

The Elementary Mathematical Works of Leonhard Euler (1707 – 1783)

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Summer 1999¹

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¹Diagrams redrawn in August, 2007.

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IA. Introduction

Euler's Opera Omnia

Series	Subject	Vol.	Pages	
I	Pure Mathematics	29	13979	Complete
II	Mechanics and Astronomy	31	11007	v.26,27 to appear
III	Physics and Miscellanea	12	4325	v.10 to appear
IV A	Correspondences	8	1811	v.3,4,7,8 to appear
IV B	Manuscripts			In progress
Total		80	31122	7 volumes

Series I – Pure Mathematics

Subject	Volumes	Pages
Number Theory	4	1955
Algebra	1	509
Combinatorics	1	577
Infinite Series	3	2022
Integraton	3	1424
Elliptic Integrals	2	750
Differential Equations	2	875
Calculus of Variations	1	342
Geometry	4	1589
Total	21	10043
Texts	8	3936
Total	29	13979

Euler: Opera Omnia
Series I – Pure Mathematics

Vol.	Pages	Year	Content
I	651	1911	<i>Elements of Algebra</i> , 1770
II	611	1915	Number Theory
III	543	1917	Number Theory
IV	431	1941	Number Theory
V	370	1944	Number Theory
VI	509	1921	Theory of Equations
VII	577	1923	Combinatorics and Probability
VIII	390	1922	Introductio in Analysin Infinitorum, I, 1748
IX	402	1945	Introductio in Analysin Infinitorum, II, 1748
X	676	1913	Differential Calculus, 1755
XI	462	1913	Integral Calculus, 1768
XII	542	1914	Integral Calculus, 1768
XIII	505	1914	Integral Calculus, 1768
XIV	617	1925	Infinite Series
XV	722	1927	Infinite Series
XVI(1)	355	1933	Infinite Series
XVI(2)	328	1935	Infinite Series
XVII	457	1914	Integration
XVIII	475	1920	Integration
XIX	492	1932	Integration
XX	370	1912	Elliptic Integrals
XXI	380	1913	Elliptic Integrals
XXII	420	1936	Differential Equations
XXIII	455	1938	Differential Equations
XXIV	308	1952	Calculus of Variations, 1744
XXV	342	1952	Calculus of Variations
XXVI	362	1953	Geometry
XXVII	400	1954	Geometry
XXVIII	381	1955	Geometry
XXIX	446	1956	Geometry

Euler's *Elements of Algebra* (1770)
Contents

Part I: Containing in Analysis of Determinate Quantities.

Section I. Of the different methods of calculating simple quantities. (23 chapters; 75 pages)

- Of impossible, or imaginary quantities, which arise from the same source.²

Section II. Of the different methods of calculating compound quantities. (13 chapters; 50 pages)

- Chapter IV. Of the summation of arithmetical progressions.
- Chapter VII. Of the greatest common divisor of two given numbers.
- Chapter XI. Of geometrical progressions.

Section III. Of ratios and proportions (13 chapters; 60 pages)

Section IV. Of Algebraic equations, and of the resolution of those equations. (16 chapters; 113 pages)

- Chapter IX. Of the nature of equations of the second degree.
- Chapter X to XII. Cubic equations; of the rule of Cardan, Or of Scipio Ferreo.
- Chapter XIII to XV. Equations of fourth degree.
- Chapter XV. Of a new method of resolving equations of the fourth degree.

Part II: Containing the Analysis of Indeterminate Quantities. (15 chapters; 164 pages)

Elementary number theory up to the solution of quadratic equations in integers.

Additions by Lagrange (9 chapters; 131 pages)

Sample exercises in *Elements of Algebra*

²Every body makes mistakes, even the masters. Here is an observation by Ivor Grattan - Guinness *The Norton History of Mathematical Sciences, The Rainbow of Mathematics*, 1997, [W.W.Norton & Company, New York, pp. 334 – 335]. on Euler's *Elements of Algebra*:

As usual, Euler gave a reliable presentation; but he gaffed in his algebraic handling of complex numbers, by misapplying the product rule for square roots

$$\sqrt{ab} = \sqrt{a}\sqrt{b},$$

to write

$$\sqrt{-2}\sqrt{-3} = \sqrt{6} \quad \text{instead of} \quad -\sqrt{6}.$$

[Art. 148, 149] The error was systematic: for example, he committed it for division as well, and it confused some later writers on the subject.

Section II, Chapter IV.

1. Required the sum of an increasing arithmetical progression, having 3 for its first term, 2 for the common difference, and the number of terms 20. (Answer: 440).

4. One hundred stones being placed on the ground, in a straight line, at the distance of a yard from each other, how far will a person travel who shall bring them one by one to a basket, which is placed one yard from the first stone? (Answer: 5 miles and 1300 yards).

Section II, Chapter VII: On finding the gcd of two numbers.

Divide the greater of the two numbers by the less; next, divide the preceding divisor by the remainder; what remains in this second division will afterwards become a divisor for a division, in which the remainder of the preceding divisor will be the dividend. we must continue this operation till we arrive at a division that leaves no remainder; and this last divisor will be the greatest common divisor of the two given numbers.

Examples: Find the gcd of (i) 252 and 576; (ii) 312 and 504; (iii) 529 and 625; (iv) 1728 and 2304.

IB. Solution of cubic equations

Euler's Paper 30: De formis radicum aequationum cuiusque ordinis coniectatio (1732/33) 1738.

§3. Resolutionem aequationis cubicae sequenti modo a quadratic pendentm considero. Sit aequation cubica

$$x^3 = ax + b$$

in qua secunda terminus deest; huius radicem x dico fore

$$= \sqrt[3]{A} + \sqrt[3]{B}$$

existentibus A et B duabus radicibus aequationes cuiusdam quadraticae

$$z^2 = \alpha z - \beta.$$

Quamobrem ex natura aequationum erit

$$A + B = \alpha \quad \text{et} \quad AB = \beta.$$

Sed ad α et β ex a et b definiendas sumo aequationem

$$x = \sqrt[3]{A} + \sqrt[3]{B},$$

quae cubice multiplicata dat

$$x^3 = A + B + 3\sqrt[3]{AB}(\sqrt[3]{A} + \sqrt[3]{B})$$

$$= 3x\sqrt[3]{AB} + A + B.$$

Quae cum proposita $x^3 = ax + b$ comparata dabit

$$a = 3\sqrt[3]{AB} = 3\sqrt[3]{\beta} \quad \text{et} \quad b = A + B = \alpha.$$

Fiet igitur

$$\alpha = b \quad \text{et} \quad \beta = \frac{a^3}{27}.$$

Huis enim radicibus cognitis A et B erit

$$x = \sqrt[3]{A} + \sqrt[3]{B}.$$

§4. Sed cum radix cubica ex quaque quantitate triplicem habeat valorem, haec formula $x = \sqrt[3]{A} + \sqrt[3]{B}$ omnes etiam radices aequationis propositae complectetur. Sint enim μ et ν praeter unitatem radices cubicae ex unitate; erit etiam

$$x = \mu\sqrt[3]{A} + \nu\sqrt[3]{B},$$

si modo sit $\mu\nu = 1$. Quamobrem μ et ν esse debent

$$\frac{-1 + \sqrt{-3}}{2} \quad \text{et} \quad \frac{-1 - \sqrt{-3}}{2}$$

vel inverse. Praeter radicem igitur

$$x = \sqrt[3]{A} + \sqrt[3]{B}$$

satisfacient quoque [aequationi] propositae hae duae alterae radices

$$x = \frac{-1 + \sqrt{-3}}{2}\sqrt[3]{A} + \frac{-1 - \sqrt{-3}}{2}\sqrt[3]{B}$$

et

$$x = \frac{-1 - \sqrt{-3}}{2}\sqrt[3]{A} + \frac{-1 + \sqrt{-3}}{2}\sqrt[3]{B}.$$

Hacque ratione aequationis cubicae etiam, in qua secundus terminus non deest, radices determinari poterunt.

IC. Solution of quartic equations

From *Elements of Algebra*, Sect. IV, Chapter XV, Of a new method of resolving equations of the fourth degree.

§774. We will suppose that the root of an equation of the *fourth* degree has the form

$$x = \sqrt{p} + \sqrt{q} + \sqrt{r}$$

in which the letters p, q, r , express the roots of an equation of the third degree, such as

$$z^3 - fz^2 + gz - h = 0;$$

so that

$$\begin{aligned} p + q + r &= f, \\ pq + pr + qr &= g, \\ pqr &= h. \end{aligned}$$

[§722.] This being laid down, we square the assumed formula, $x = \sqrt{p} + \sqrt{q} + \sqrt{r}$, and we obtain

$$x^2 = p + q + r + 2\sqrt{pq} + 2\sqrt{pr} + 2\sqrt{qr};$$

and, since $p + q + r = f$, we have

$$x^2 - f = 2\sqrt{pq} + 2\sqrt{pr} + 2\sqrt{qr}.$$

We again take the squares, and find

$$x^4 - 2fx^2 + f^2 = 4pq + 4pr + 4qr + 8\sqrt{p^2qr} + 8\sqrt{pq^2r} + 8\sqrt{pqr^2}.$$

Now, $4pq + 4pr + 4qr = 4g$; so that the equation becomes

$$x^4 - 2fx^2 + f^2 - 4g = 8\sqrt{pqr}(\sqrt{p} + \sqrt{q} + \sqrt{r});$$

but $\sqrt{p} + \sqrt{q} + \sqrt{r} = x$, and $pqr = h$, or $\sqrt{pqr} = \sqrt{h}$; wherefore we arrive at this equation of the fourth degree,

$$x^4 - 2fx^2 - 8x\sqrt{h} + f^2 - 4g = 0,$$

³ one of the roots of which is $x = \sqrt{p} + \sqrt{q} + \sqrt{r}$; and in which p, q, r , are the roots of the equation of the third degree,

$$z^3 - fz^2 + gz - h = 0.$$

§775. The equation of the fourth degree, at which we have arrived, may be considered as general, although the second term x^3y is wanting; for we shall afterwards shew, that every complete equation may be transformed into another, from which the second term has been taken away.

