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Conic Construction of a Triangle From Its Incenter, Nine-point Center, and a Vertex

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Abstract. We construct a triangle given its incenter, nine-point center and a vertex by locating the circumcenter as an intersection of two rectangular hyperbolas. Some special configurations leading to solutions constructible with ruler and compass are studied. The related problem of construction of a triangle given its circumcenter, incenter, and one vertex is revisited, and it is established that such a triangle exists if and only if the incenter lies inside the cardioid relative to the circumcircle.

Key Words: triangle geometry, construction problems, nine-point center, rectangular hyperbolas

MSC 2010: 51M15, 51M04

1. Introduction

In this note we solve the construction problem of a triangle ABC given its incenter I, its nine-point center N, and one vertex A. This is Problem 35 in Harold Connelly's list of construction problems [1]. We shall show that the triangle is in general not constructible with ruler and compass, but can be easily obtained by intersecting conics. Some special configurations of I, N, A from which the triangle can be constructible with ruler and compass are studied in details in §§ 2, 6 − 8.

2. The isosceles case

We assume equal lengths $AB = AC$. In this case, I, N, A are collinear. The incircle and the nine-point circle are tangent to each other internally at the midpoint of the base BC. We set up a rectangular coordinate system with N at the origin, I at $(p, 0)$, and A at $(a, 0)$, and

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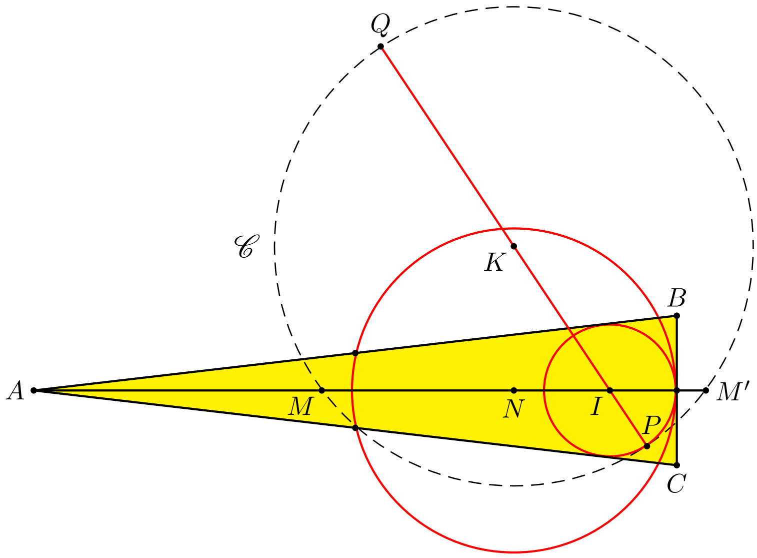

Figure 1: Construction in the isosceles case with collinear $\{I, N, A\}$

seek the inradius r. We shall assume $p > 0$ so that it is the distance between I and N. The circumradius is $R = 2(p+r)$. On the other hand, $\sin\frac{A}{2} = \frac{r}{p-a}$. The circumradius is also given by

$$R = \frac{2(p-a+r)\tan\frac{A}{2}}{2\sin A} = \frac{p-a+r}{2\cos^2\frac{A}{2}} = \frac{(p-a+r)}{2\left(1-\left(\frac{r}{p-a}\right)^2\right)} = \frac{(p-a)^2}{2(p-a-r)}.$$

Equating this to $2(p+r)$, and simplifying, we have

$$f(r) := 4r^2 + 4ar + (a-p)(a+3p) = 0. \tag{1}$$

This equation has two real roots if and only if $a \le \frac{3p}{2}$.

(i) If $p \le a \le \frac{3p}{2}$, both roots are negative.

(ii) If $-3p < a < p$, there is exactly one positive root r. Since $f(0) < 0$ and $f(p-a) = (p-a)^2 > 0$, the positive root $r < p-a$. There is a unique solution.

(iii) If $a < -3p$, this equation has two positive roots. Since

$$f(0) = f(-a) = (a-p)(a+3p) > 0 \quad \text{and} \quad f\left(\frac{a}{2}\right) = p(2a-3p) < 0,$$

the two roots are in the range $0 < r < -a$. Rewriting (1) as

$$r(-a-r) = \frac{p-a}{2}\cdot\frac{-a-3p}{2},$$

we are led to the following construction:

Let M be the midpoint of IA, and M' the reflection of M in N.

(1) Construct a circle $\mathscr{C}$ of diameter $-a$ (> 0) passing through M and M'. Its center is a point K on the perpendicular bisector of MM'.

(2) Join IK to intersect the circle $\mathscr{C}$ at P and Q (see Fig. 1).

Each of the circles, center I, passing through P (respectively Q), is the incircle of an isosceles triangle with nine-point center N.

