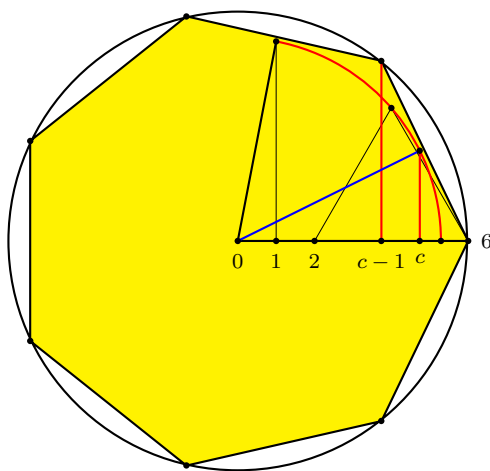


The Regular Heptagon by Angle Trisection and Other Constructions

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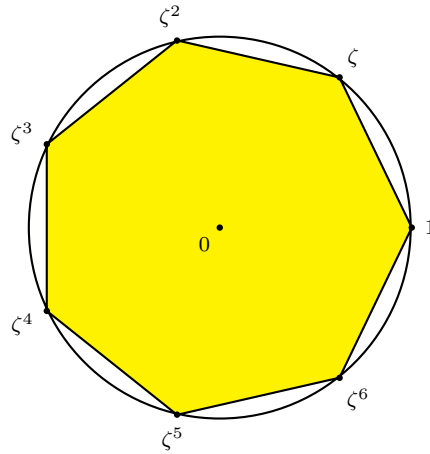
April 24, 2007



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1 Which regular n -gons are constructible?



$\zeta^k, k = 1, 2, \dots, 6$, are the roots of

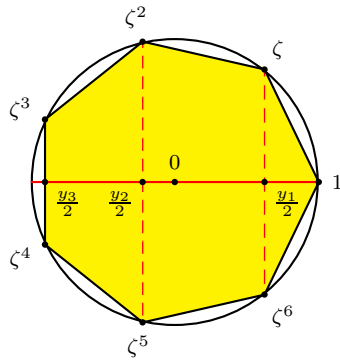
$$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0.$$

These are constructible with ruler and compass if and only if $y = x + \frac{1}{x} = 2 \cos k \cdot \frac{2\pi}{7}$ are.

$$y^3 + y^2 - 2y - 1 = 0.$$

Theorem 1. Let F be a field.

The roots of a **cubic** polynomial $p(x) \in F[x]$ are constructible by ruler and compass from F if and only if $p(x)$ has a root in F .



Corollary 2. The regular 7-gon is not constructible by ruler and compass.

Proof. The cubic polynomial $y^3 + y^2 - 2y - 1$ has no rational root, (the only possible candidates being ± 1).

□

More examples

1. $\sqrt[3]{2}$ cannot be constructed by ruler and compass.

Proof. $x^3 - 2$ does not have a rational root. \square

2. The 60° angle cannot be trisected by ruler and compass.

Proof. Since $\cos 60^\circ = \frac{1}{2}$, the trisection of 60° is equivalent to the solution of $\cos 3\theta = \frac{1}{2}$. With $x = \cos \theta$, this is

$$4x^3 - 3x = \frac{1}{2},$$

or $8x^3 - 6x - 1 = 0$. It is clear that this does not have rational roots. This shows that 60° cannot be trisected with ruler and compass. \square

Theorem 3. *A regular n -gon is constructible if and only if $n = 2^a \prod_j p_j$ where*

(i) $a \geq 0$,

(ii) if $j \geq 1$, each p_j is a Fermat prime of the form $2^{2^k} + 1$,

(iii) if $j \geq 2$, the Fermat primes p_j are all distinct.

Proof. (1) If a regular n -gon is constructible, then so is a regular $2n$ -gon.

(2) If a regular n -gon is constructible, and $m|n$, then a regular m -gon is constructible.

(3) If p is an odd prime and a regular p -gon is constructible, then $p - 1$ is a power of 2.

(4) If p is an odd prime of the form $2^k + 1$, then a regular p -gon is constructible (Gauss).

(5) If $\gcd(m, n) = 1$ and regular m - and regular n -gons are constructible, then so is a regular mn -gon.

Find integers a and b such that $am + bn = 1$, and note that

$$\frac{2\pi}{mn} = b \cdot \frac{2\pi}{m} + a \cdot \frac{2\pi}{n}.$$

It remains to show why the Fermat primes may not repeat.

(6) Let p be an odd prime divisor of n . If a regular n -gon is constructible, then p^2 cannot divide n .

□

A regular n -gon is constructible if and only if the numbers $\tan \frac{k \cdot 2\pi}{n}$, $k = 1, 2, \dots, n-1$, are constructible. These are the roots of the polynomial

$$P_n(x) = \sum_{2j \leq n-1} (-1)^{j-1} \binom{n}{2j+1} x^{2j}.$$

Proof.

$$\begin{aligned} \tan n\theta &= \frac{\sum_{2j+1 \leq n} (-1)^j \binom{n}{2j+1} \tan^{2j+1} \theta}{\sum_{2j \leq n} (-1)^j \binom{n}{2j} \tan^{2j} \theta} \\ &= \frac{\tan \theta \sum_{2j+1 \leq n} (-1)^j \binom{n}{2j+1} \tan^{2j} \theta}{\sum_{2j \leq n} (-1)^j \binom{n}{2j} \tan^{2j} \theta}. \end{aligned}$$

□

If n is odd, we may take P_n to be a monic polynomial (with leading coefficient 1).

(6) Let p be an odd prime divisor of n . If a regular n -gon is constructible, then p^2 cannot divide n .

Proof. $P_{p^2}(x) = P_p(x)Q(x)$ for a polynomial $Q(x)$.

(i) $Q(x) \in \mathbb{Z}$, and is monic.

(ii) Each coefficient of $Q(x)$, apart from the leading one, is divisible by p .

(iii) The constant term is not divisible by p^2 .

Therefore, $Q(x)$ is irreducible by Eisenstein's criterion.

Example: For $p = 3$, $P_3(x) = x^2 - 3$,

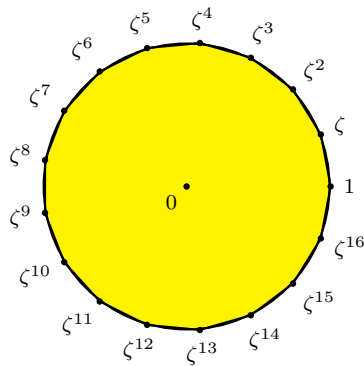
$$P_9(x) = x^8 - 36x^6 + 126x^4 - 84x^2 + 9,$$

$$Q(x) = \frac{P_9(x)}{P_3(x)} = x^6 - 33x^4 + 27x - 3.$$

Since $\deg Q = p^2 - p = p(p-1)$ is not a power of 2, the roots of $Q(x)$ are not ruler-and-compass constructible.

□

2 Gauss' Construction of the regular 17-gon

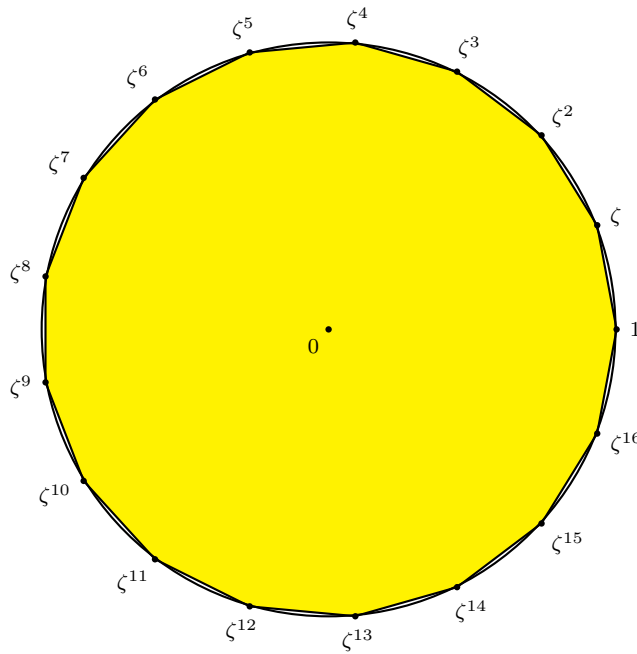


First entry of Gauss' diary:

[1] The principles upon which the division of the circle depend, and geometrical divisibility of the same into seventeen parts, etc.

[1796] March 30 Brunswick

The details appeared in Section VII of his *Disquisitiones Arithmeticae* (1801).