Let there be proposed the equation

$$x^4 - ax^2 - bx - c = 0,$$

³The English translation prints the term f^2 twice.

in order to determine one of its roots. We will first compare it with the formula,

$$x^4 - 2x^2 - 8x\sqrt{h} + f^2 - 4g = 0,$$

in order to obtain the values of f , g , and h ; and we shall have

1. $2f = a$, and consequently $f = \frac{a}{2}$;
2. $8\sqrt{h} = b$, so that $h = \frac{b^2}{64}$;
3. $f^2 - 4g = -c$, or (as $f = \frac{a}{2}$),

$$a^2 - 4g + c = 0,$$

or $\frac{1}{4}a^2 + c = 4g$; consequently, $g = \frac{1}{16}a^2 + \frac{1}{4}c$.

IIA. Factorization of a quartic as a product of two real quadratics

§777. This method appears at first to furnish only one root of the given equation; but if we consider that every sign $\sqrt{\quad}$ may be taken negatively, as well as positively, we immediately perceive that this formula contains all the four roots. Farther, if we chose to admit all the possible changes of the signs, we should have eight different values of x , and yet four only can exist. But it is to be observed, that the product of those three terms, or \sqrt{pqr} , must be equal to $\sqrt{h} = \frac{1}{8}b$, and that if $\frac{1}{8}b$ be positive, the product of the terms \sqrt{p} , \sqrt{q} , \sqrt{r} ; must likewise be positive, so that all the variations that can be admitted are reduced to the four following:

$$\begin{aligned} x &= \sqrt{p} + \sqrt{q} + \sqrt{r}, \\ x &= \sqrt{p} - \sqrt{q} - \sqrt{r}, \\ x &= -\sqrt{p} + \sqrt{q} - \sqrt{r}, \\ x &= -\sqrt{p} - \sqrt{q} + \sqrt{r}. \end{aligned}$$

In the same manner, when $\frac{1}{8}b$ is negative, we have only the four following values of x :

$$\begin{aligned} x &= \sqrt{p} + \sqrt{q} - \sqrt{r}, \\ x &= \sqrt{p} - \sqrt{q} + \sqrt{r}, \\ x &= -\sqrt{p} + \sqrt{q} + \sqrt{r}, \\ x &= -\sqrt{p} - \sqrt{q} - \sqrt{r}. \end{aligned}$$

This circumstance enables us to determine the four roots in all cases; as may be seen in the following example.

§778. Solve the equation

$$x^4 - 25x^2 + 60x - 36 = 0.$$

Hint: The resulting cubic equation has a root 9.

Solve the quartic equations

- (1) $y^4 - 4y^3 - 3y^2 - 4y + 1 = 0$.
- (2) $x^4 - 3x^2 - 4x = 3$.

Answer: (1) $\frac{-1 \pm \sqrt{-3}}{2}$, and $\frac{5 \pm \sqrt{21}}{2}$; (2) $\frac{1 \pm \sqrt{13}}{2}$, and $\frac{-1 \pm \sqrt{-3}}{2}$.

Paper 170: *Recherches sur les Racines Imaginaires des Equations* (1749) 1751.

In §§9 – 13, Euler solved explicitly the fourth degree equation:

$$x^4 + 2x^3 + 4x^2 + 2x + 1 = 0.$$

He factored this into

$$(x^2 + (1 + i)x + 1)(x^2 + (1 - i)x + 1) = 0,$$

⁴ and obtained the four simple factors

$$\begin{aligned} x + \frac{1}{2}(1 + i) + \frac{1}{2}\sqrt{2i - 4}, \\ x + \frac{1}{2}(1 + i) - \frac{1}{2}\sqrt{2i - 4}, \\ x + \frac{1}{2}(1 - i) + \frac{1}{2}\sqrt{-2i - 4}, \\ x + \frac{1}{2}(1 - i) - \frac{1}{2}\sqrt{-2i - 4}. \end{aligned}$$

Then he proceeded to rewrite these ... by setting

$$\sqrt{2i - 4} = u + vi \quad \text{and} \quad \sqrt{-2i - 4} = u - vi.$$

From these,

$$v^2 - u^2 = 4 \quad \text{and} \quad uv = 1.$$

Now, it is easy to find

$$v = \sqrt{\sqrt{5} + 2} \quad \text{and} \quad u = \sqrt{\sqrt{5} - 2}.$$

In terms of u and v , Euler rewrote the above four simple factors, and presented the product of the first and the third factors as

$$\left(x + \frac{1}{2}(1 + u)\right)^2 + \frac{1}{4}(1 + v)^2,$$

while that of the second and fourth as

$$\left(x + \frac{1}{2}(1 - u)\right)^2 + \frac{1}{4}(1 - v)^2.$$

⁴When did Euler begin to use i for $\sqrt{-1}$? According to F. Cajori, *History of Mathematical Notations*, §498: It was Euler who first used the letter i for $\sqrt{-1}$. He gave it in a memoir presented in 1777 to the Academy at St. Petersburg, and entitled *De formulis [differentialibus angularibus, 1777, I.19,129-140; p.130]*, but it was not published until 1794 after the death of Euler. As far as is now known, the symbol i for $\sqrt{-1}$ did not again appear in print for seven years, until 1801. In that year Gauss [in his *Disq. Arith.*] began to make systematic use of it; ...

Symmetric equations

Then he discussed (§§14 – 18) the general problem of factoring a quartic of the form

$$x^4 + ax^3 + (b + 2)x^2 + ax + 1 = 0$$

into two quadratic factors with *real* coefficients. The case $a^2 > 4b$ is easy. In §18, he gave an explicit factorization for the case $a^2 < 4b$.

This is the beginning of a series of examples for what he aimed at to establish finally.

Theorem 4 (§27). Every quartic equation can be decomposed into two quadratic factors of real coefficients.

Euler's 1748 Introductio in Analysin Infinitorum, I

§31. *If Q is the real product of four complex linear factors, then this product can also be represented as the product of two real quadratic factors.*

Now Q has the form $z^4 + Az^3 + Bz^2 + Cz + D$ and if we suppose the Q cannot be represented as the product of two real quadratic factors, then we show that $i[t]$ can be represented as the product of two complex quadratic factors having the following forms:

$$z^2 - 2(p + qi)z + r + si$$

and

$$z^2 - 2(p - qi)z + r - si.$$

No other form is possible, since the product is real, namely, $z^4 + Az^3 + Bz^2 + Cz + D$. From these complex quadratic factors we derive the following four complex linear factors:

$$\begin{aligned} I. & \quad z - (p + qi) + \sqrt{p^2 + 2pqi - q^2 - r - si}, \\ II. & \quad z - (p + qi) - \sqrt{p^2 + 2pqi - q^2 - r - si}, \\ III. & \quad z - (p - qi) + \sqrt{p^2 - 2pqi - q^2 - r + si}, \\ IV. & \quad z - (p - qi) - \sqrt{p^2 - 2pqi - q^2 - r + si}. \end{aligned}$$

For the sake of brevity we let $t = p^2 - q^2 - r$ and $u = 2pq - s$. When the first and third of these factors are multiplied, the product is equal to

$$z^2 + (2p - \sqrt{2t + 2\sqrt{t^2 + u^2}})z + p^2 + q^2 - p\sqrt{2t + 2\sqrt{t^2 + u^2}} + \sqrt{t^2 + u^2} + q\sqrt{-2t + 2\sqrt{t^2 + u^2}},$$

which is real. In like manner the product of the second and fourth factors is the real

$$z^2 - (2p + \sqrt{2t + 2\sqrt{t^2 + u^2}})z + p^2 + q^2 + p\sqrt{2t + 2\sqrt{t^2 + u^2}} + \sqrt{t^2 + u^2} + q\sqrt{-2t + 2\sqrt{t^2 + u^2}}.$$

Thus the proposed product Q , which we supposed could not be expressed as two real factors, can be expressed as the product of two real quadratic factors.

II B. Euler's proof of the FTA for quartic polynomials

Paper 170. Theoreme 4

§27. Toute équation du quatrième degré, comme

$$x^4 + Ax^3 + Bx^2 + Cx + d = 0$$

se peut toujours décomposer en deux facteurs réels du second degré.

Démonstration

On said que posant $x = y - \frac{1}{4}A$, cette équation se change dans une autre du même degré, où le second terms manque; et comme cette transformation se peut toujours faire, supposons que dans l'équation proposée le second manque déjà, et que nous ayons cette équation

$$x^4 + Bx^2 + Cx + D = 0$$

à résoudre en deux facteurs réels du second degré; et il est d'abord clair que ces deux facteurs seront de cette forme

$$(x^2 + ux + \alpha)(x^2 - ux + \beta) = 0,$$

dont comparant le produit avec l'équation proposée, nous aurons

$$B = \alpha + \beta - u^2, \quad C = (\beta - \alpha)u, \quad D = \alpha\beta,$$

d'où nous tirerons

$$\alpha + \beta = B + u^2, \quad \beta - \alpha = \frac{C}{u}$$

et partant

$$2\beta = u^2 + B + \frac{C}{u}, \quad 2\alpha = u^2 + B - \frac{C}{u};$$

ayant donc $4\alpha\beta = 4D$, nous obtiendrons cette équation

$$u^4 + 2Bu^2 + B^2 - \frac{C^2}{u^2} = 4D$$

ou bien

$$u^6 + 2Bu^4 + (B^2 - 4D)u^2 - C^2 = 0,$$

d'où il faut chercher la valeur de u . Or puisque le terme absolu $-C^2$ est essentiellement négatif, nous venons de démontrer, que cette équation a au moins deux racines réelles; prenant donc l'une ou l'autre pour u , les valeurs α et β seront également réelles, et par conséquent les deux facteurs supposés du second degré $x^2 + ux + \alpha$ et $x^2 - ux + \beta$ seront réels. C.Q.F.D.