3. Construction of triangle from O, I, A

In the general case we shall solve the construction problem of triangle ABC from I, N, A by locating the circumcenter O. In the present section we revisit the construction problem of a triangle from its circumcenter O, incenter I, and one vertex A. The problem has been solved, for example, in [5, § 4.1]; see also [3]. Here, we study the locus of I (relative to O and A) for which an inscribed triangle exists, and distinguish between the possibilities of I as the incenter or one of the excenters of the triangle. We adopt the notation $P(Q)$ for the circle, center P, passing through the point Q.

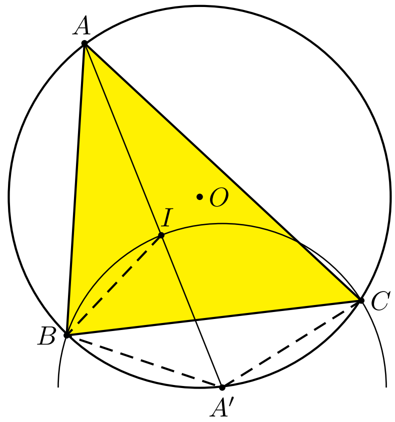

Figure 2: Illustration to Proposition 1

Proposition 1. *Given distinct points O, A, and I, construct*

(1) *the circle $O(A)$ to intersect the line AI at a point A' different from A,*

(2) *the circle $A'(I)$.*

If the two circles intersect at two distinct points B and C, then I is the incenter or an excenter of triangle ABC according as it lies inside or outside the circle $O(A)$.

Proof: We shall only prove the case when I is inside the circle. With reference to Fig. 2, we have

(i) $\angle BAI = \angle CAI = \frac{A}{2}$ since $A'B = A'C$,

(ii) $\angle A'BC = \angle A'AC = \angle A'AB = \frac{A}{2}$, and $\angle A'BI = \angle A'IB = \frac{A+B}{2}$.

From these,

$$\angle CBI = \angle A'BI - \angle A'BC = \angle A'IB - \angle A'AB = \angle ABI.$$

Therefore, BI bisects angle B. Similarly, CI bisects angle C, and I is the incenter of triangle ABC. □

Proposition 2. *Given distinct points O and A, and a point I not on the circle $O(A)$, there is a triangle ABC with circumcenter O and incenter (or excenter) I if and only if I lies inside the cardioid with cusp A relative to the circle $O(A)$.*

Proof: Set up a rectangular coordinate system with A at the origin and O at $(0, -b)$, $b < 0$. The circle $O(A)$ has equation

$$x^2 + (y + b)^2 = b^2.$$

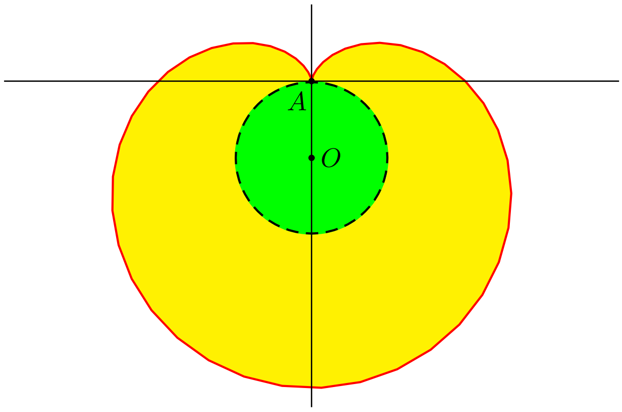

Figure 3: Given O, A, I, there is a triangle ABC with circumcenter O and incenter or excenter I if and only if I lies inside the cardioid.

Consider a point I with coordinates (p, q), not on the given circle. The line AI intersects the circle again at

$$A' = \left(-\frac{2bpq}{p^2+q^2},\ -\frac{2bq^2}{p^2+q^2}\right).$$

The circle $A'(I)$ has equation

$$(p^2+q^2)(x^2+y^2) + 4bpqx + 4bq^2y - (p^2+q^2)(p^2+4bq+q^2) = 0.$$

The radical axis of the two circles is the line

$$4bpqx - 2b(p^2-q^2)y - (p^2+q^2)(p^2+4bq+q^2) = 0.$$

The two circles intersect at two real points if and only if the square distance from $(0, -b)$ to the radical axis is less than b^2:

$$\begin{aligned} 0 &> \frac{(2b^2(p^2-q^2) - (p^2+q^2)(p^2+4bq+q^2))^2}{(4bpq)^2 + (2b(p^2-q^2))^2} - b^2 \\ &= \frac{(p^2+2bq+q^2)^2}{4b^2(p^2+q^2)^2} \cdot ((p^2+q^2)^2 + 4bq(p^2+q^2) - 4b^2p^2). \end{aligned}$$