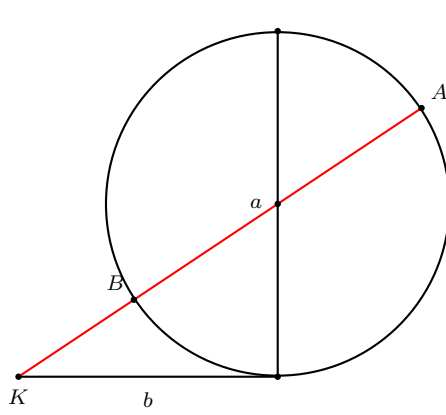


$\zeta, \zeta^2, \dots, \zeta^{15}, \zeta^{16}$ are the roots of the equation

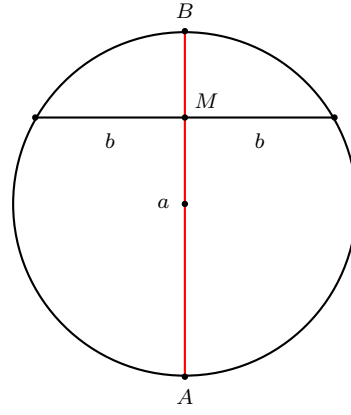
$$\frac{x^{17} - 1}{x - 1} = x^{16} + x^{15} + \dots + x + 1 = 0.$$

Gauss' construction depends on two key ideas.

(1) Solution of quadratic equations by ruler and compass:



(a) $x^2 = ax + b^2$:
 $x_1 = KA, x_2 = -KB$



(b) $x^2 + b^2 = ax$:
 $x_1 = AM, x_2 = BM$

(2) Gauss observed that, modulo 17, all the residues are powers of 3 (which is a **primitive root** for the prime 17):

k	0	2	4	6	8	10	12	14
3^k	1	9	13	15	16	8	4	2
k	1	3	5	7	9	11	13	15
3^k	3	10	5	11	14	7	12	6

If we take

$$y_1 = \zeta + \zeta^9 + \zeta^{13} + \zeta^{15} + \zeta^{16} + \zeta^8 + \zeta^4 + \zeta^2,$$

$$y_2 = \zeta^3 + \zeta^{10} + \zeta^5 + \zeta^{11} + \zeta^{14} + \zeta^7 + \zeta^{12} + \zeta^6,$$

then $y_1 + y_2 = -1$. More importantly,

the product y_1y_2 does *not* depend on the choice of ζ .

$$\begin{aligned} y_1 &= \zeta + \zeta^9 + \zeta^{13} + \zeta^{15} + \zeta^{16} + \zeta^8 + \zeta^4 + \zeta^2, \\ y_2 &= \zeta^3 + \zeta^{10} + \zeta^5 + \zeta^{11} + \zeta^{14} + \zeta^7 + \zeta^{12} + \zeta^6. \end{aligned}$$

This table gives $\zeta^h \cdot \zeta^k = \zeta^m$, $m \equiv h + k \pmod{17}$.

	1	9	13	15	16	8	4	2
3	4	12	16	1	2	11	7	5
10	11	2	6	8	9	1	14	12
5	6	14	1	3	4	13	9	7
11	12	3	7	9	10	2	15	13
14	15	6	10	12	13	5	1	16
7	8	16	3	5	6	15	11	9
12	13	4	8	10	11	3	16	14
6	7	15	2	4	5	14	10	8

Each of $1, 2, \dots, 16$ appears **exactly** 4 times.

$$\begin{aligned}y_1 + y_2 &= \zeta + \zeta^2 + \cdots + \zeta^{16} = -1, \\y_1 y_2 &= 4(\zeta + \zeta^2 + \cdots + \zeta^{16}) = -4.\end{aligned}$$

It follows that y_1 and y_2 are the roots of

$$y^2 + y - 4 = 0,$$

and are constructible. We may take

$$y_1 = \frac{-1 + \sqrt{17}}{2}, \quad y_2 = \frac{-1 - \sqrt{17}}{2}.$$

Recall

$$\begin{aligned}y_1 &= \zeta + \zeta^9 + \zeta^{13} + \zeta^{15} + \zeta^{16} + \zeta^8 + \zeta^4 + \zeta^2, \\y_2 &= \zeta^3 + \zeta^{10} + \zeta^5 + \zeta^{11} + \zeta^{14} + \zeta^7 + \zeta^{12} + \zeta^6.\end{aligned}$$

Now separate each of y_1, y_2 into two “groups” of four:

$$\begin{aligned}z_1 &= \zeta + \zeta^{13} + \zeta^{16} + \zeta^4, \\z_2 &= \zeta^9 + \zeta^{15} + \zeta^8 + \zeta^2; \\z_3 &= \zeta^3 + \zeta^5 + \zeta^{14} + \zeta^{12}, \\z_4 &= \zeta^{10} + \zeta^{11} + \zeta^7 + \zeta^6.\end{aligned}$$

Then,

$$\begin{aligned}z_1 z_2 &= (\zeta + \zeta^{13} + \zeta^{16} + \zeta^4)(\zeta^9 + \zeta^{15} + \zeta^8 + \zeta^2) \\&= \zeta + \zeta^2 + \cdots + \zeta^{16} \\&= -1; \\z_3 z_4 &= -1.\end{aligned}$$

Numbers	quadratic equation
z_1, z_2	$z^2 - y_1z - 1 = 0$
z_3, z_4	$z^2 - y_2z - 1 = 0$

Finally, further separating $z_1 = \zeta + \zeta^{13} + \zeta^{16} + \zeta^4$ into

$$t_1 = \zeta + \zeta^{16}, \quad t_2 = \zeta^{13} + \zeta^4,$$

we obtain

$$t_1 + t_2 = z_1,$$

$$t_1 t_2 = (\zeta + \zeta^{16})(\zeta^{13} + \zeta^4)$$

$$= \zeta^{14} + \zeta^5 + \zeta^{12} + \zeta^3$$

$$= z_3.$$

Therefore, $t_1 = \zeta + \zeta^{16}$ and $t_2 = \zeta^{13} + \zeta^4$ are the roots of

$$t^2 - z_1 t + z_3 = 0,$$

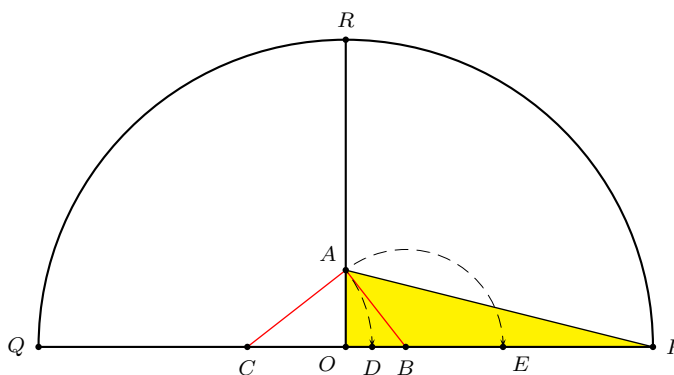
and are constructible.

2.1 An explicit construction of a regular 17-gon

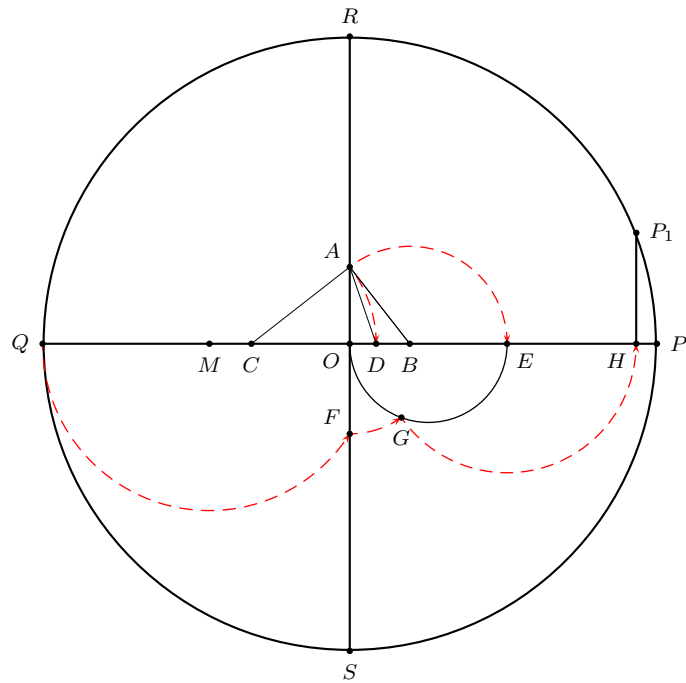
$$\cos \frac{2\pi}{17} = \frac{1}{2} (\zeta + \zeta^{16}) = \frac{1}{16} \left(\sqrt{17} - 1 + \sqrt{34 - 2\sqrt{17}} \right) + \frac{1}{8} \sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}}}.$$

Circle (O) with perpendicular diameters PQ and RS .

- (1) A on OR with $OA = \frac{1}{4}OP$.
- (2) Construct the bisectors OB and OC of angle OAP .
- (3) $D = OP \cap C(A)$ and $E = OP \cap B(A)$.