Fundamental Theorem of Algebra as a theorem in Complex Analysis

Every nonconstant polynomial with complex coefficients must have a complex zero.

Liouville's Theorem A bounded, analytic function in the whole complex plane must be constant.

Liouville's Theorem \implies FTA

Let $f(z)$ be a polynomial of degree $n \geq 1$.

It is clearly nonconstant.

If $f(z)$ is never zero, then $\frac{1}{f(z)}$ is an analytic function.

Since $f(z) \rightarrow \infty$ as $z \rightarrow \infty$, $\frac{1}{f(z)} \rightarrow 0$.

Since its absolute value is continuous on the Riemann sphere (= complex plane together with the point at infinity), which is compact, the function $\frac{1}{f(z)}$ is bounded.

By Liouville's theorem, $\frac{1}{f(z)}$ must be constant.

This contradicts the assumption that $f(z)$ is non-constant.

II.C. Relations between roots and coefficients

General polynomial equation of degree n :

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - Ex^{n-5} + \cdots \pm Mx \mp N = 0 = 0.$$

5

Roots $\alpha, \beta, \dots, \nu$.

Relations between roots and coefficients:

$$\begin{aligned} A &= \alpha + \beta + \gamma + \delta + \cdots = \text{sum of roots} \\ B &= \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \cdots = \text{sum of products of roots taken **two** at a time} \\ C &= \alpha\beta\gamma + \cdots = \text{sum of products of roots taken **three** at a time} \\ D &= \alpha\beta\gamma\delta + \cdots = \text{sum of products of roots taken **four** at a time} \\ E &= \alpha\beta\gamma\delta\epsilon + \cdots = \text{sum of products of roots taken **five** at a time} \end{aligned}$$

Consider the sums of powers of the roots:

⁵Euler wrote $\pm N$ instead of $\pm Mx \mp$.

$$\begin{aligned}
\sum \alpha &= \alpha + \beta + \gamma + \delta + \epsilon + \cdots + \nu, \\
\sum \alpha^2 &= \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \epsilon^2 + \cdots + \nu^2, \\
\sum \alpha^3 &= \alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \epsilon^3 + \cdots + \nu^3, \\
\sum \alpha^4 &= \alpha^4 + \beta^4 + \gamma^4 + \delta^4 + \epsilon^4 + \cdots + \nu^4, \\
\sum \alpha^5 &= \alpha^5 + \beta^5 + \gamma^5 + \delta^5 + \epsilon^5 + \cdots + \nu^5, \\
\sum \alpha^6 &= \alpha^6 + \beta^6 + \gamma^6 + \delta^6 + \epsilon^6 + \cdots + \nu^6, \\
&\vdots
\end{aligned}$$

Newton's Theorem

$$\begin{aligned}
\sum \alpha &= A, \\
\sum \alpha^2 &= A \sum \alpha - 2B, \\
\sum \alpha^3 &= A \sum \alpha^2 - B \sum \alpha + 3C, \\
\sum \alpha^4 &= A \sum \alpha^3 - B \sum \alpha^2 + C \sum \alpha - 4D, \\
\sum \alpha^5 &= A \sum \alpha^4 - B \sum \alpha^3 + C \sum \alpha^2 - D \sum \alpha + 5E, \\
\sum \alpha^6 &= A \sum \alpha^5 - B \sum \alpha^4 + C \sum \alpha^3 - D \sum \alpha^2 + E \sum \alpha - 6F, \\
&\vdots
\end{aligned}$$

Euler's first proof (§§5 – 7).

Write

$$Z = x^n - Ax^{n-1} + Bx^{n-2} + \cdots \pm N$$

which, when set to zero, gives the n roots $\alpha, \beta, \gamma, \dots, \nu$.

$$Z = (x - \alpha)(x - \beta)(x - \gamma)(x - \delta) \cdots (x - \nu),$$

and taking logarithms,

$$\log Z = \log(x - \alpha) + \log(x - \beta) + \log(x - \gamma) + \log(x - \delta) + \cdots + \log(x - \nu),$$

⁶ Differentiating,

$$\frac{dZ}{Z} = \frac{dx}{x - \alpha} + \frac{dx}{x - \beta} + \frac{dx}{x - \gamma} + \frac{dx}{x - \delta} + \cdots + \frac{dx}{x - \nu}.$$

⁶Euler used ℓ for natural (hyperbolic) logarithms.

Converting into infinite geometric series

$$\begin{aligned}\frac{1}{x-\alpha} &= \frac{1}{x} + \frac{\alpha}{x^2} + \frac{\alpha^2}{x^3} + \frac{\alpha^3}{x^4} + \frac{\alpha^4}{x^5} + \frac{\alpha^5}{x^6} + \dots \\ \frac{1}{x-\beta} &= \frac{1}{x} + \frac{\beta}{x^2} + \frac{\beta^2}{x^3} + \frac{\beta^3}{x^4} + \frac{\beta^4}{x^5} + \frac{\beta^5}{x^6} + \dots \\ \frac{1}{x-\gamma} &= \frac{1}{x} + \frac{\gamma}{x^2} + \frac{\gamma^2}{x^3} + \frac{\gamma^3}{x^4} + \frac{\gamma^4}{x^5} + \frac{\gamma^5}{x^6} + \dots \\ &\vdots \\ \frac{1}{x-\nu} &= \frac{1}{x} + \frac{\nu}{x^2} + \frac{\nu^2}{x^3} + \frac{\nu^3}{x^4} + \frac{\nu^4}{x^5} + \frac{\nu^5}{x^6} + \dots\end{aligned}$$

Adding these, we have

$$\frac{dZ}{Zdx} = \frac{n}{x} + \frac{1}{x^2} \sum \alpha + \frac{1}{x^3} \sum \alpha^2 + \frac{1}{x^4} \sum \alpha^3 + \frac{1}{x^5} \sum \alpha^4 + \frac{1}{x^6} \sum \alpha^5 + \dots$$

§6. From the other expression of Z , namely,

$$Z = x^n - Ax^{n-1} + Bx^{n-2} + \dots \pm N$$

we have

$$\frac{dZ}{dx} = nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + (n-4)Dx^{n-5} - \dots$$

This should be the same as the product of

$$\frac{n}{x} + \frac{1}{x^2} \sum \alpha + \frac{1}{x^3} \sum \alpha^2 + \frac{1}{x^4} \sum \alpha^3 + \frac{1}{x^5} \sum \alpha^4 + \frac{1}{x^6} \sum \alpha^5 + \dots$$

and

$$x^n - Ax^{n-1} + Bx^{n-2} + \dots \pm N.$$

Therefore,

$$\begin{aligned}& nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + (n-4)Dx^{n-5} - \dots \\ = & nx^{n-1} + x^{n-2} \sum \alpha + x^{n-3} \sum \alpha^2 + x^{n-4} \sum \alpha^3 + x^{n-5} \sum \alpha^4 + \dots \\ & - nAx^{n-2} - Ax^{n-3} \sum \alpha - Ax^{n-4} \sum \alpha^2 - Ax^{n-5} \sum \alpha^3 - \dots \\ & + nBx^{n-3} + Bx^{n-4} \sum \alpha - Bx^{n-5} \sum \alpha^2 + \dots \\ & - nCx^{n-4} - Cx^{n-5} \sum \alpha - \dots \\ & + nDx^{n-5} - \dots\end{aligned}$$

Therefore,

$$\begin{aligned}
 -(n-1)A &= \sum \alpha - nA \\
 +(n-2)B &= \sum \alpha^2 - A \sum \alpha + nB, \\
 -(n-2)C &= \sum \alpha^3 - A \sum \alpha^2 - B \sum \alpha - nC, \\
 +(n-3)D &= \sum \alpha^4 - A \sum \alpha^3 + B \sum \alpha^2 - C \sum \alpha + nD, \\
 &\vdots
 \end{aligned}$$

From these,

$$\begin{aligned}
 \sum \alpha &= A, \\
 \sum \alpha^2 &= A \sum \alpha - 2B \\
 \sum \alpha^3 &= A \sum \alpha^2 - B \sum \alpha + 3C, \\
 \sum \alpha^4 &= A \sum \alpha^3 - B \sum \alpha^2 + C \sum \alpha - 4D, \\
 \sum \alpha^5 &= A \sum \alpha^4 - B \sum \alpha^3 + C \sum \alpha^2 - D \sum \alpha + 5E, \\
 &\vdots
 \end{aligned}$$

From the rightmost column we have

$$\sum \alpha^{n-1} = A \sum \alpha^{n-2} - B \sum \alpha^{n-3} + C \sum \alpha^{n-4} - \dots \pm (n-1)M$$

In a later paper (406), *Observationes circa radices aequationum*, 1770, Euler went further and wrote these explicitly in terms of A, B, C etc.