This is equivalent to

$$(p^2+q^2)^2 + 4bq(p^2+q^2) - 4b^2p^2 < 0.$$

In polar coordinates, we write $(p, q) = (r\cos\theta,\ r\sin\theta)$ and obtain

$$\begin{gathered} r^2(r^2 + 4br\sin\theta - 4b^2\cos^2\theta) < 0, \\ r^2(r + 2b\sin\theta + 2b)(r + 2b\sin\theta - 2b) < 0, \\ -2b(1+\sin\theta) < r < 2b(1-\sin\theta). \end{gathered}$$

This is the region, shown in Fig. 3, bounded by the *cardioid* $r = 2b(1-\sin\theta)$. $\square$

Remarks. (1) A point I inside the cardioid, but outside the circle $O(A)$, is the A-excenter of a triangle ABC if and only if I and O are on the same side of the tangent to the circle at A.
(2) Figure 4 shows two examples when I lies outside $O(A)$, and realized as an excenter of a triangle ABC inscribed in the given circle.

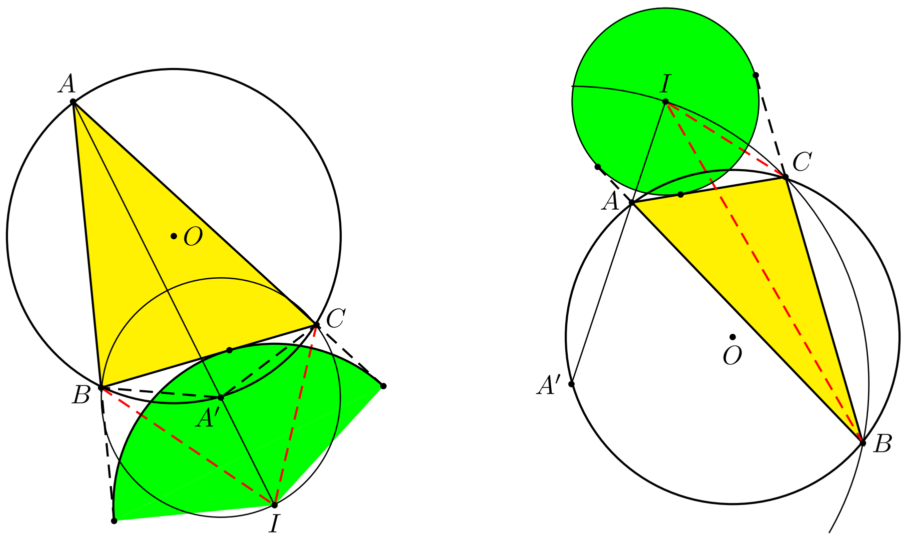

Figure 4: Two examples where point I lies inside the cardioid and outside $O(A)$, hence I is an excenter of a triangle ABC.

4. Circumcenter from I, N, A

Given three distinct points I, N, A, we set up a rectangular coordinate system with N at the origin, I at (p, q), and A at (a, q), assuming without loss of generality that $q \geq 0$. We establish Proposition 3 below by making use of three well known conditions (in §§ 4.1–3) to construct equations governing the coordinates of O.

Proposition 3. *If triangle ABC is nonisosceles, the circumcenter O is an intersection of the two curves*

$$\mathscr{H}: \quad xy = aq,$$

and

$$\mathscr{K}: \quad (x-2p)^2 - (y-2q)^2 = 3p^2 - 2ap - q^2.$$

These curves are rectangular hyperbolas except when

(i) $a = 0$, i.e., $\angle IAN = 90°$,

(ii) $q = 0$, *or*

(iii) $q^2 = p(3p - 2a)$.

In these cases, the triangle is constructible with ruler and compass (see §§ 6 – 8 below).

The proof of Proposition 3 consists of several steps, worked out in §§ 4.1–3 below. We begin by noting that if the circumcenter O has coordinates (u, v), then the orthocenter H has coordinates $(-u, -v)$, since N is the midpoint of OH.

4.1. I is equidistant from AO and AH

The incenter I is equidistant from the lines AO and AH, since these lines make equal angles with the sidelines AC and AB (see Fig. 5). Now, the line AO is represented by the equation

$$(q - v)x - (a - u)y = qu - av.$$

The square distance from I to the line AO is

$$\frac{((q-v)p-(a-u)q-(qu-av))^2}{(a-u)^2+(q-v)^2}. \tag{2}$$

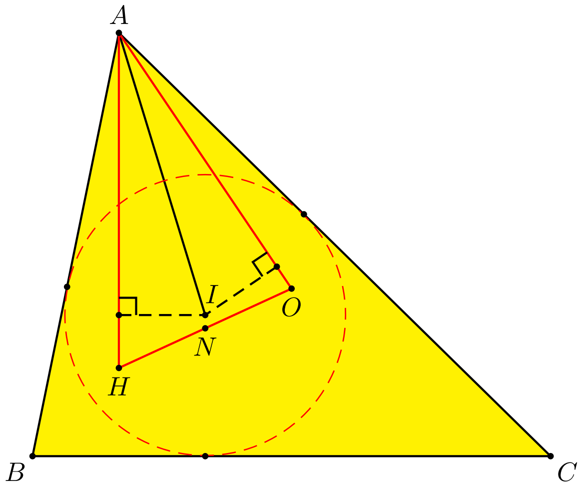

Figure 5: The incenter I is equidistant from the lines AO and AH

By replacing (u, v) by $(-u, -v)$, we obtain the square distance from I to the line AH as