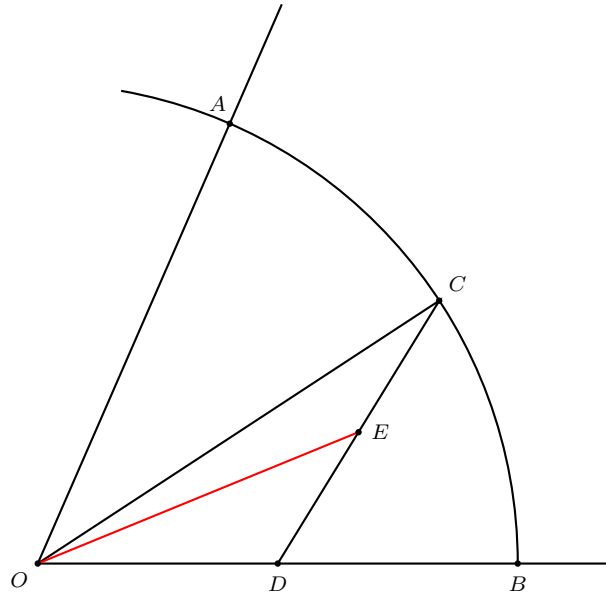


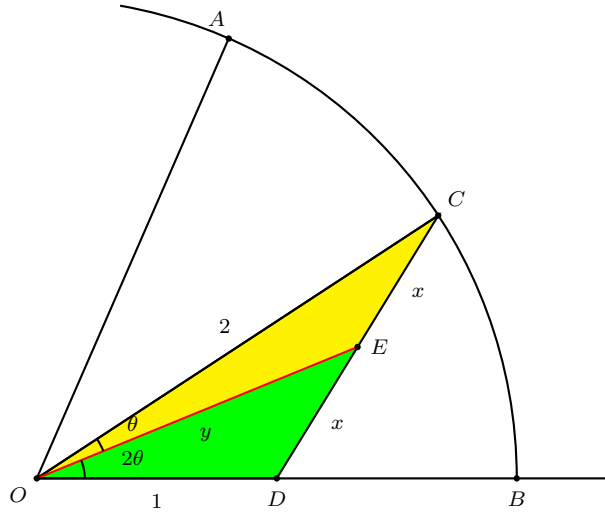
- (4) $M = \text{midpoint } M \text{ of } QD.$
- (5) $F = OS \cap M(Q).$
- (6) G on semicircle on OE , with $OG = OF.$
- (7) $H = OP \cap E(G).$
- (8) $P_1H \perp OP.$



3 Angle trisection

3.1 A false trisection





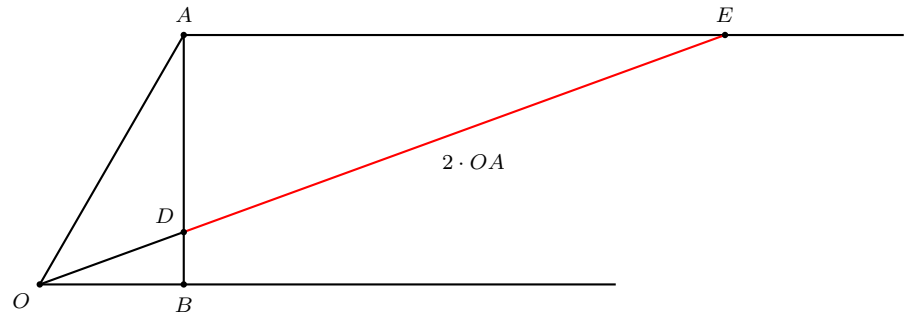
$$\begin{aligned} \text{Equal areas} &\Rightarrow y \sin 2\theta = 2y \sin \theta \\ &\Rightarrow \cos \theta = 1 \quad !!! \end{aligned}$$

In this example, $\cos AOB = \frac{2}{5}$, $\angle AOB \approx 66.422^\circ$,

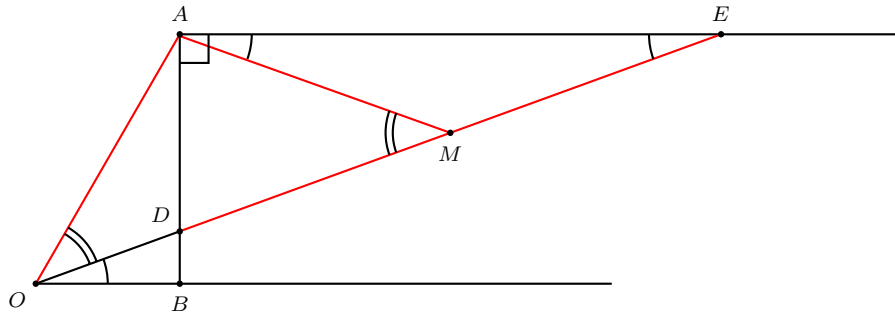
$$2\theta - \frac{1}{3}\angle AOB < 0.142^\circ.$$

3.2 Constructions with a marked ruler

(1) Pass a line through O such that the intercept between the parallel and the perpendicular at A to the line OB is $2 \cdot OA$. Then this line is a trisector of angle AOB .



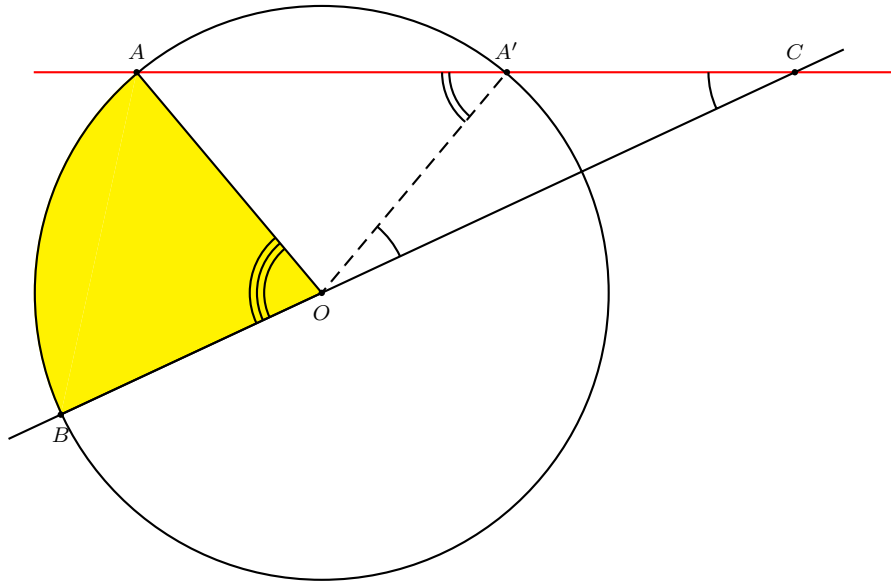
Proof.



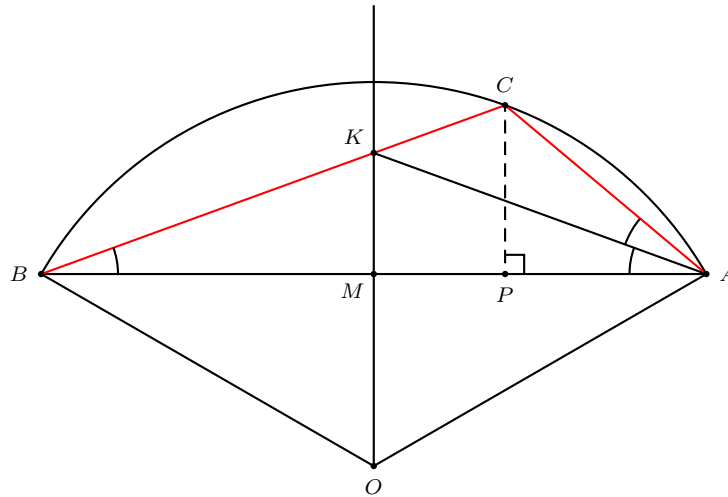
(2) Archimedes [Book of Lemmas, Proposition 8]

Construct a line through A such that the intercept between the circle and the line BO has the same length as the radius of the circle.

Then $\angle A'OC = \frac{1}{3}\angle AOB$.



3. The use of conics



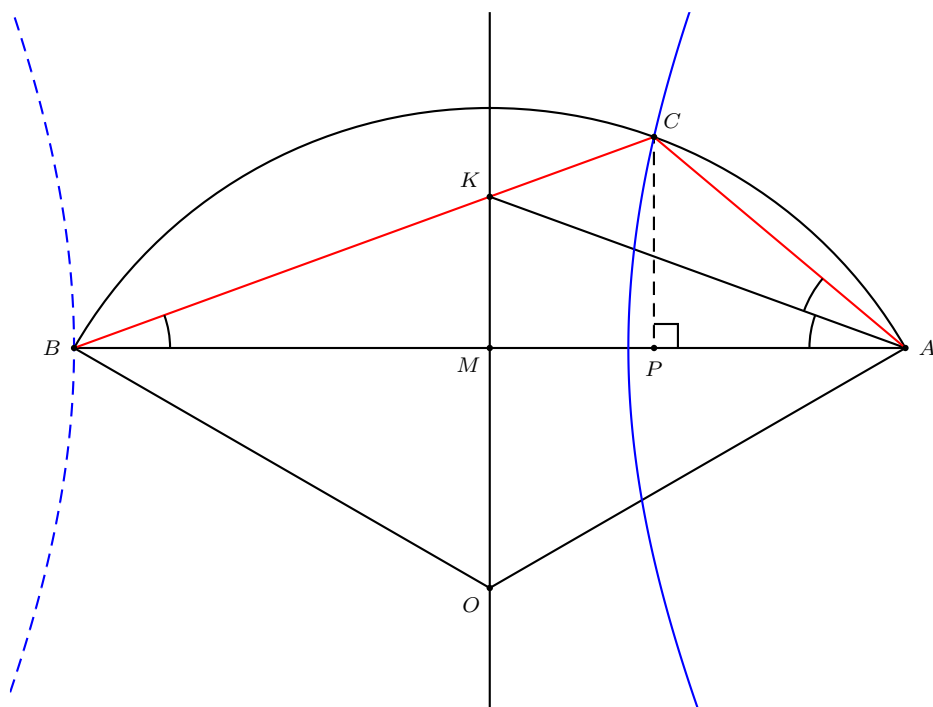
If $\angle AOB = 3\angle AOC$, then

$$AB : AC = BK : CK = BM : PM.$$

Since $AB = 2 \cdot BM$, we have $AC = 2 \cdot PM$.