$$\begin{aligned}
 \sum \alpha &= A, \\
 \sum \alpha^2 &= A^2 + 2B \\
 \sum \alpha^3 &= A^3 + 3AB + 3C, \\
 \sum \alpha^4 &= A^4 + 4A^2B + 4AC + 4D \\
 &\quad + 2B^2 \\
 \sum \alpha^5 &= A^5 + 5A^3B + 5A^2C + 5AD + 5E \\
 &\quad + 5AB^2 + 5BC \\
 \sum \alpha^6 &= A^6 + 6A^4B + 6A^3C + 6A^2D + 6AE + 6F \\
 &\quad + 9A^2B^2 + 12ABC + 6BD \\
 &\quad + 2B^3 + 3C^2 \\
 \sum \alpha^7 &= A^7 + 7A^5B + 7A^4C + 7A^3D + 7A^2E + 7AF + 7G \\
 &\quad + 14A^3B^2 + 21A^2BC + 14ABD + 7BE \\
 &\quad + 7AB^3 + 7AC^2 + 7CD \\
 &\quad + 7B^2C
 \end{aligned}$$

Euler's second proof

§11. Euler showed that it is clear from the equation

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - Ex^{n-5} + \dots \mp Mx \mp N = 0$$

that

$$\sum \alpha^n = A \sum \alpha^{n-1} - B \sum \alpha^{n-2} + C \sum \alpha^{n-3} - D \sum \alpha^{n-4} + \dots \mp M \sum \alpha \pm nN.$$

Multiplying by successive powers of x , we have

$$\begin{aligned} \sum \alpha^{n+1} &= A \sum \alpha^n - B \sum \alpha^{n-1} + C \sum \alpha^{n-2} - D \sum \alpha^{n-3} + \dots \mp M \sum \alpha^2 \pm N \sum \alpha, \\ \sum \alpha^{n+2} &= A \sum \alpha^{n+1} - B \sum \alpha^n + C \sum \alpha^{n-1} - D \sum \alpha^{n-2} + \dots \mp M \sum \alpha^3 \pm N \sum \alpha^2, \\ \sum \alpha^{n+3} &= A \sum \alpha^{n+2} - B \sum \alpha^{n+1} + C \sum \alpha^n - D \sum \alpha^{n-1} + \dots \mp M \sum \alpha^4 \pm N \sum \alpha^3, \\ &\vdots \end{aligned}$$

More generally, for any positive integer m ,

$$\sum \alpha^{n+m} = A \sum \alpha^{n+m-1} - B \sum \alpha^{n+m-2} + C \sum \alpha^{n+m-3} - D \sum \alpha^{n+m-4} + \dots \mp M \sum \alpha^{m+1} \pm N \sum \alpha^m.$$

It remains to determine sums

$$\sum \alpha, \sum \alpha^2, \sum \alpha^3, \dots, \sum \alpha^{n-1}.$$

§13–15. Euler gave the proof for $n = 5$. The same method naturally applies to a general n .

Given the equation

$$x^5 - Ax^4 + bx^3 - Cx^2 + Dx - E = 0,$$

consider the lower degree equations “formed by retaining its coefficients”:

I.	$x - A = 0,$	roots p
II	$x^2 - Ax + B = 0,$	roots q
III.	$x^3 - Ax^2 + Bx - C = 0,$	roots r
IV.	$x^4 - Ax^3 + Bx^2 - Cx + D = 0,$	roots s .

Earlier in §3, Euler deduced

$$\sum \alpha^2 = A^2 - B$$

from the obvious identity

$$\begin{aligned} &(\alpha + \beta + \gamma + \delta + \dots)^2 \\ &= \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \epsilon^2 + \dots \\ &\quad + 2\alpha\beta + 2\alpha\gamma + 2\alpha\delta + 2\beta\gamma + 2\beta\delta + \dots \end{aligned}$$

Quarum aequationum radices, etiamsi inter se maxime discrepent, tamen in his singulis aequationibus eandem consituent summan = A . Deinde remota prima summa productorum ex binis radicibus ubique erit eadem = B . Tum summa productorum ex ternis radicibus ubique erit = C , praeter aequationes scilicet I et II , ubi C non occurrit. Similiter in IV etc. proposita summa productorum ex quaternis radicibus erit eadem = D .

§14 In quibus autem aequationibus non solum summa radicum est eadem, set etiam summa productorum ex binis radicibus, ibi quoque summa quadratorum radicum est eadem. Sin atuem praeterea summa productorum ex ternis radicibus fuerit eadem, tum summa quoque cuborum omnium radicum erit eadem. Atque si insuper summa productorum ex quaternis radicibus fuerit eadem, tum quoque summa biquadratorum omnium radicum erit eadem, atque ita porro. Hic scilicet assumo quod facile concedetur, summam quadratorum per summam radicum et summam productorum ex binis determinari; summam cuborum autem praeterea requirere summam factorum ex ternis radicibus; ac summam biquadratorum praeterea summam factorum ex quaternis radicibus, et ita porro; quod quidem demonstratu non esset difficile.

§15 In aequationibus ergo inferiorum graduum, quarum radices denotantur respective per litteras p, q, r, s , dum ipsius propositae quinti gradus quaelibet radix littera α indicatur, erit

$$\begin{aligned}\sum \alpha &= \sum s = \sum r = \sum q = \sum p, \\ \sum \alpha^2 &= \sum s^2 = \sum r^2 = \sum q^2, \\ \sum \alpha^3 &= \sum s^3 = \sum r^3, \\ \sum \alpha^4 &= \sum s^4.\end{aligned}$$

But,

$$\begin{aligned}\sum p &= A, \\ \sum q^2 &= A \sum q - 2B, \\ \sum r^3 &= A \sum r^2 - B \sum r + 3C, \\ \sum s^4 &= A \sum s^3 - B \sum s^2 + C \sum s - 4D.\end{aligned}$$

It follows that

$$\begin{aligned}\sum \alpha &= A, \\ \sum \alpha^2 &= A \sum \alpha - 2B, \\ \sum \alpha^3 &= A \sum \alpha^2 - B \sum \alpha + 3C, \\ \sum \alpha^4 &= A \sum \alpha^3 - B \sum \alpha^2 + C \sum \alpha - 4D.\end{aligned}$$

IIIA. Partial fraction decomposition

Partial fraction decomposition

Chapter 2 of Introductio, I

§40. A rational function $\frac{M}{N}$ can be resolved into as many simple fractions of the form $\frac{A}{p-qz}$ as there are different linear factors in the denominator N .

Example (method of undetermined coefficients):

$$\frac{1+z^2}{z-z^3} = \frac{1}{z} + \frac{1}{1-z} - \frac{1}{1+z}.$$

§41. Since each linear factor of the denominator N gives rise to a simple fraction in the resolution of the given function $\frac{M}{N}$, it is shown how, from the knowledge of a linear factor of the denominator N , the corresponding simple fraction can be found.

Same example: $N = (z-a)S$

a	$z-a$	S	$A = \frac{M}{S}$	A at $z = a$
0	z	$1-z^2$	$\frac{1+z^2}{1-z^2}$	1
1	$1-z$	$z+z^2$	$\frac{1+z^2}{z+z^2}$	1
-1	$1+z$	$z-z^2$	$\frac{1+z^2}{z-z^2}$	-1

§42. A rational function with the form $\frac{P}{(p-qz)^n}$, where the degree of the numerator P is less than the degree of the denominator $(p-qz)^n$, can be transformed into the sum of partial fractions of the following form:

$$\frac{A}{(p-qz)^n} + \frac{B}{(p-qz)^{n-1}} + \frac{C}{(p-qz)^{n-2}} + \cdots + \frac{K}{p-qz},$$

where all the numerators are constants.

§43. If the denominator N of the rational function $\frac{M}{N}$ has a factor $(p-qz)^2$, the partial fractions arising from this factor are found in the following way.

Suppose $N = (p-qz)^2 S$. Write

$$\frac{M}{N} = \frac{A}{(p-qz)^2} + \frac{B}{p-qz} + \frac{P}{S}$$

where $\frac{P}{S}$ stands for the sum of all the simple fractions which arise from the factor S .

$$\frac{P}{S} = \frac{M - AS - B(p-qz)S}{(p-qz)^2 S},$$

and

$$P = \frac{M - AS - B(p - qz)S}{(p - qz)^2}$$

is a polynomial. Therefore, $M - AS - B(p - qz)S$ is a polynomial divisible by $(p - qz)^2$. Putting $z = \frac{p}{q}$, we have $M - AS = 0$, and

$$A = \frac{M}{S} \quad \text{at } z = \frac{p}{q}.$$

Now,

$$\frac{M - AS}{p - qz} - BS$$

is divisible by $p - qz$. From this,

$$B = \frac{M - AS}{(p - qz)S} = \frac{\frac{M}{S} - A}{p - qz} \quad \text{at } z = \frac{p}{q}.$$

Here, the division must first be carried out before substituting $\frac{p}{q}$ for z .

Examples. 1. $\frac{1-z^2}{z^2(1+z^2)}$. Only one partial fraction $\frac{1}{z^2}$ corresponding to z^2 .

$$2. \frac{z^3}{(1-z)^2(1+z^4)}.$$

Partial fraction corresponding to $(1 - z)^2$: $\frac{1}{2(1-z)^2} - \frac{1}{2(1-z)}$.

$$\S 44. \text{ Example: } \frac{z^2}{(1-z)^3(1+z^2)}.$$

Partial fractions corresponding to $(1 - z)^3$:

$$\frac{1}{2(1-z)^3} - \frac{1}{2(1-z)^2} - \frac{1}{4(1-z)}.$$

\S 45. If the denominator N of the rational function $\frac{M}{N}$ has a factor $(p - qz)^n$, then the partial fractions

$$\frac{A}{(p - qz)^n} + \frac{B}{(p - qz)^{n-1}} + \frac{C}{(p - qz)^{n-2}} + \cdots + \frac{K}{p - qz}$$

therefrom are calculated in the following way.