$$\frac{((q+v)p-(a+u)q+(qu-av))^2}{(a+u)^2+(q+v)^2}. \tag{3}$$

Equating the expressions in (2) and (3), and simplifying, we obtain, after clearing denominator and cancelling $4(a-p)^2$,

$$(qu-av)(uv-aq)=0.$$

Note that $qu-av \neq 0$, for otherwise, $O=(u, v)$ lies on the line AN, and the triangle is isosceles. This means that $uv-aq=0$, and the circumcenter O lies on the curve $\mathscr{H}$.

4.2. Euler's formula and Feuerbach's theorem

The famous Euler's formula relates the centers and radii of the circumcircle and the incircle:

$$OI^2 = R(R-2r).$$

By the Feuerbach theorem, the nine-point circle is tangent internally to the incircle, *i.e.*, $IN = \frac{R}{2} - r$. It follows that $OI^2 = 2R \cdot IN$. This means

$$\left((u-p)^2+(v-q)^2\right)^2 = 4((u-a)^2+(v-q)^2)(p^2+q^2).$$

Now,

$$\begin{aligned}&((u-p)^2+(v-q)^2)^2-4((u-a)^2+(v-q)^2)(p^2+q^2)\\ =\ &4(uv-aq)(uv-2qu-2pv+(2p+a)q)+f_1(u,v)\cdot f_2(u,v).\end{aligned}$$

where

$$\begin{aligned} f_1(u,v) &= (u-2p)^2-(v-2q)^2-3p^2+2ap+q^2, \\ f_2(u,v) &= u^2-v^2+p^2-2ap+q^2. \end{aligned} \tag{4}$$

Since $uv = aq$, we conclude that $f_1(u,v)=0$ or $f_2(u,v)=0$.

4.3. A third relation

Since $AH = 2R\cos A$ and $\sin\frac{A}{2} = \frac{r}{AI}$, we also have

$$\begin{aligned} AO\cdot AH &= 2R^2\cos A = 2R^2\left(1-2\sin^2\frac{A}{2}\right)\\ &= 2R^2 - \frac{(2Rr)^2}{(a-p)^2} = 2R^2 - \frac{(R^2-OI^2)^2}{(a-p)^2}\\ &= 2((u-a)^2+(v-q)^2) - \frac{(u-a)^2+(v-q)^2-(u-p)^2-(v-q)^2}{(a-p)^2}\\ &= 2(u-p)^2-2(v-q)^2-(a-p)^2. \end{aligned}$$

Squaring, we obtain

$$((u-a)^2+(v-q)^2)((u+a)^2+(v+q)^2) = (2(u-p)^2-2(v-q)^2-(a-p)^2)^2.$$

Now, it can be verified that

$$\begin{aligned} &((u-a)^2+(v-q)^2)((u+a)^2+(v+q)^2) - (2(u-p)^2-2(v-q)^2-(a-p)^2)^2\\ =\ &4(uv-aq)^2 - f_1(u,v)\cdot f_3(u,v) \end{aligned}$$

where $f_1(u,v)$ is given in (4) above, and

$$f_3(u,v) = 3u^2-3v^2-4pu+4qv+p^2+2ap-q^2-2a^2.$$

Since $uv = aq$, we conclude that $f_1(u,v)=0$ or $f_3(u,v)=0$.
Proposition 3 follows from the combined results of §§ 4.1–3.

5. Solutions of the construction problem from I, N, A

We shall assume $a \neq 0$ and $q^2 \neq a(3a-2p)$ so that the curves $\mathscr{H}$ and $\mathscr{K}$ are rectangular hyperbolas. Counting multiplicity, the number of real intersections is between 2 and 4, since elimination of y leads to the quartic equation

$$f(x) = x^4 - 4px^3 + (p^2+2ap-3q^2)x^2 + 4aq^2x - a^2q^2 = 0. \tag{5}$$

5.1. Nonconstructibility

The intersections of two rectangular hyperbolas are in general not constructible using ruler and compass. For example, with $a=3$, $p=1$, $q=1$, the quartic polynomial in (5) becomes $x^4-4x^3+4x^2+12x-9$. It cannot be factored into the product of two quadratic polynomials in x (with rational coefficients), nor can it be expressed as a quadratic polynomial in a polynomial in x.