Therefore, C lies on the **hyperbola** with focus A , directrix OM , and eccentricity 2.

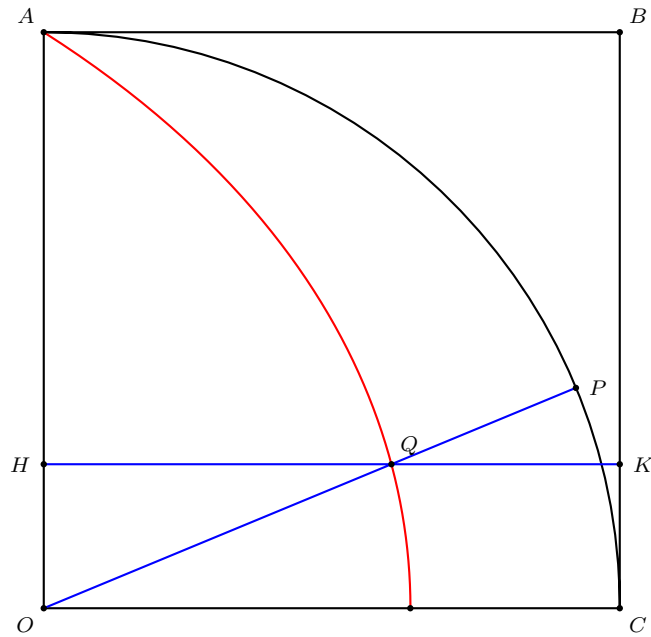
$AC = 2 \cdot PM \Rightarrow C$ lies on the **hyperbola** with focus A , directrix OM , and eccentricity 2.

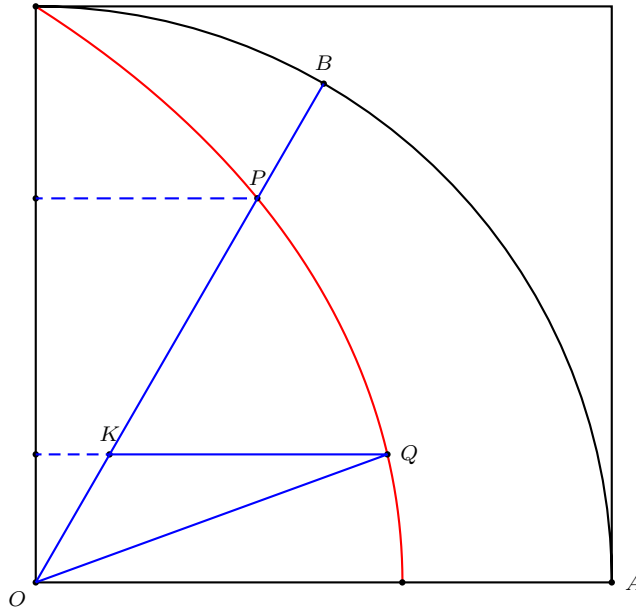


3.3 The quadratrix

A horizontal line HK (with initial position AB) falls vertically, and a radius OP (with initial position OA) rotates about O , both uniformly and arrive at OC at the same time.

The locus of the intersection $Q = HK \cap OP$ is the **quadratrix**.

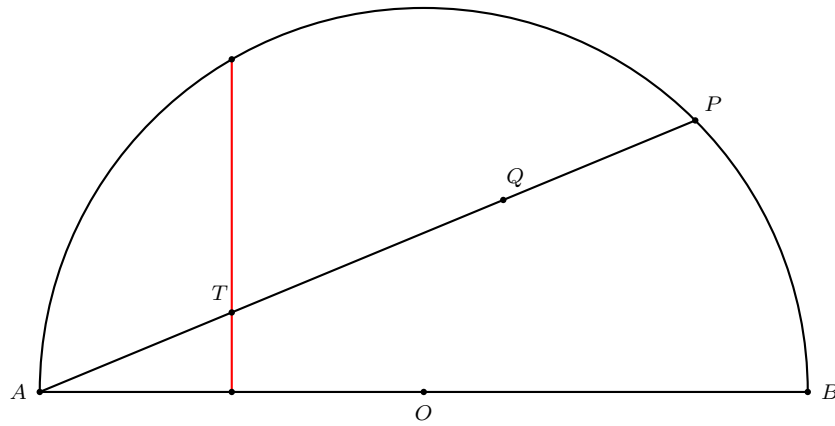




To trisect angle AOB , let OB intersect the quadratrix at P . Trisect the **segment** OP at K . Construct the parallel through K to OA to intersect the quadratrix at Q .

Then OQ is a trisector of angle AOB .

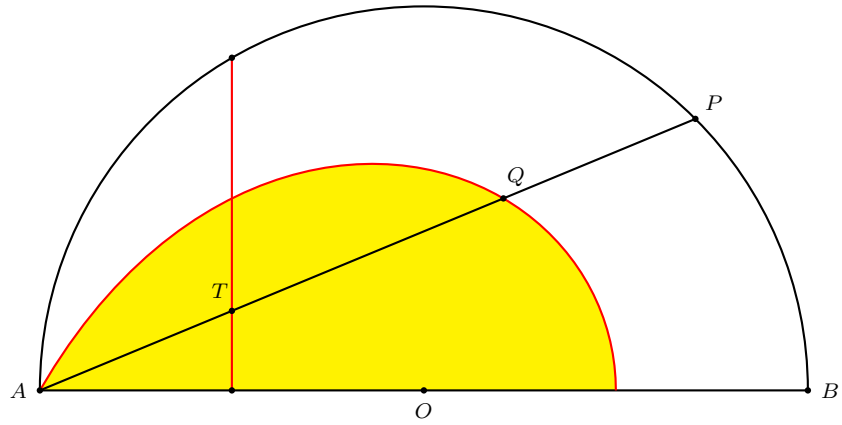
3.4 The trisectrix (MacLaurin)



$T = AP \cap$ perpendicular bisector of OA .

$AT = QP$.

The locus of Q is the **trisectrix**.

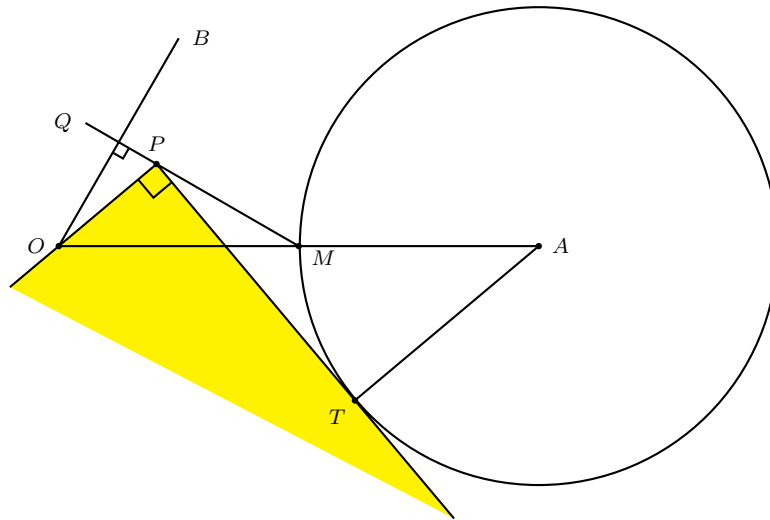


OP is a trisector of angle QOB .

3.5 A recent trisection

D. A. Brooks, A new method of trisection, *College Math. Journal*, 38 (2007) 78–81.

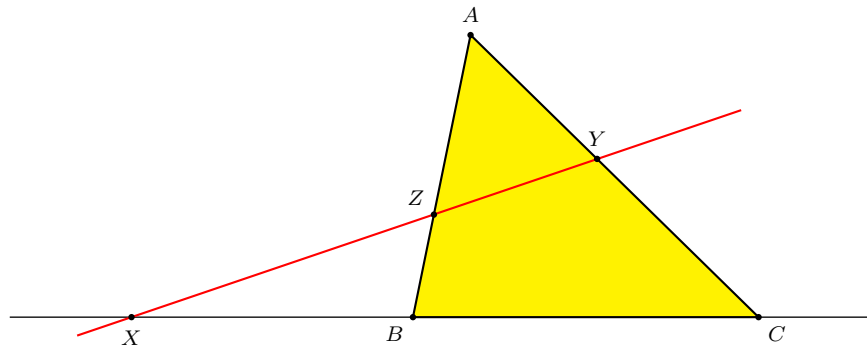
Given an acute angle AOB , let M be the midpoint of OA . Construct (i) the circle, center M , passing through A , (ii) the perpendicular MQ from M to OB . If P is a point on MQ such that the tangent from P to the circle makes a right angle with OP , then OP is a trisector of angle OAB .