Let the denominator $N = (p - qz)^n Z$. Then

$A = \frac{M}{Z}$ when $z = \frac{p}{q}$, and

$P = \frac{M - AZ}{p - qz}$ gives $B = \frac{P}{Z}$ when $z = \frac{p}{q}$;

$Q = \frac{P - BZ}{p - qz}$ gives $C = \frac{Q}{Z}$ when $z = \frac{p}{q}$;

$$R = \frac{Q-CZ}{p-qz} \text{ gives } D = \frac{R}{Z} \text{ when } z = \frac{p}{q};$$

$$S = \frac{R-DZ}{p-qz} \text{ gives } E = \frac{S}{Z} \text{ when } z = \frac{p}{q};$$

etc., and so we find all of the partial fractions which arise from the factor $(p - qz)^n$ of the denominator N .

§46. Given any rational function whatsoever $\frac{M}{N}$, it can be resolved into parts and transformed into its simplest form in the following way.

First, one obtains all the linear factors, **whether real or complex**.

Of these factors, those which are not repeated are treated individually, and from each of them a partial is obtained from §41.

If a linear factor occurs two or more times, then these are taken together, and from their product, which will be of the form $(p - qz)^n$, we obtain the corresponding partial fractions from §45.

In this way, since for each of the linear factors partial fractions have been found, the sum of all these partial fractions will equal the given function $\frac{M}{N}$ *unless it is improper*.

If it is improper, then the polynomial part must be found and then added to the computed partial fractions in order to obtain the function $\frac{M}{N}$ expressed in its simplest form.

This is the form whether the polynomial part is extracted before or after the partial fractions are obtained, since the same partial fraction arises from an individual factor of the denominator N whether the numerator M itself is used or M increased or diminished by some multiples of N .

III.B.Introductio in Analysin Infinitorum, I (1748)
Contents

1. On functions in general
2. On the transformation of functions
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4. On the development of functions in infinite series
5. Concerning functions of two or more variables
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7. Exponentials and logarithms expressed through series
8. On transcendental quantities which arise from the circle
9. On trinomial factors
10. On the use of the discovered factors to sum infinite series
11. On other infinite expressions for arcs and sines
12. On the development of real rational functions
13. On recurrent series
14. On the multiplication and division of angles
15. On series which arise from products
16. On the partition of numbers
17. Using recurrent series to find roots of equations
18. On continued fractions.

Chapter V of Introductio, I

On the development of functions into infinite series

§60. By a **continued division procedure** the rational function $\frac{a}{\alpha + \beta z}$ can be expressed as the infinite series

$$\frac{a}{\alpha} - \frac{a\beta}{\alpha^2}z + \frac{a\beta^2}{\alpha^3}z^2 - \frac{a\beta^3}{\alpha^4}z^3 + \frac{a\beta^4}{\alpha^5}z^5 - \dots$$

This series can **also** be found by setting

$$\frac{a}{\alpha + \beta z} = A + Bz + Cz^2 + Dz^3 + Ez^4 + \dots$$

and then find the coefficients A, B, C, D, \dots which give equality.

§61. In a similar way by means of a continued division procedure the rational function

$$\frac{a + bz}{\alpha + \beta z + \gamma z^2}$$

can be converted into an infinite series.

§64. A recurrent series deserves special attention if the denominator of the fraction which gives rise to it happens to be a power.

Thus, if the rational function $\frac{a+bz}{(1-\alpha z)^2}$ is expressed as a series, the result is

$$\begin{aligned} a &+ 2\alpha az &+ 3\alpha^2 az^2 &+ 4\alpha^3 az^3 &+ 5\alpha^4 az^4 &+ \dots \\ &+ bz &+ 2\alpha bz^2 &+ 3\alpha^2 bz^3 &+ 4\alpha^3 bz^4 &+ \dots \end{aligned}$$

in which the coefficient of z^n will be

$$(n+1)\alpha^n a + n\alpha^{n-1}b.$$

But this is a recurrent series, since each term is determined by the two preceding terms. The law of formation is clearly seen when the denominator is expanded to $1 - 2\alpha z + \alpha^2 z^2$. If we let $\alpha = 1$ and $z = 1$, the series becomes a general arithmetic progression

$$a + (2a + b) + (3a + 2b) + (4a + 3b) + \dots$$

whose terms have a constant difference. Thus, every arithmetic progression is a recurrent series, for if

$$A + B + C + D + E + F + \dots$$

is an arithmetic progression then

$$C = 2B - A, \quad D = 2C - B, \quad E = 2D - C, \dots$$

III. Exponentials and logarithms

Chapter VI of Introductio, I

On exponentials and logarithms

§106. When they are transcendental, logarithms can be only approximately represented by decimal fractions. The discrepancy is less to the extent that more decimal places are used in the approximation. In the following way we can find an approximation for a logarithm by only extracting square roots.

Let $\log y = z$ and $\log v = x$, then $\log \sqrt{vy} = \frac{x+z}{2}$.

If the proposed number b lies between a^2 and a^3 , whose logarithms are 2 and 3 respectively, we look for the value of $a^{\frac{5}{2}}$... and then b lies either between a^2 and $a^{\frac{5}{2}}$ or between $a^{\frac{5}{2}}$ and a^3 . Whichever is the case, we then take the geometric mean of these two and we have closer bounds.

We repeat the process and the lengths of the intervals between which b lies decreases. In this way we eventually arrive at the value of b with the desired number of decimal places. Since the logarithm of the bounds have been computed, we finally find the logarithm of b .

Example Let $a = 10$ be the base of the logarithm, which is usually the case in the computed tables. We seek an approximate logarithm of the number 5. Since 5 lies between 1 and 10, whose logarithms are 0 and 1, in the following manner we take successive square roots until we arrive at the number 5 exactly.

$A = 1.000000;$	$\log A = 0.0000000;$	so that
$B = 10.000000;$	$\log B = 1.0000000;$	$C = \sqrt{AB}$
$C = 3.162277;$	$\log C = 0.5000000;$	$D = \sqrt{BC}$
$D = 5.623413;$	$\log D = 0.7500000;$	$E = \sqrt{CD}$
$E = 4.216964;$	$\log E = 0.6250000;$	$F = \sqrt{DE}$
$F = 4.869674;$	$\log F = 0.6875000;$	$G = \sqrt{DF}$
$G = 5.232991;$	$\log G = 0.7187500;$	$H = \sqrt{FG}$
$H = 5.048065;$	$\log H = 0.7031250;$	$I = \sqrt{FH}$
$I = 4.958069;$	$\log I = 0.6953125;$	$K = \sqrt{HI}$
$K = 5.002865;$	$\log K = 0.6992187;$	$L = \sqrt{KI}$
$L = 4.980416;$	$\log L = 0.6972656;$	$M = \sqrt{KL}$
$M = 4.991627;$	$\log M = 0.6982421;$	$N = \sqrt{KM}$
$N = 4.997242;$	$\log N = 0.6987304;$	$O = \sqrt{KN}$
$O = 5.000052;$	$\log O = 0.6989745;$	$P = \sqrt{NO}$
$P = 4.998647;$	$\log P = 0.6988525;$	$Q = \sqrt{OP}$
$Q = 4.999350;$	$\log Q = 0.6989135;$	$R = \sqrt{OQ}$
$R = 4.999701;$	$\log R = 0.6989440;$	$S = \sqrt{OR}$
$S = 4.999876;$	$\log S = 0.6989592;$	$T = \sqrt{OS}$
$T = 4.999963;$	$\log T = 0.6989668;$	$V = \sqrt{OT}$
$V = 5.000008;$	$\log V = 0.6989707;$	$W = \sqrt{TV}$
$W = 4.999984;$	$\log W = 0.6989687;$	$X = \sqrt{WV}$
$X = 4.999997;$	$\log X = 0.6989697;$	$Y = \sqrt{VX}$
$Y = 5.000003;$	$\log Y = 0.6989702;$	$Z = \sqrt{XY}$
$Z = 5.000000;$	$\log Z = 0.6989700.$	

Thus the geometric means finally converge to $z = 5.000000$ and so the logarithm of 5 is 0.6989700 when the base is 10.

§110. Example III.

Since after the flood all men descended from a population of six, if we suppose that the population after two hundred years was 1,000,000, we would like to find the annual rate of growth. We suppose that each year the increase is $\frac{1}{x}$, so that after two hundred years the population is

$$\left(\frac{1+x}{x}\right)^{200} \cdot 6 = 1,000,000.$$

It follows that

$$\frac{1+x}{x} = \left(\frac{1000000}{6}\right)^{\frac{1}{200}},$$

and so

$$\log\left(\frac{1+x}{x}\right) = \frac{1}{200} \log \frac{1000000}{6} = \frac{1}{200} \cdot 5.2218487 = 0.0261092.$$

From this, we have $\frac{1+x}{x} = \frac{1061963}{1000000}$, and so $1000000 = 61963x$ and finally x is approximately 16. We have shown that if each year the population increases by $\frac{1}{16}$, the desired result takes place. Now, if the same rate holds over an interval of four hundred years, then the population becomes

$$1000000 \cdot \frac{1000000}{6} = 166666666666.$$

However, the whole earth would never be able to sustain that population.

Chapter VII of Introductio, I Exponential and logarithms expressed through series

§114. Euler starts with $a > 1$, and write

$$a^\omega = 1 + k\omega$$

for “infinitely small” numbers ω . He immediately gave an example to illustrate how k depends on a .

Let $a = 10$. From the table of common logarithms, we look for the logarithm of a number which exceeds 1 by the smallest possible amount, for instance, $1 + \frac{1}{1000000}$, so that $k\omega = \frac{1}{1000000}$. Then

$$\log\left(1 + \frac{1}{1000000}\right) = \log \frac{1000001}{1000000} = 0.00000043429 = \omega.$$

From this, ..., $k = 2.30258$.