Remark. The quartic equation

$$a_0x^4+a_1x^3+a_2x^2+a_3x+a_4=0$$

is solvable by quadratic equations in two stages if and only if

$$a_1^3 = 4a_0(a_1a_2-2a_0a_3).$$

5.2. Number of intersections of the rectangular hyperbolas

The quartic polynomial (5) has multiple roots if and only if

$$3p^4 - 18p^2q^2 + 27q^4 + 16ap^3 + 144apq^2 + 24a^2p^2 - 72a^2q^2 - 16a^4 = 0.$$

Let a be a fixed positive number. The number of real intersections of the rectangular hyperbolas is 2, 3, or 4 according as I lies inside, on, or outside the region

$$3x^4 - 18x^2y^2 + 27y^4 + 16ax^3 + 144axy^2 + 24a^2x^2 - 72a^2y^2 - 16a^4 \leq 0, \quad y \geq 0. \tag{6}$$

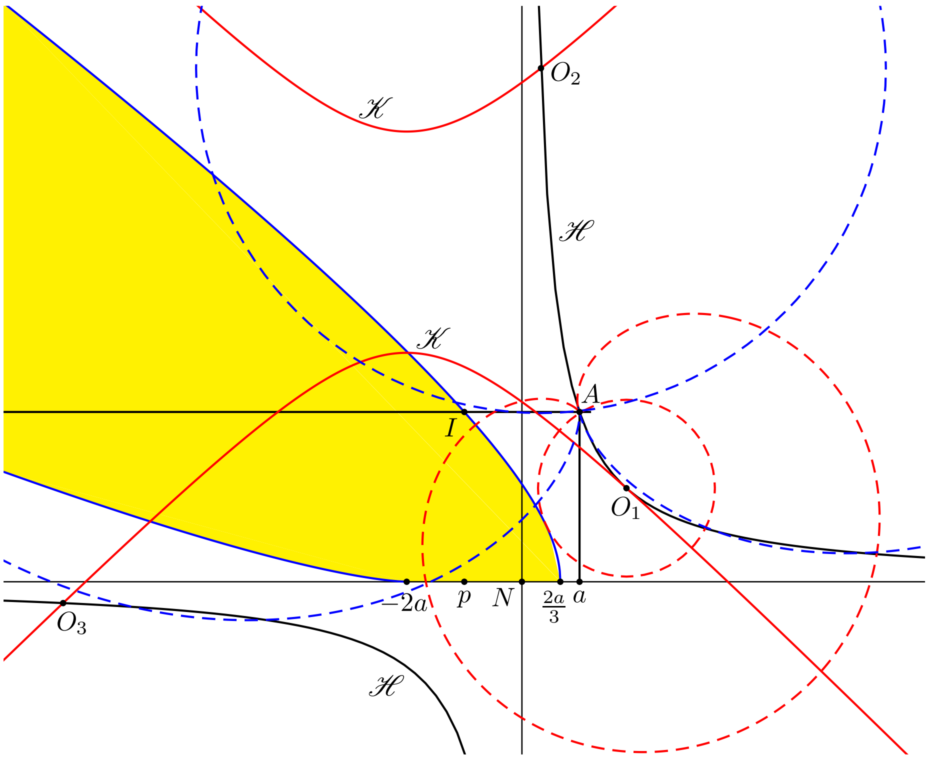

Figure 6: Case with three real intersections of the rectangular hyperbolas

Figure 6 shows a point I on the boundary of the region defined by (6), with the corresponding three locations O_1, O_2, and O_3 for the circumcenter. The point O_1 is a double point defined by the rectangular hyperbolas $\mathscr{H}$ and $\mathscr{K}$. Since I lies outside the cardioid of the circle $O_1(A)$, there is no solution with O_1 as circumcenter. On the other hand, since I is outside the circle $O_2(A)$ and inside the corresponding cardioid, there is a triangle ABC with nine-point center N, circumcenter O_2, and excenter I. Finally, O_3 yields a triangle with incenter I since I clearly lies inside the circle $O_3(A)$.

6. Constructible case $q = 0$

Suppose $q = 0$. This means that I, N, A are collinear. We shall avoid the isosceles case, which has been treated in § 2. In this case, the circumcenter is an intersection of the line $x = 0$ and the rectangular hyperbola

$$\mathscr{K}_0 : \quad (x - 2p)^2 - y^2 = 3p^2 - 2ap.$$

By symmetry, we need only consider

$$O = \left(0,\ \sqrt{p(p+2a)}\right)$$

for $a > -\frac{p}{2}$. This can be constructed as follows. Let I' be the reflection of I in N. Construct the circle $A(I')$, and the perpendicular to NI at N. One of these intersections can be taken as O. This leads to a triangle ABC with

$$\begin{aligned} B &= \left(-\frac{a}{2} - \frac{\sqrt{3p(p+2a)}}{2},\ \frac{\sqrt{3}a}{2} + \frac{\sqrt{p(p+2a)}}{2}\right), \\ C &= \left(-\frac{a}{2} + \frac{\sqrt{3p(p+2a)}}{2},\ -\frac{\sqrt{3}a}{2} + \frac{\sqrt{p(p+2a)}}{2}\right), \end{aligned}$$

and sidelengths

$$a = \sqrt{3}(a+p), \quad b = \left|\sqrt{3}a - \sqrt{p(p+2a)}\right|, \quad c = \sqrt{3}a + \sqrt{p(p+2a)}.$$