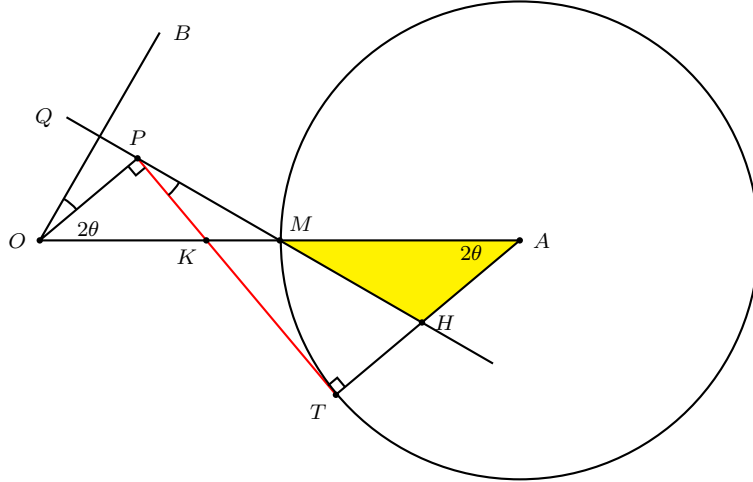


Consider a triangle ABC with points X, Y, Z on the side lines BC, CA, AB respectively.

Theorem 4 (Menelaus). *The points X, Y, Z are collinear if and only if*

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1.$$





Proof. Let $\angle POA = \angle OAT = 2\theta$. We show that $\angle HPT = \theta$. It will follow that $\angle BOP = \theta$, and OP is a trisector of angle AOB .

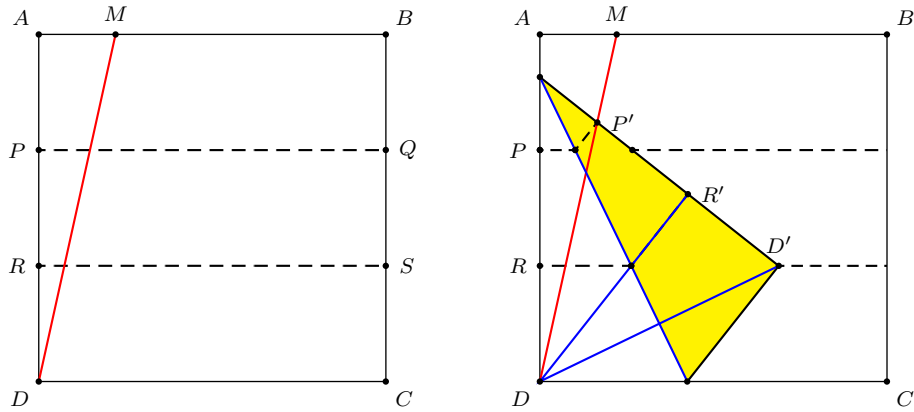
Assume unit radius for the circle. Applying Menelaus' theorem to triangle MHA with transversal TKP , we have $\frac{HT}{TA} \cdot \frac{AK}{KM} \cdot \frac{MP}{PH} = -1$. Therefore,

$$\frac{HT}{TA} = -\frac{PH}{MP} \cdot \frac{KM}{AK} = -\frac{OA}{MO} \cdot \frac{AK - AM}{AK} = 2 \cdot \frac{\sec 2\theta - 1}{\sec 2\theta} = 2(1 - \cos 2\theta) = 4 \sin^2 \theta.$$

Since $TA = 1$, we have $TH = 4 \sin^2 \theta$. Now, $PT = 2 \sin 2\theta = 4 \sin \theta \cos \theta$. This gives $\tan HPT = \frac{TH}{PT} = \tan \theta$. \square

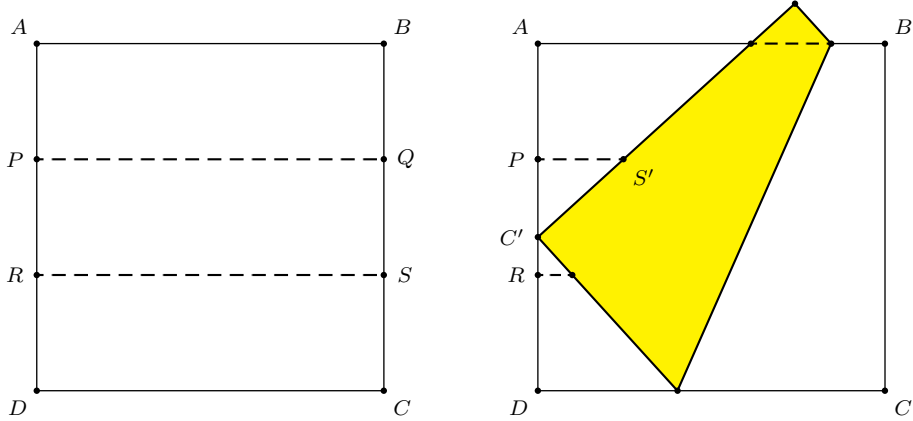
3.6 Angle trisection by paper folding

B. Casselman, If Euclid had been Japanese, *Notices of AMS*, 54 (2007) May issue, 626–628.



(2) Cubic root of 2 by paper folding

A paper square $ABCD$ is divided into three strips of equal area by the parallel lines PQ and RS . The square is then folded so that C falls on AD and S falls on PQ (as C' in the second diagram). Then $\frac{AC'}{C'D} = \sqrt[3]{2}$.

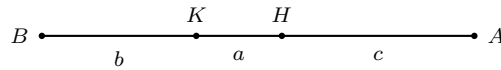


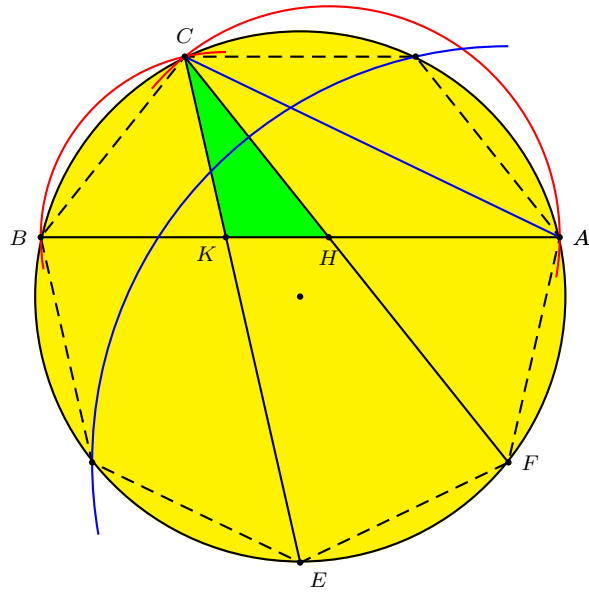
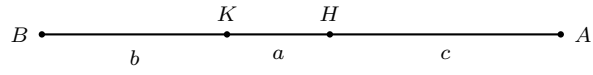
4 Archimedes' construction of the regular heptagon (Arabic tradition)

4.1 Archimedes' construction

Construct a segment AB with division points H and K such that

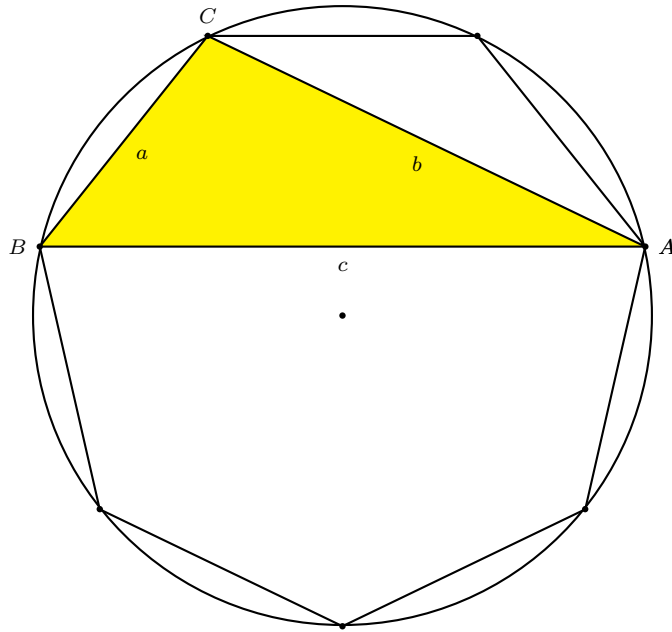
$$b^2 = a(c + a), \quad c^2 = (a + b)b.$$