§115. Since $a^\omega = 1 + k\omega$, we have $a^{j\omega} = (1 + k\omega)^j$, whatever value we assign to j .

$$a^{j\omega} = 1 = \frac{j}{1}k\omega + \frac{j(j-1)}{1 \cdot 2}k^2\omega^2 + \frac{j(j-1)(j-2)}{1 \cdot 2 \cdot 3}k^3\omega^3 + \dots$$

Note: Here, Euler is not using the general binomial theorem, but rather the usual binomial theorem for a large exponent j , and then think of j as large, resulting in an infinite series.

If now we let $j = \frac{z}{\omega}$, where z is any finite number, since ω is infinitely small, then j is infinitely large. Then we have $\omega = \frac{z}{j}$, ...

$$a^z = \left(1 + \frac{kz}{j}\right)^j = 1 + \frac{1}{1}kz + \frac{1(j-1)}{1 \cdot 2j}k^2z^2 + \frac{1(j-1)(j-2)}{1 \cdot 2j \cdot 3j}k^3z^3 + \frac{1(j-1)(j-2)(j-3)}{1 \cdot 2j \cdot 3j \cdot 4j}k^4z^4 + \dots$$

This equation is true provided an infinitely large number is substituted for j , but then k is a finite number depending on a .

§116. Since j is infinitely large, $\frac{j-1}{j} = 1$, and the larger the number we substitute for j , the closer the value of the fraction $\frac{j-1}{j}$ comes to 1. Therefore, if j is a number larger than any assignable number, then $\frac{j-1}{j}$ is equal to 1. For the same reason, $\frac{j-2}{j} = 1$, $\frac{j-3}{j} = 1$, and so forth. It follows that

$$\frac{j-1}{2j} = \frac{1}{2}, \quad \frac{j-2}{3j} = \frac{1}{3}, \quad \frac{j-3}{4j} = \frac{1}{4}, \dots$$

When we substitute these values, we obtain

$$a^z = 1 + \frac{kz}{1} + \frac{k^2z^2}{1 \cdot 2} + \frac{k^3z^3}{1 \cdot 2 \cdot 3} + \frac{k^4z^4}{1 \cdot 2 \cdot 3 \cdot 4} + \dots$$

This equation expresses a relationship between the numbers a and k , since when we let $z = 1$, we have

$$a = 1 + \frac{k}{1} + \frac{k^2}{1 \cdot 2} + \frac{k^3}{1 \cdot 2 \cdot 3} + \frac{k^4}{1 \cdot 2 \cdot 3 \cdot 4} + \dots$$

When $a = 10$, then k is necessarily approximately equal to 2.30258 as we have already seen.

§119. Since we have let $(1 + k\omega)^j = 1 + z$, we have

$$1 + k\omega = (1 + x)^{\frac{1}{j}} \quad \text{and} \quad k\omega = (1 + x)^{\frac{1}{j}} - 1$$

so that

$$j\omega = \frac{j}{k}((1 + x)^{\frac{1}{j}} - 1).$$

Since $j\omega = \log(1 + x)$, it follows that

$$\log(1 + x) = \frac{j}{k}(1 + x)^{\frac{1}{j}} - \frac{j}{k}$$

where j is a number infinitely large. But we have

$$(1 + x)^{\frac{1}{j}} = 1 + \frac{1}{j}x - \frac{1(j-1)}{j \cdot 2j}x^2 + \frac{1(j-1)(2j-1)}{j \cdot 2j \cdot 3j}x^3 - \frac{1(j-1)(2j-1)(3j-1)}{j \cdot 2j \cdot 3j \cdot 4j}x^4 + \dots$$

Since j is an infinite number,

$$\frac{j-1}{2j} = \frac{1}{2}, \quad \frac{2j-1}{3j} = \frac{2}{3}, \quad \frac{3j-1}{4j} = \frac{3}{4}, \dots$$

Now it follows that

$$j(1+x)^{\frac{1}{j}} = j + \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

As a result we have

$$\log(1+x) = \frac{1}{k} \left(\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \right),$$

where a is the base of the logarithm and

$$a = 1 + \frac{k}{1} + \frac{k^2}{1 \cdot 2} + \frac{k^3}{1 \cdot 2 \cdot 3} + \frac{k^4}{1 \cdot 2 \cdot 3 \cdot 4} + \dots$$

§122. Since we are free to choose the base a for the system of logarithms, we now choose a in such a way that $k = 1$. Suppose that $k = 1$, then the series found above

$$1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \dots$$

is equal to a . If the terms are represented as decimal fractions and summed, we obtain the value for

$$a = 2.71828182845904523536028 \dots$$

When this base is chosen, the logarithms are called natural or hyperbolic. The latter name is used since the quadrature of a hyperbola can be expressed through these logarithms. For the sake of brevity for this number 2.718281828459... we will use the symbol e , which we denote the base for natural or hyperbolic logarithms, ...

§120. Euler ran into the paradox

$$2.30258 \dots = \frac{9}{1} - \frac{9^2}{2} + \frac{9^3}{3} - \frac{9^4}{4} + \dots$$

He remarked "but it is difficult to see how this can be since the terms of this series continually grow larger and the sum of several terms does not seem to approach any limit. We will soon have to answer to this paradox."

In the following section (§121), Euler displayed the series

$$\log \frac{1+x}{1-x} = \frac{2}{k} \left(\frac{x}{1} + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots \right)$$

and used this to get

$$k = 2 \left(\frac{9}{11} + \frac{9^3}{3 \cdot 11^3} + \frac{9^5}{5 \cdot 11^5} + \frac{9^7}{7 \cdot 11^7} + \dots \right)$$

“and the terms of this series decrease in a reasonable way so that soon a satisfactory approximation for k can be obtained.

§123. Natural logarithms have the property that the logarithm of $1 + \omega$ is equal to ω , where ω is an infinitely small quantity. From this it follows that $k = 1$ and the natural logarithm of all numbers can be found. Let e stand for the number found above. Then

$$e^z = 1 + \frac{z}{1} + \frac{z^2}{1 \cdot 2} + \frac{z^3}{1 \cdot 2 \cdot 3} + \frac{z^4}{1 \cdot 2 \cdot 3 \cdot 4} + \dots,$$

and the natural logarithms themselves can be found these series, where

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \dots$$

and

$$\log \frac{1+x}{1-x} = \frac{2x}{1} + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \frac{2x^9}{9} + \dots$$

This last series is strongly convergent if we substitute an extremely small fraction for x .

§123 (continued) Euler applied this last series to find

$$\begin{aligned} \log \frac{3}{2} &= \frac{2}{1 \cdot 5} + \frac{2}{3 \cdot 5^3} + \frac{2}{5 \cdot 5^5} + \frac{2}{7 \cdot 5^7} + \dots \\ \log \frac{4}{3} &= \frac{2}{1 \cdot 7} + \frac{2}{3 \cdot 7^3} + \frac{2}{5 \cdot 7^5} + \frac{2}{7 \cdot 7^7} + \dots \\ \log \frac{5}{4} &= \frac{2}{1 \cdot 9} + \frac{2}{3 \cdot 9^3} + \frac{2}{5 \cdot 9^5} + \frac{2}{7 \cdot 9^7} + \dots \end{aligned}$$

Then he made use of these to find the logarithms of integers from 1 to 10:

$$\begin{aligned} \log 2 &= 0.69314\ 71805\ 59945\ 30941\ 72321 \\ \log 3 &= 1.09861\ 22886\ 68109\ 69139\ 52452 \\ \log 4 &= 1.38629\ 43611\ 19890\ 61883\ 44642 \\ \log 5 &= 1.60943\ 79124\ 34100\ 37460\ 07593 \\ \log 6 &= 1.79175\ 94692\ 28055\ 00081\ 24773 \\ \log 7 &= 1.94591\ 01490\ 55313\ 30510\ 54639 \\ \log 8 &= 2.07944\ 15416\ 79835\ 92825\ 16964 \end{aligned}$$

$$\begin{aligned}\log 9 &= 2.19722\ 45773\ 36219\ 38279\ 04905 \\ \log 10 &= 2.30258\ 50929\ 94045\ 68401\ 79914\end{aligned}$$

Euler commented that “[a]ll of these logarithms are computed from the above three series, with the exception of $\log 7$, which can be found as follows.”

When in the last series we let $x = \frac{1}{99}$, we obtain

$$\log \frac{100}{98} = \log \frac{50}{49} = 0.02020\ 27073\ 17519\ 44840\ 78230.$$

When this is subtracted from

$$\log 50 = 2 \log 5 + \log 2 = 3.91202\ 30054\ 28146\ 05861\ 87508$$

we obtain $\log 49$. But $\log 7 = \frac{1}{2} \log 49$.

IVA. Trigonometric functions

Chapter VIII of *Introductio*, I: On transcendental quantities which arise from the circle

§126. After having considered logarithms and exponentials, we must now turn to circular arcs with their sines and cosines. This is not only because these are further genera of transcendental quantities, but also since they arise from logarithms and exponentials when complex values are used.

Then Euler gives the value of π to 127 places after the decimal point:

$$\begin{array}{r} \pi = 3.14159 \ 26535 \ 89793 \ 23846 \ 26433 \ 83279 \ 50288 \ 41971 \ 69399 \ 37510 \\ \quad \quad 58209 \ 74944 \ 59230 \ 78164 \ 06286 \ 20899 \ 86280 \ 34825 \ 34211 \ 70679 \\ \quad \quad 82148 \ 08651 \ 32723 \ 06647 \ 09384 \ 46 \dots \end{array}$$

Note that the 113th digit after the decimal point should be 8 instead of 7. Here, as in Paper 74, Euler cited the same value, adopted from Lagny (1660 – 1734).