Here are the various possibilities of the triangle depending on the range of a:

range of a	b	angle A	I
$-\frac{p}{2} < a < -\frac{p}{3}$	$\sqrt{p(p+2a)} - \sqrt{3}a$	120°	$B-$excenter
$-\frac{p}{3} < a < p$	$\sqrt{3}a - \sqrt{p(p+2a)}$	120°	$B-$excenter
$a > p$	$\sqrt{p(p+2a)} - \sqrt{3}a$	60°	incenter

The observation that angle A is either 60° or 120° suggests the following simpler construction of the triangle, given $a > -\frac{p}{2}$ (see Fig. 7).

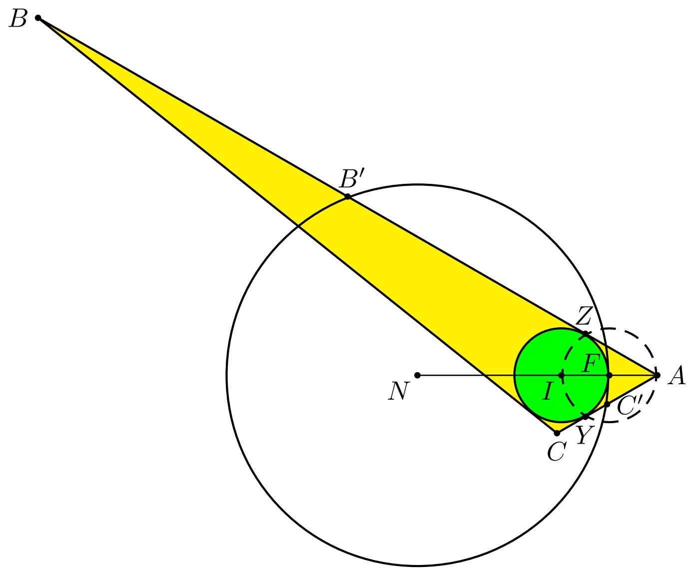

Figure 7: Constructible case $q = 0$ with $a > -\frac{p}{2}$

Given I, N, A on a line, let F be the midpoint of IA. Construct

(1) the circle $I(F)$,

(2) the circle $F(A)$ to intersect the circle $I(F)$ at Y and Z,

(3) the circle $N(F)$ to intersect the line AY at C' and the extension of AZ at B' (or AZ at B' and the extension of AY at C'),

(4) B on AZ such that $AB = 2 \cdot AB'$ and C on AY such that $AC = 2 \cdot AC'$.

ABC is the triangle with I as the incenter or B-excenter.

7. Constructible case $a = 0$

Suppose $a = 0$. This means that $\angle IAN = 90°$. In this case, $uv = 0$. If $u = 0$, then I, N, A are collinear, a case we have treated in §6. If $v = 0$, then $(u - 2p)^2 - (2q)^2 = 3p^2 - q^2$; $u = 2p \pm \sqrt{3(p^2 + q^2)}$. There are two possible positions for the circumcenter:

$$O = \left(2p + \sqrt{3(p^2 + q^2)},\ 0\right), \quad O' = \left(2p - \sqrt{3(p^2 + q^2)},\ 0\right).$$

These points can be constructed as follows.

(1) Construct the line ℓ through N parallel to AI, and the point N' on ℓ such that $IN' = IN$.

(2) Extend IN' to I' such that $II' = 2 \cdot IN'$.

(3) Construct an equilateral triangle $II'P$.

(4) Construct the circle $N'(P)$ to intersect ℓ at O and O' (see Fig. 8).

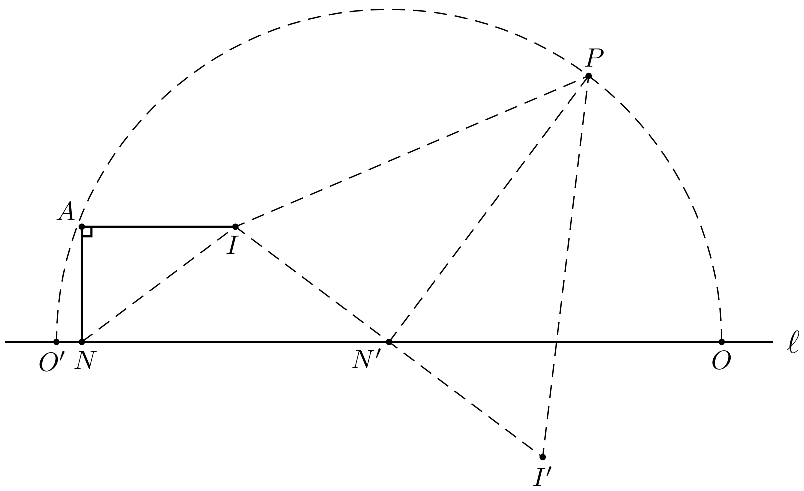

Figure 8: Construction of O and O'