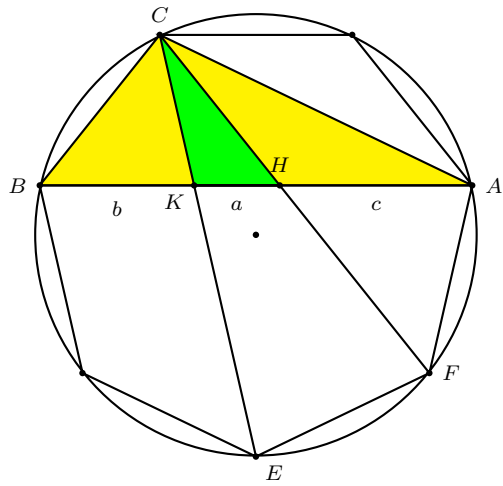




4.2 The heptagonal triangle

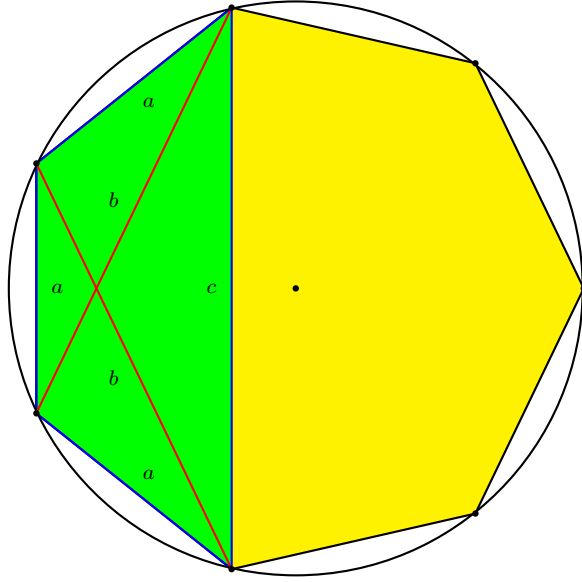
$$A : B : C = 1 : 2 : 4.$$



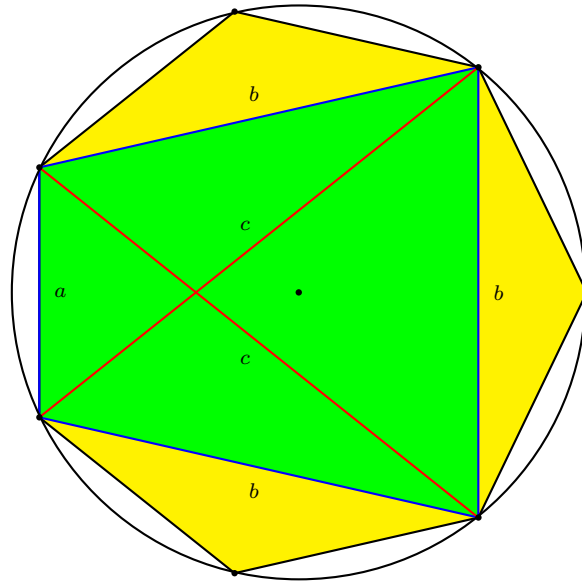


$$b^2 = a(c + a), \quad c^2 = (a + b)b.$$

Proof. Applications of Ptolemy's theorem. □



$$b^2 = a(c + a).$$



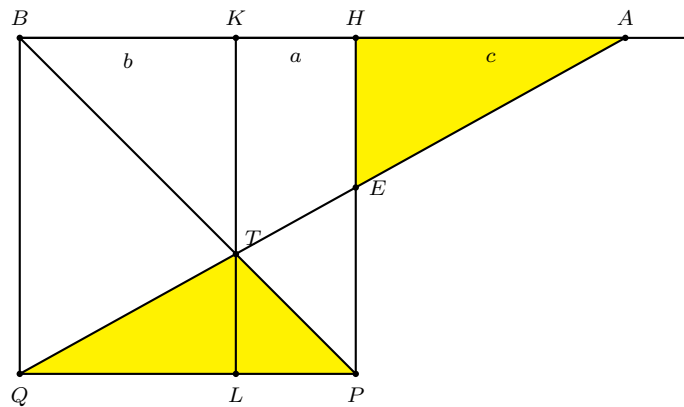
$$c^2 = (a + b)b.$$

4.3 Division of segment

Let $BHPQ$ be a square, with one side BH sufficiently extended.

Draw the diagonal BP . Place a ruler through Q , intersecting the diagonal BP at T , and the side HP at E , and the line BH at A such that the triangles AHE and TPQ have equal areas. Then,

$$b^2 = a(c + a), \quad c^2 = (a + b)b.$$



GSP

5 Construction of regular 7-gon by angle trisection

$y^3 + py + q = 0$ with **discriminant**

$$D := \frac{p^3}{27} + \frac{q^2}{4}.$$

The cubic equation $y^3 + py + q = 0$ has

- (i) three distinct real roots if $D < 0$,
- (ii) three real roots with one of multiplicity ≥ 2 if $D = 0$,
- (iii) one real root and two imaginary roots if $D > 0$.

The **geometric** solution of a nonsingular cubic equation reduces to one of the two problems:

- (1) extraction of a real cube root if $D > 0$;
- (2) angle trisection: if $D < 0$, then the equation $y^3 + py + q = 0$ can be converted into

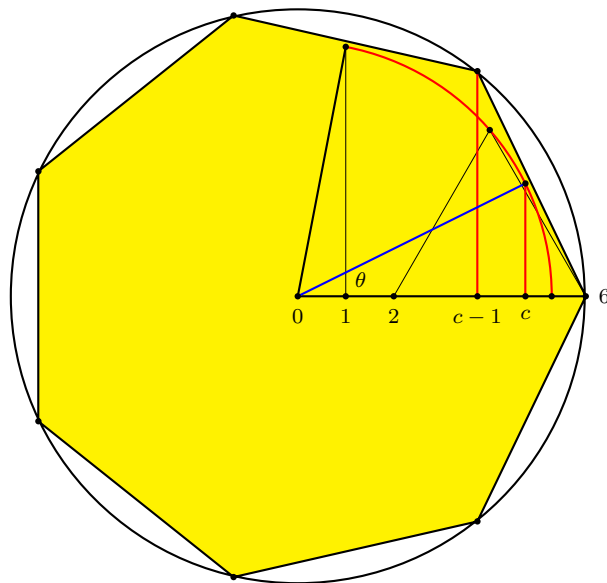
$$\cos 3\theta = f(p, q).$$

Construction of regular 7-gon by angle trisection

$$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0.$$

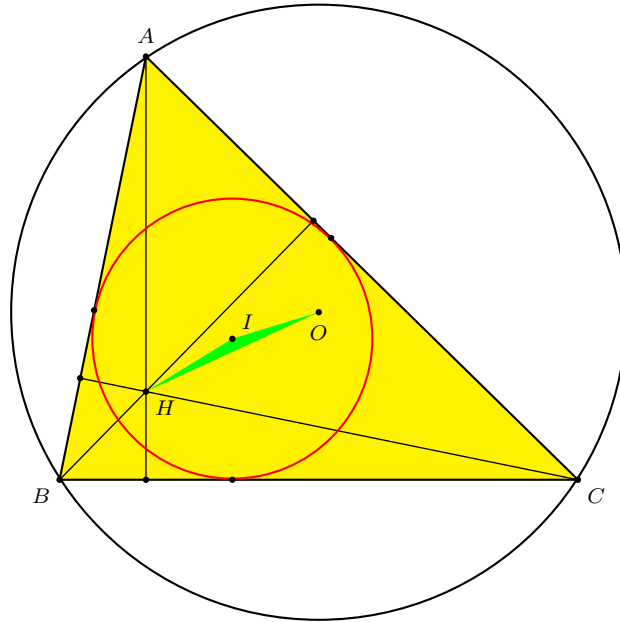
transformation	equation
$y = x + \frac{1}{x}$	$y^3 + y^2 - 2y - 1 = 0$
$y = \frac{t-1}{3}$	$t^3 - 21t - 7 = 0$
$t = r \cos \theta$	$r^3 \cos^3 \theta - 21r \cos \theta = 7$
$t = 2\sqrt{7} \cos \theta$	$\cos 3\theta = \frac{1}{2\sqrt{7}}$

$$\cos \frac{2\pi}{7} = \frac{2\sqrt{7} \cos \theta - 1}{6}, \quad \cos 3\theta = \frac{1}{2\sqrt{7}}.$$



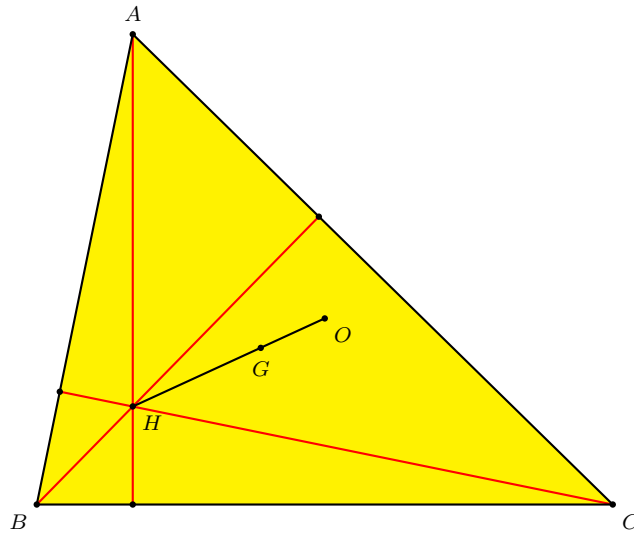
6 Euler's problem of construction of ABC from OHI

