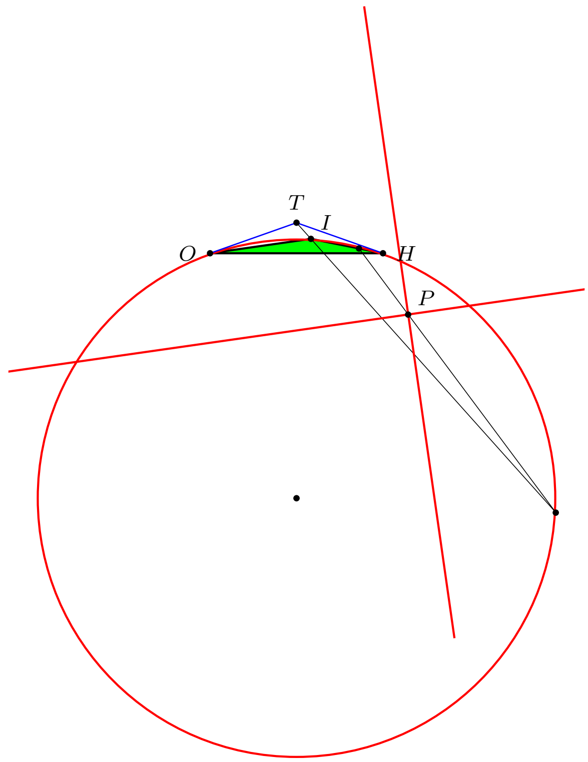


T := intersection of tangents of circumcircle of OHI at O and H,

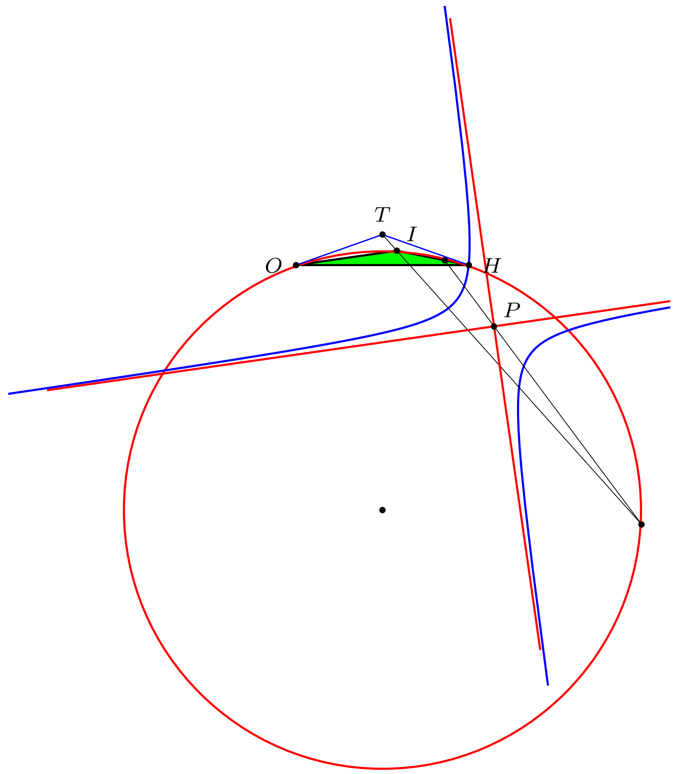
TI intersects the circle again at Q,

M is a trisection point of IH,

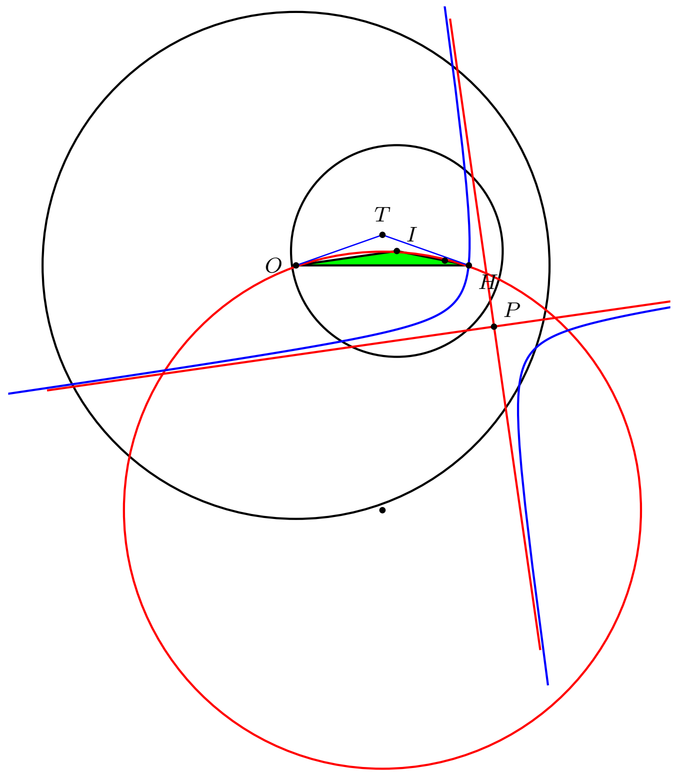
P divides MQ in the ratio $1:3$.