(i) For O, we have

$$B = \left(\frac{2p + \sqrt{3}q + \sqrt{3(p^2 + q^2)}}{2},\ -\frac{2\sqrt{3}p + q + \sqrt{3(p^2 + q^2)}}{2}\right)$$

and

$$C = \left(\frac{2p - \sqrt{3}q + \sqrt{3(p^2 + q^2)}}{2},\ \frac{2\sqrt{3}p - q + \sqrt{3(p^2 + q^2)}}{2}\right).$$

The triangle ABC has sidelengths

$$\begin{aligned} a &= 3p + 2\sqrt{3(p^2+q^2)}\,, \\ b &= 2p - \sqrt{3}q + \sqrt{3(p^2+q^2)}\,, \\ c &= 2p + \sqrt{3}q + \sqrt{3(p^2+q^2)}\,, \end{aligned}$$

and $A = 120°$, with circumradius $R = \sqrt{3}p + 2\sqrt{p^2+q^2}$. In this case, I is the incenter, with inradius $r = \frac{\sqrt{3}}{2}\,p$.

(ii) For O', we have

$$B' = \left(\frac{2p - \sqrt{3}q - \sqrt{3(p^2+q^2)}}{2},\ \frac{2\sqrt{3}p - q - \sqrt{3(p^2+q^2)}}{2}\right)$$

and

$$C' = \left(\frac{2p + \sqrt{3}q - \sqrt{3(p^2+q^2)}}{2},\ -\frac{2\sqrt{3}p + q - \sqrt{3(p^2+q^2)}}{2}\right).$$

The triangle ABC has sidelengths

$$\begin{aligned} a' &= -3p + 2\sqrt{3(p^2+q^2)}, \\ b' &= 2p + \sqrt{3}q - \sqrt{3(p^2+q^2)}, \\ c' &= \left|-2p + \sqrt{3}q + \sqrt{3(p^2+q^2)}\right|, \end{aligned}$$

and $A = 60°$, with circumradius $R = -\sqrt{3}p + 2\sqrt{p^2+q^2}$. I is the A- or B-excenter according as $p > 4\sqrt{3}\,q$ or $p < 4\sqrt{3}\,q$. The corresponding exradius is $\frac{\sqrt{3}}{2}\,p$. If $p = 4\sqrt{3}\,q$, the point B' coincides with A.

Figure 9 illustrates the case $q = \sqrt{3}\,p$.

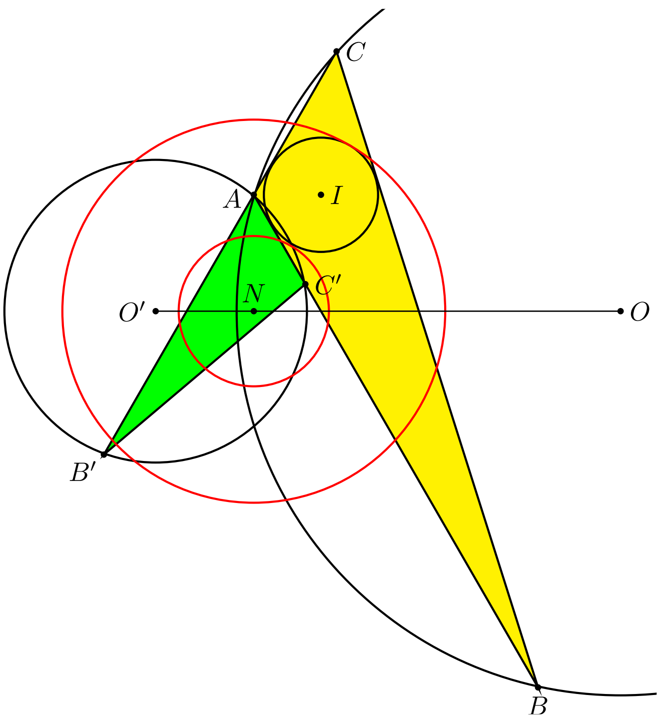

Figure 9: Constructible case $q = \sqrt{3}\,p$

8. Constructible case $3p^2 - 2ap - q^2 = 0$

Given N and I, we first determine the position of A satisfying this condition, so that the curve $\mathscr{K}$ is the union of the two lines $y - 2q = \pm(x - 2p)$. Recall that the origin is at N. We may assume I in the first quadrant, with coordinates (p, q). Complete the rectangle $NI'IN'$ with $I' = (p, 0)$ and $N' = (0, q)$. Construct

(i) the perpendicular to $N'I'$ at I', to intersect the line $N'I$ at D,

(ii) the midpoint E of $N'D$,

(iii) the reflection A of E in I (see Fig. 10).