6.1 Euler's fundamental results



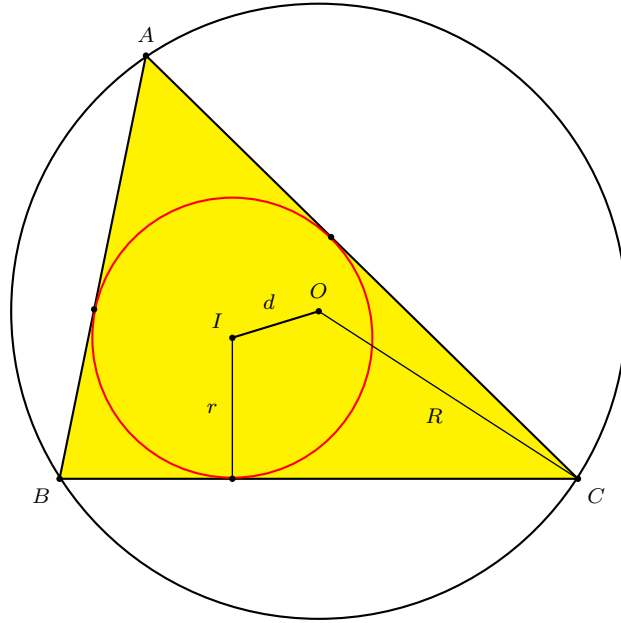
Euler line

O , H , and G are collinear, and $HG : GO = 2 : 1$.



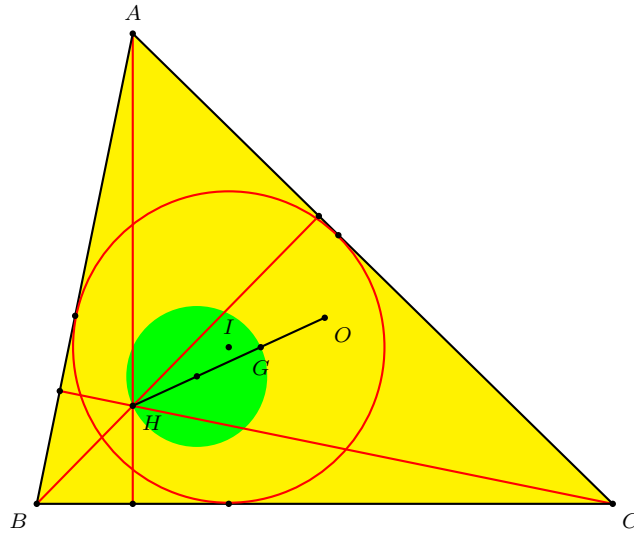
Euler's formula

$$OI^2 = R(R - 2r).$$



Condition for existence

There is a triangle ABC with given O, H, I if and only if I lies in the interior of the **orthocentroidal disk** (except the midpoint of OH).



6.2 Euler's examples

Given O , H , and I , Euler found a cubic equation whose roots are the lengths of the sides of triangle ABC .

$$(1) HI^2 = 3, IO^2 = 2, HO^2 = 9.$$

The sides of the triangle are the roots of

$$z^3 - \sqrt{71}z^2 + 22z - 2\sqrt{71} = 0.$$

Euler noted that the roots of this equation is equivalent to the trisection of the angle

$$\alpha = \arccos \sqrt{\frac{71}{125}} \approx 41^\circ 5' 30''.$$

The side lengths are

$$\frac{\sqrt{71}}{3} - \frac{2\sqrt{5}}{3} \cdot \cos \frac{1}{3}\alpha \quad \text{and} \quad \frac{\sqrt{71}}{3} + \frac{2\sqrt{5}}{3} \cdot \cos(60^\circ \pm \frac{1}{3}\alpha).$$

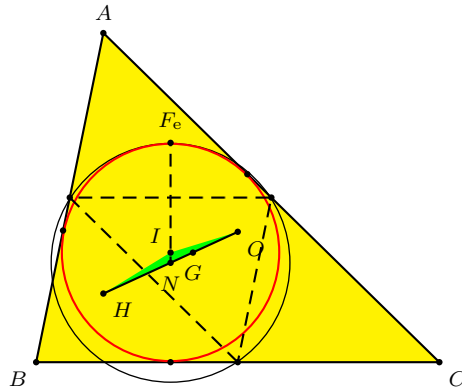
(2) Euler also showed that if $OI = IH$, the cubic polynomial factors. This means that the triangle can be constructed by ruler and compass.

6.3 Conic solution of Euler's problem

Given O , H , I , the circumcircle and the incircle of the required triangle can be constructed, making use of

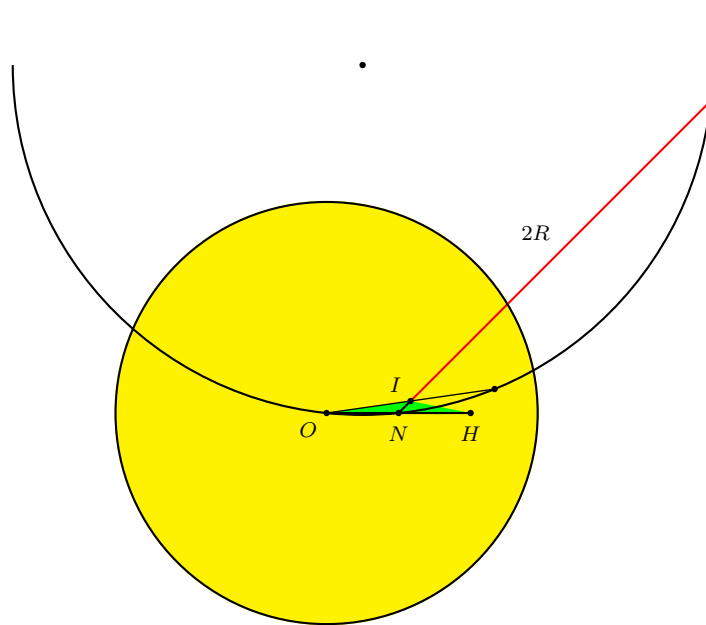
(1) Euler's formula $OI^2 = R(R - 2r)$,

(2) **Feuerbach theorem**: The nine-point circle (through the midpoints of the sides and with center the midpoint of OH) is tangent internally to the incircle.

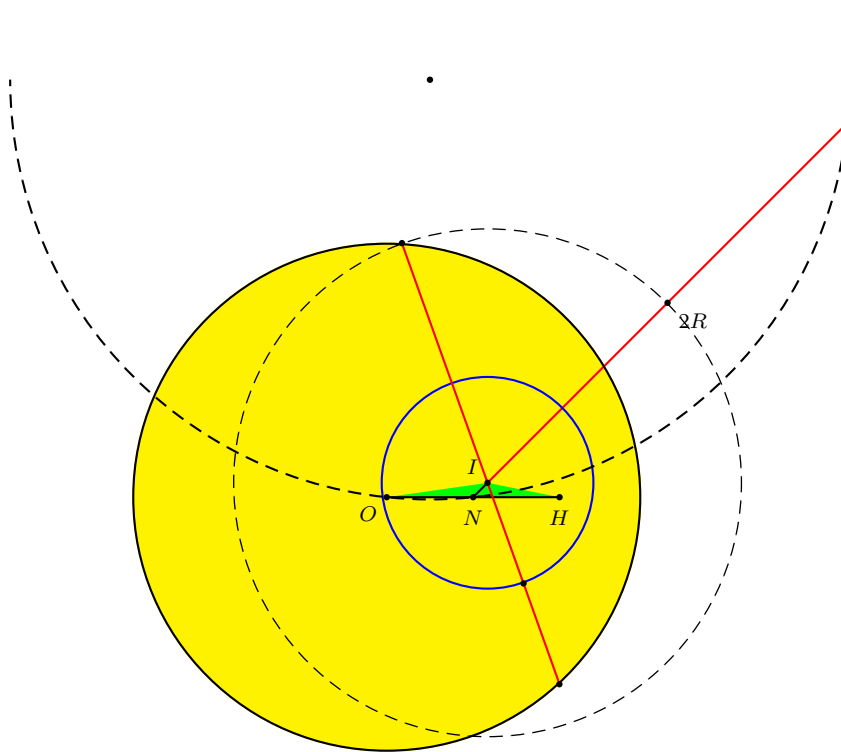


Circumradius

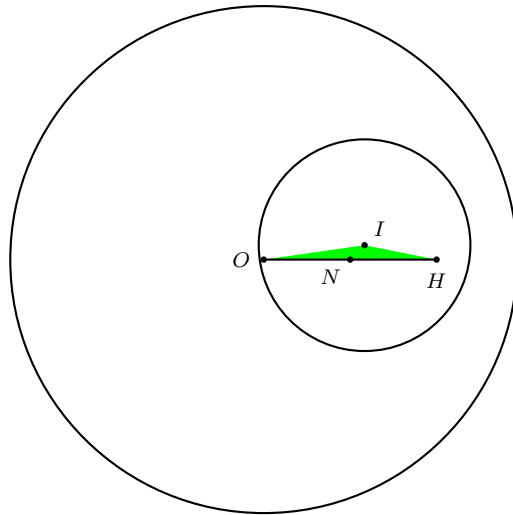
$$OI^2 = R(R - 2r) \text{ and } IN = \frac{1}{2}(R - 2r)$$
$$\Rightarrow OI^2 = 2R \cdot IN.$$