Footnote 2 of paper 74:

Figuram centesimam decimam tertiam, qualem Eulerus auctores Th. F. de Lagny tradiderat, falsam esse, quippe quae 8, non 7, esse debeat, adnotavit G. de Vega (1756 – 1802) in libro, qui inscribitur *Thesaurus logarithmorum completus*, Lipsiae 1794, p.633.

§130. Recall addition formulas

$$\begin{aligned} \sin(y \pm z) &= \sin y \cos z \pm \cos y \sin z, \\ \cos(y \pm z) &= \cos y \cos z \mp \sin y \sin z. \end{aligned}$$

Convert products into sums:

$$\begin{aligned} \sin y \cos z &= \frac{1}{2}[\sin(y+z) + \sin(y-z)] \\ \cos y \sin z &= \frac{1}{2}[\sin(y+z) - \sin(y-z)] \\ \cos y \cos z &= \frac{1}{2}[\cos(y+z) + \cos(y-z)] \\ \sin y \sin z &= \frac{-1}{2}[\cos(y+z) - \cos(y-z)]. \end{aligned}$$

Also establish the half angle formulas:

$$\cos \frac{v}{2} = \sqrt{\frac{1 + \cos v}{2}}, \quad \sin \frac{v}{2} = \sqrt{\frac{1 - \cos v}{2}}.$$

§131. Conversion of sums into products:

$$\begin{aligned}\sin a + \sin b &= 2 \sin \frac{a+b}{2} \cos \frac{a-b}{2}, \\ \sin a - \sin b &= 2 \sin \frac{a-b}{2} \cos \frac{a+b}{2}, \\ \cos a + \cos b &= 2 \cos \frac{a+b}{2} \cos \frac{a-b}{2}, \\ \cos a - \cos b &= -2 \sin \frac{a+b}{2} \sin \frac{a-b}{2}.\end{aligned}$$

§132. Since $\sin^2 z + \cos^2 z = 1$, we have the factors $(\cos z + i \sin z)(\cos z - i \sin z) = 1$. Although these factors are complex, still they are quite useful in combining and multiplying arcs.

Then Euler established the formula

$$(\cos x \pm i \sin x)(\cos y \pm i \sin y)(\cos z \pm i \sin z) = \cos(x + y + z) \pm i \sin(x + y + z).$$

⁷ and deduce (§133) de Moivre's theorem

$$(\cos z \pm i \sin z)^n = \cos nz \pm i \sin nz.$$

§133 (continued) It follows that

$$\begin{aligned}\cos nz &= \frac{1}{2}[(\cos z + i \sin z)^n + (\cos z - i \sin z)^n], \\ \sin nz &= \frac{1}{2}[(\cos z + i \sin z)^n - (\cos z - i \sin z)^n].\end{aligned}$$

Expanding the binomials we obtain the following series

$$\begin{aligned}\cos nz &= (\cos z)^n - \frac{n(n-1)}{1 \cdot 2}(\cos z)^{n-2}(\sin z)^2 + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4}(\cos z)^{n-4}(\sin z)^4 \\ &\quad - \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}(\cos z)^{n-6}(\sin z)^6 + \dots\end{aligned}$$

and

⁷[sic] The correct formula should read

$$(\cos x \pm i \sin x)(\cos y \pm i \sin y)(\cos z \pm i \sin z) = \cos(x \pm y \pm z) + i \sin(x \pm y \pm z).$$

$$\begin{aligned}\sin nz &= \frac{n}{1}(\cos z)^{n-1} \sin z - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}(\cos z)^{n-3}(\sin z)^3 \\ &\quad + \frac{n(n-1)(n-2)(n-3)(n-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}(\cos z)^{n-5}(\sin z)^5 - \dots\end{aligned}$$

§134. Let the arc z be infinitely small, then $\sin z = z$ and $\cos z = 1$. If n is an infinitely large number, so that nz is a finite number, say $nz = v$, then, since $\sin z = z = \frac{v}{n}$, we have

$$\begin{aligned}\cos v &= 1 - \frac{v^2}{2!} + \frac{v^4}{4!} - \frac{v^6}{6!} + \dots, \\ \sin v &= v - \frac{v^3}{3!} + \frac{v^5}{5!} - \frac{v^7}{7!} + \dots\end{aligned}$$

It follows that if v is a given arc, by means of these series, the sine and cosine can be found.

§135. Once sines and cosines have been computed, tangents and cotangents can be found in the ordinary way. However, since the multiplication and division of such gigantic numbers is so inconvenient, a different method of expressing these functions is desirable.

§138. Once again we use the formulas in §133, where we let z be an infinitely small arc and let n be an infinitely large number j , so that jz has a finite value v . Now we have $nz = v$ and $z = \frac{v}{j}$, so that $\sin z = \frac{v}{j}$ and $\cos z = 1$. With these substitutions,

$$\cos v = \frac{(1 + \frac{iv}{j})^j + (1 - \frac{iv}{j})^j}{2}$$

and

$$\sin v = \frac{(1 + \frac{iv}{j})^j - (1 - \frac{iv}{j})^j}{2i}.$$

In the preceding chapter we saw that $(1 + \frac{z}{j})^j = e^z$ where e is the base of the natural logarithm. When we let $z = iv$ and then $z = -iv$ we obtain

$$\cos v = \frac{e^{iv} + e^{-iv}}{2} \quad \text{and} \quad \sin v = \frac{e^{-v} - e^{-iv}}{2i}.$$

From these equations we understand how complex exponentials can be expressed by real sines and cosines, since

$$e^{iv} = \cos v + i \sin v \quad \text{and} \quad e^{-iv} = \cos v - i \sin v.$$

IVB. Logarithms of complex numbers

Dunham, pp.98 – 102 on the controversy of Johann Bernoulli and Leibniz on the logarithms of negative numbers.

Paper 170 *Recherches sur les racines ...* (1749)⁸

Probleme 1. §79. *Un quantité imaginaire étant élevée à une puissance dont l'exposant est une quantité réelle quelconque, déterminer la forme imaginaire qui en résulte.*

Solution: Soit $a + bi$ la quantité imaginaire, et m l'exposant réel de la puissance, de sorte qu'il s'agit de déterminer M et N pour qu'il soit

$$(a + bi)^m = M + Ni.$$

Posons

$$\sqrt{a^2 + b^2} = c$$

et c sera toujours une quantité réelle et positive, car nous ne regardons pas ici l'ambiguïté du signe $\sqrt{}$. Ensuite cherchons l'angle φ tel que son sinus soit $= \frac{b}{c}$ et le cosinus $= \frac{a}{c}$, ayant ici égard à la nature des quantités a et b , si elles sont affirmatives ou négatives; et il est certain, qu'on pourra toujours assigner et angle φ , quelles que soient les quantités a , b , pourvu qu'elles soient réelles, comme nous le supposons. Or ayant trouvé cet angle φ , qui sera toujours réel, on aura en même tems tous les autres angles dont le sinus $\frac{b}{c}$ et le cosinus $\frac{a}{c}$ sont les mêmes; car posant π l'angle de 180° , tous ces angles seront

$$\varphi, \quad 2\pi + \varphi, \quad 4\pi + \varphi, \quad 6\pi + \varphi, \quad 8\pi + \varphi, \dots$$

auxquels on peut ajouter ceux-cy

$$-2\pi + \varphi, \quad -4\pi + \varphi, \quad -6\pi + \varphi, \quad -8\pi + \varphi, \dots$$

Cela posé il sera

$$a + bi = c(\cos \varphi + i \sin \varphi),$$

et la puissance proposée

$$(a + bi)^m = c^m(\cos \varphi + i \sin \varphi)^m,$$

où c^m aura toujours une valeur réelle positive, qu'il faut lui donner préférablement à toutes les autres valeurs, qu'il pourroit avoir. Ensuite il est démontré que

$$(\cos \varphi + i \sin \varphi)^m = \cos m\varphi + i \sin m\varphi;$$

où il faut remarquer, que puisque m est une quantité réelle, l'angle $m\varphi$ sera aussi réel; et partant aussi son sinus et son cosinus. Donc nous aurons

$$(a + bi)^m = c^m(\cos m\varphi + i \sin m\varphi),$$

⁸See also Paper 168: *De la Controverse Entre Mrs. Leibniz et Bernoulli sur les Logarithmes des Nombres Negatifs et Imaginaires* (1749) and Paper 807: *Sur les logarithmes des nombres negatifs et imaginaires* (1749)

ou bien la puissance $(a + bi)^m$ est contenue dans la forme $M + Ni$, en prenant

$$M = c^m \cos m\varphi \quad \text{et} \quad N = c^m \sin m\varphi,$$

où il y a

$$c = \sqrt{a^2 + b^2} \quad \text{et} \quad \cos \varphi = \frac{a}{c} \quad \text{et} \quad \sin \varphi = \frac{b}{c}.$$

Probleme 2. §87. *Une quantité réelle positive étant élevée à une puissance dont l'exposant est une quantité imaginaire, trouver la valeur imaginaire de cette puissance.*

Solution. Soit la quantité réelle positive et $m + ni$ l'exposant de la puissance, de sorte qu'il faut chercher la valeur imaginaire de a^{m+ni} . Soit donc $a^{m+ni} = x + yi$, et il sera

$$(m + ni) \log a = \log(x + yi),$$

dont prenant les différentiels en posant a , x et y variable, nous aurons

$$\frac{mda}{a} + \frac{nda}{a}i = \frac{dx + idy}{x + yi} = \frac{xdx + ydy}{x^2 + y^2} + \frac{xdy - ydx}{x^2 + y^2}i.$$