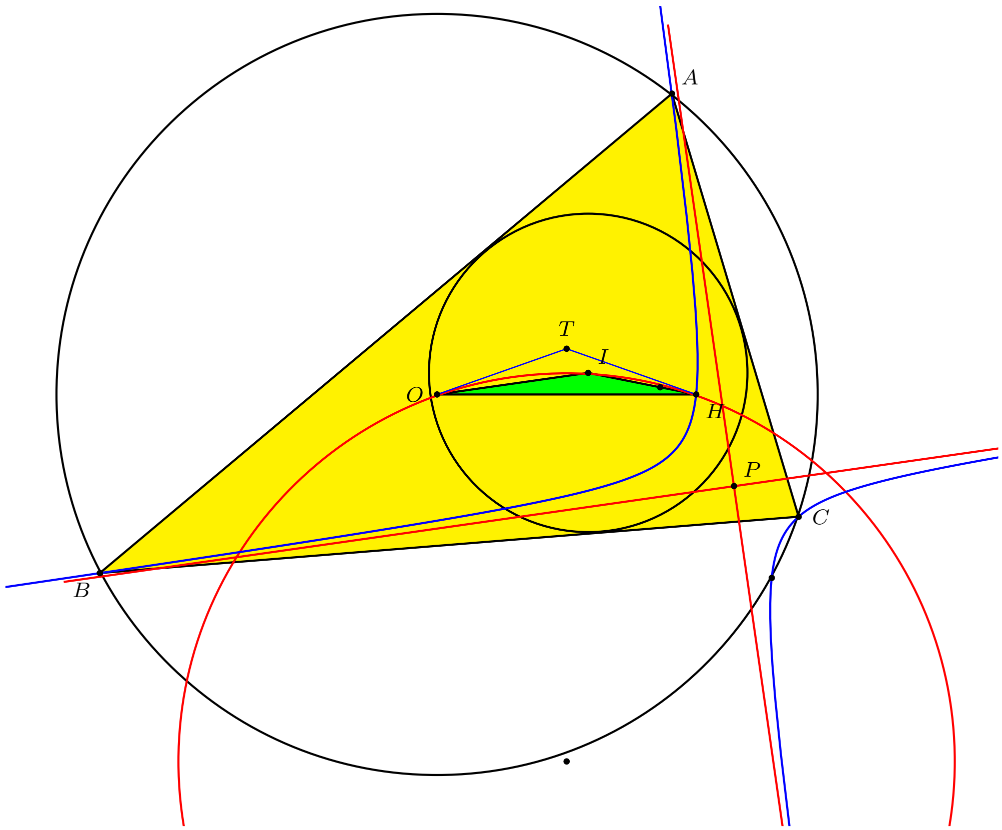

T
I
O
H
P

The rectangular hyperbola through H



The circumcircle and incircle

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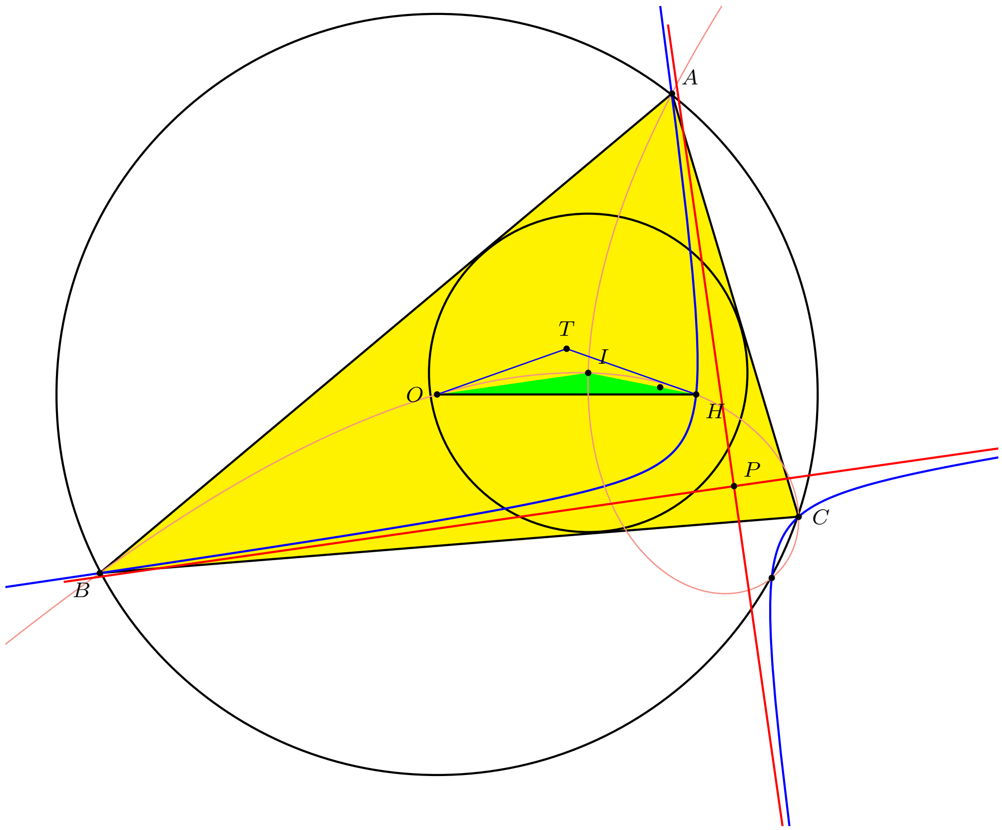
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