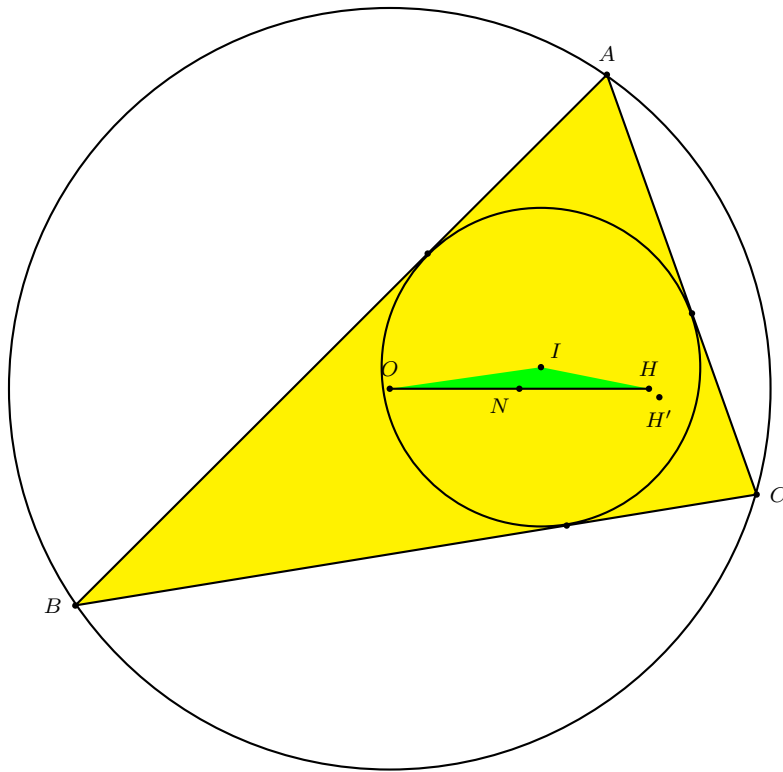
Circumcircle and incircle



Thus, given O , H , I , we can have constructed the circumcircle and the incircle.



In fact, starting with an arbitrary point A on (O) , by drawing tangents, we can complete a triangle ABC with incircle (I) and circumcircle (O) .



The orthocenter, however, is not always correct!

GSP

Note that $\angle HAI = \angle OAI$. If we put

$$O = (0, 0), \quad H = (k, 0), \quad I = (p, q),$$

the locus of point P for which $\angle HPI = \angle OPI$ is the curve

$$\begin{aligned} F &:= 2qx^3 - (2p - k)x^2y + 2qxy^2 - (2p - k)y^3 \\ &\quad - 2(p + k)qx^2 + 2(p^2 - q^2)xy + 2(p - k)qy^2 \\ &\quad + 2kpqx - k(p^2 - q^2)y \\ &= 0. \end{aligned}$$

Note that $F(k, 0) = 0$, *i.e.*, F contains the point H .
The circumcircle has equation

$$G := x^2 + y^2 - \frac{(p^2 + q^2)^2}{(2p - k)^2 + 4q^2}.$$

We find a linear function L such that $F - L \cdot G$ has no third degree terms. With $L = 2qx - (2p - k)y - 2qk$, we have

$$F - L \cdot G = E,$$

where

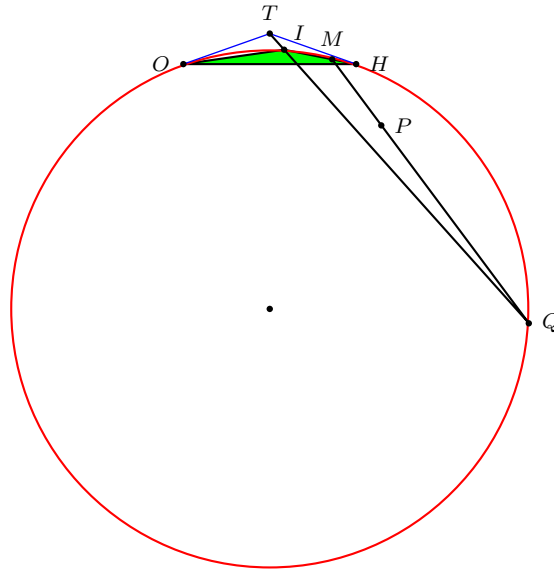
$$\begin{aligned} E := & -2pqx^2 + 2(p^2 - q^2)xy + 2pqy^2 \\ & - \frac{k^2(k - 4p)(p^2 - q^2) + k(3p^2 - 5q^2)(p^2 + q^2) + 2p(p^2 + q^2)^2}{(2p - k)^2 + 4q^2} \cdot x \\ & + \frac{2q(kp((2p - k)^2 + 4q^2) + (p^2 + q^2)^2)}{(2p - k)^2 + 4q^2} \cdot y \\ & - \frac{2k(p^2 + q^2)^2q}{(2p - k)^2 + 4q^2}. \end{aligned}$$

The quadratic part factors as $-2(px + qy)(qx - py)$. This means that E represents a rectangular hyperbola with asymptotes with slopes $\frac{q}{p}$ and $-\frac{p}{q}$.

Note that L represents a line through $H = (k, 0)$.

Therefore, the conic $E = 0$ also contains H .

The center of the rectangular hyperbola is a point which can be described geometrically as follows.

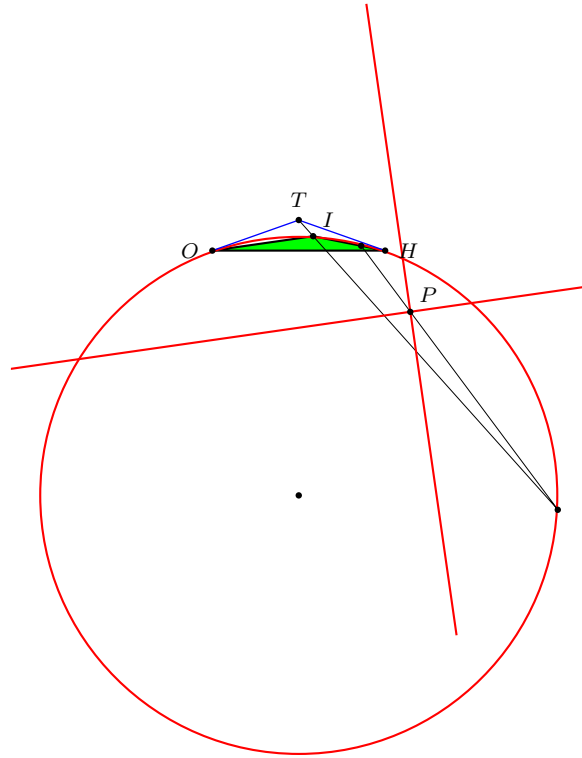


$T :=$ intersection of tangents of circumcircle of OHI at O and H ,

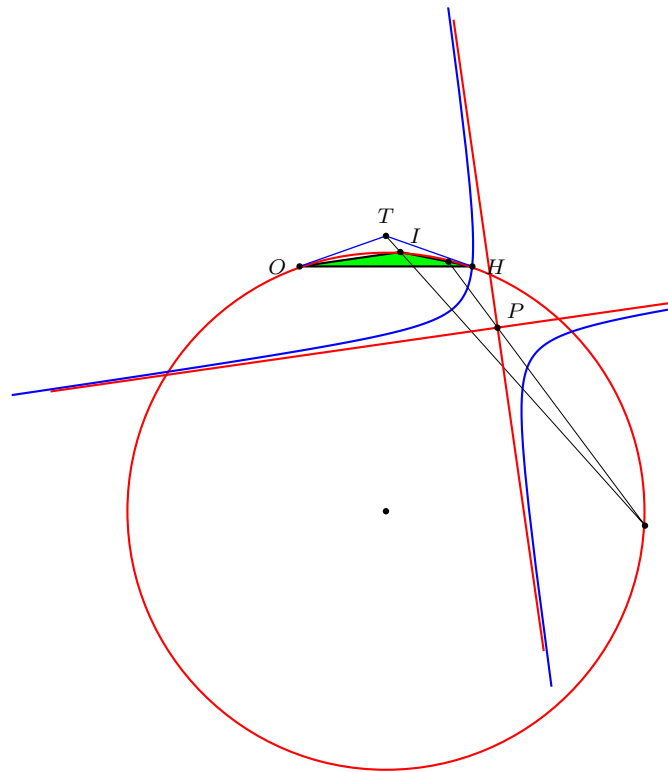
TI intersects the circle again at Q ,

M is a trisection point of IH ,

P divides MQ in the ratio $1 : 3$.



The rectangular hyperbola through H



The circumcircle and incircle