If $A = (a, q)$, then $E = (2p - a, q)$, and $D = (4p - 2a, q)$. Since $N'I \cdot I'D = II'^2$, we have $p(3p - 2a) = q^2$.

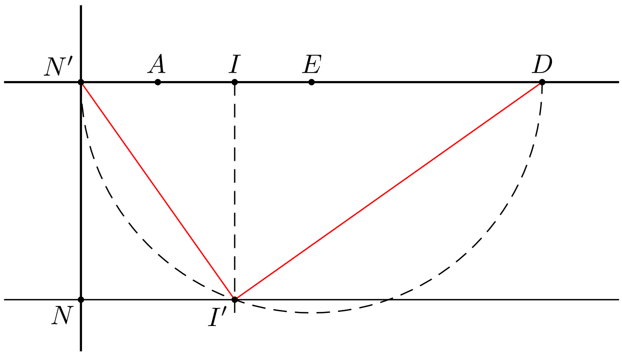

Figure 10: Construction of A satisfying $3p^2 - 2ap - q^2 = 0$

(1) The line $y - 2q = -(x - 2p)$ intersects the hyperbola

$$\mathscr{H}_0 : \quad xy = \frac{(3p^2 - q^2)q}{2p}$$

at two real points

$$O_1 = \left(p + q + \sqrt{\frac{(2p+q)(p^2+q^2)}{2p}},\ p + q - \sqrt{\frac{(2p+q)(p^2+q^2)}{2p}}\right),$$

$$O_2 = \left(p + q - \sqrt{\frac{(2p+q)(p^2+q^2)}{2p}},\ p + q + \sqrt{\frac{(2p+q)(p^2+q^2)}{2p}}\right).$$

(2) If $2p > q$, the line $y - 2q = (x - 2p)$ intersects the hyperbola $\mathscr{H}$ at two real points

$$O_3 = \left(p - q + \sqrt{\frac{(2p-q)(p^2+q^2)}{2p}},\ -p + q + \sqrt{\frac{(2p-q)(p^2+q^2)}{2p}}\right),$$

$$O_4 = \left(p - q - \sqrt{\frac{(2p-q)(p^2+q^2)}{2p}},\ -p + q - \sqrt{\frac{(2p-q)(p^2+q^2)}{2p}}\right).$$

If $2p = q$, $O_3 = O_4 = (-p, p)$. In this case, $A = \left(-\frac{p}{2},\ 2p\right)$, and $I = (p, 2p)$. Figure 11 illustrates this case, when there are three locations of the circumcenter. Clearly, there is a

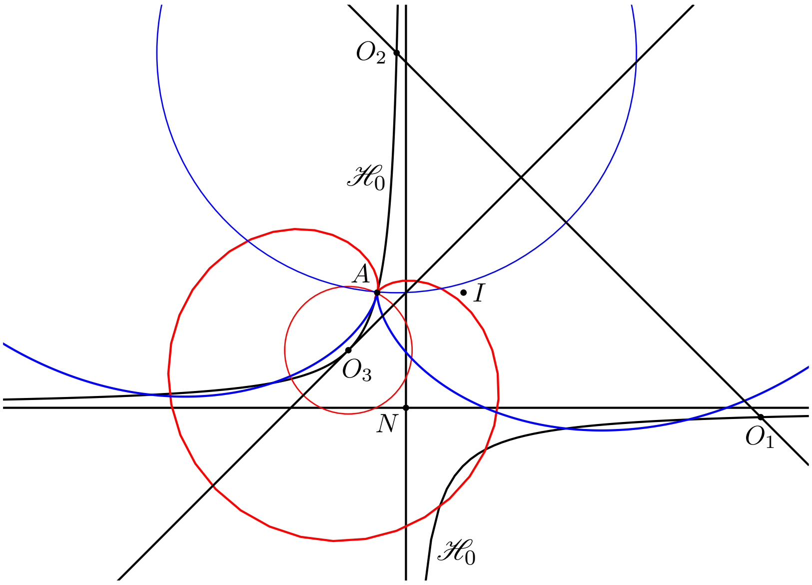

Figure 11: Three locations of circumcenter for $2p = q$

triangle with circumcenter O_1 and incenter I. For O_2, there is one with excenter I since I lies in the cardioid for the circle $O_2(A)$. But there is no such triangle with circumcenter O_3, the point of tangency of the hyperbola $xy = -p^2$ and the line $x - y + 2p = 0$, since I lies outside the cardioid corresponding to $O_3(A)$.

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