Egalant donc séparément ensemble les membres réels et imaginaires, nous aurons ces deux équations

$$\frac{mda}{a} = \frac{xdx + ydy}{x^2 + y^2} \quad \text{et} \quad \frac{nda}{a} = \frac{xdy - ydx}{x^2 + y^2},$$

dont les intégrales prises, comme il faut, seront

$$\sqrt{x^2 + y^2} = a^m, \quad \text{et} \quad \arctan \frac{y}{x} = n \log a \quad \text{ou} \quad \frac{y}{x} = \tan n \log a,$$

où la maque le logarithme hyperbolique de la quantité réelle positive a , lequel aura par conséquent aussi une valeur réelle. Prenant donc dans un cercle dont le rayon = 1, un arc = $n \log a$, à cause de $\sqrt{x^2 + y^2} = a^m$, nous obtiendrons

$$x = a^m \cos n \log a \quad \text{et} \quad y = a^m \sin n \log a,$$

et ces valeurs étant posées pour x et y , on aura

$$a^{m+ni} = x + yi.$$

Probleme 3. §93. *Une quantité imaginaire étant élevée à une puissance dont l'exposant est aussi imaginaire, trouver la valeur imaginaire de cette puissance.*

Solution: To express $(a + bi)^{m+ni}$ in the form $x + yi$ for real x and y , write $a + bi = c(\cos \varphi + i \sin \varphi)$. Then,

$$\begin{aligned}x &= c^m e^{-n\varphi} \cos(m\varphi + n \log c), \\y &= c^m e^{-n\varphi} \sin(m\varphi + n \log c).\end{aligned}$$

Probleme 4. §100. *Une nombre imaginaire quelconque étant proposé, trouver son logarithme hyperbolique.*

Solution:

$$\begin{aligned}\log(a + bi) &= \log \sqrt{a^2 + b^2} + i \arccos \frac{a}{\sqrt{a^2 + b^2}}, \quad \text{ou} \\ \log(a + bi) &= \log \sqrt{a^2 + b^2} + i \arcsin \frac{b}{\sqrt{a^2 + b^2}}.\end{aligned}$$

§101. (Corollaire 1) Puisqu'il y a une infinité d'angles auxquels répond de même sinus $\frac{b}{\sqrt{a^2+b^2}}$ et cosinus $\frac{a}{\sqrt{a^2+b^2}}$, chaque nombre, tant réel qu'imaginaire, a une infinité des logarithmes, dont tous sont imaginaires à l'exception d'un seul, lorsque $b = 0$ et a un nombre positif.

§102. (Corollaire 2) Si nous posons $\sqrt{a^2 + b^2} = c$, et l'angle trouvé = φ , à cause de $a = c \cos \varphi$ et $b = c \sin \varphi$, il sera $\log c(\cos \varphi + i \sin \varphi) = \log c + i\varphi$; où au lieu de φ il est permis de mettre

$$\pm 2\pi + \varphi, \pm 4\pi + \varphi, \pm 6\pi + \varphi, \text{ etc.}$$

le caractère π marquant la somme de deux angles droits. On aura donc

$$\log(\cos \varphi + i \sin \varphi) = \varphi i.$$

Probleme 5. §103. *Un logarithme imaginaire étant donné, trouver le nombre qui lui convient.*

Solution: If $\log(x + yi) = a + bi$, then

$$x = e^a \cos b \quad y = e^a \sin b.$$

Probleme 6. §106. *Un angle ou arc de cercle imaginaire quelconque étant proposé, trouver son sinus et cosinus et tangente.*

Solution:

$$\cos bi = 1 + \frac{b^2}{1 \cdot 2} + \frac{b^4}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots = \frac{e^b + e^{-b}}{2},$$

$$\sin bi = bi + \frac{b^3 i}{1 \cdot 2 \cdot 3} + \frac{b^5 i}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \cdots = \frac{e^b - e^{-b}}{2} i.$$

From these,

$$\begin{aligned}\sin(a + bi) &= \frac{1}{2}(e^b + e^{-b}) \sin a + \frac{i}{2}(e^b - e^{-b}) \cos a, \\ \cos(a + bi) &= \frac{1}{2}(e^b + e^{-b}) \cos a - \frac{i}{2}(e^b - e^{-b}) \sin a.\end{aligned}$$

Probleme 7. §112. *Le sinus d'un angle étant réel, mais plus grand que le sinus total, de sorte que l'angle soit imaginaire, trouver la valeur de cet angle.*

Probleme 8. §113 *Le cosinus d'un angle étant réel, mais plus grand que le sinus total = 1, trouver la valeur de cet angle.*

Probleme 9. §114[a] *Le sinus d'un angle étant imaginaire, trouver la valeur de l'angle ou l'arc imaginaire qui lui répond.*

$$\sin(a + bi) = p + qi.$$

Probleme 10. §116. *Le cosinus d'un angle étant imaginaire, trouver la valeur de l'angle ou l'arc imaginaire qui lui répond.*

$$\cos(a + bi) = p + qi.$$

Probleme 11 §119. *Une tangente imaginaire étant donnée, trouver la valeur imaginaire de l'angle ou de l'arc qui lui répond.*

Solution. If $\tan(a + bi) = p + qi$, then

$$\begin{aligned}\sin 2a &= \frac{2p}{\sqrt{4p^2 + (1 - p^2 - q^2)^2}}, \\ \cos 2a &= \frac{1 - p^2 - q^2}{\sqrt{4p^2 + (1 - p^2 - q^2)^2}}, \\ b &= \frac{1}{2} \log \frac{p^2 + (1 + q)^2}{\sqrt{4p^2 + (1 - p^2 - q^2)^2}}.\end{aligned}$$

VA. Euler's works on infinite series Volumes I.14, 15, 16 (2 parts) of Series prima, Opera Omnia.

- Euler's summation formula, the series $\sum \frac{1}{n^{2k}}$ and the Bernoulli numbers, (14 papers).
- On trigonometric functions and trigonometric series (13 papers).
- Infinite products and continued fractions, (11 papers)
- On the calculation of π , (7 papers).
- The binomial series and the binomial coefficients, (9 papers)
- Interpolation, the Gamma function, and the Euler constant, (10 papers).
- various functions and series, (17 papers).

Paper 25. Methodus Generalis Summandi Progressiones (1732/3) 1738

§4 begins with a review on the summation of a geometric progression

$$x^a + x^{a+b} + x^{a+2b} + \dots + x^{a+(n-1)b} = \frac{x^a - x^{a+nb}}{1 - x^b}$$

[Just write s for the sum.

$$s - x^a = x^{a+b} + x^{a+2b} + \dots + x^{a+(n-1)b}.$$

Adding x^{a+nb} and dividing by x^b , we get s again. From this, s can be found.]

§5 considers summation of a more general series

$$x^a + 2x^{a+b} + 3x^{a+2b} + \dots + nx^{a+(n-1)b}.$$

Write s for the sum.

(1) Write down $s - x^a$.

(2) Add the "next" term $(n+1)x^{a+nb}$ to it and divide by x^b .

(3) Compare the result with s and obtain

$$\frac{s - x^a + (n+1)x^{a+nb}}{x^b} - s = \frac{x^a - x^{a+nb}}{1 - x^b}.$$

Solving for s :

$$s = \frac{x^a - x^{a+nb}}{(1 - x^b)^2} - \frac{nx^{a+nb}}{1 - x^b}.$$

§7 sums a special case by a different method:

$$s = x + 2x^2 + 3x^3 + 4x^4 + \dots + nx^n$$

Divide by x and then integrate:

$$\begin{aligned}\int \frac{s}{x} dx &= \int (1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1}) dx \\ &= x + x^2 + \cdots + x^n \\ &= \frac{x - x^{n+1}}{1 - x}.\end{aligned}$$

Differentiating,

$$\frac{s}{x} = \frac{d}{dx} \left(\frac{x - x^{n+1}}{1 - x} \right) = \frac{(1 - (n+1)x^n)(1 - x) - (x - x^{n+1})}{(1 - x)^2}.$$

From this,

$$s = \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1 - x)^2}.$$

§19. Euler considered the more general series [hypergeometric series]

$$\frac{a+b}{\alpha+\beta}x + \frac{(a+b)(a+2b)}{(\alpha+\beta)(\alpha+2\beta)}x^2 + \cdots + \frac{(a+b)(a+2b)\cdots(a+(n-1)b)}{(\alpha+\beta)(\alpha+2\beta)\cdots(\alpha+(n-1)\beta)}x^n + \cdots$$

and obtained the sum as

$$s = \frac{(b+a) \int x^{\frac{\beta}{\alpha}} (\alpha - ax)^{\frac{b}{\alpha} - \frac{\beta}{\alpha}} dx}{x^{\frac{\beta}{\alpha}} (\alpha - ax)^{\frac{b}{\alpha} - \frac{\beta}{\alpha} + 1}}.$$

§20 considers a particular example in detail:

$$s := x + \frac{3}{2!}x^2 + \frac{5}{3!}x^3 + \cdots + \frac{2n-1}{n!}x^n.$$

Multiply by px^q and integrate:

$$p \int x^q s dx = \frac{p}{(q+2)}x^{q+2} + \frac{3p}{(q+3)2!}x^{q+3} + \cdots + \frac{(2n-1)p}{(q+n+1)n!}x^{q+n+1}$$

Now choose p and q so that

$$(2k-1)p = q + k + 1$$

for each $k = 1, \dots, n$. This requires $p = \frac{1}{2}$ and $q = -\frac{3}{2}$. From this we obtain

$$\frac{1}{2} \int x^{-\frac{3}{2}} s dx = \frac{x^{\frac{1}{2}}}{1} + \frac{x^{\frac{3}{2}}}{2!} + \frac{x^{\frac{5}{2}}}{3!} + \cdots + \frac{x^{n-\frac{1}{2}}}{n!}.$$

Multiply by $x^{\frac{1}{2}}$:

$$\frac{1}{2}x^{\frac{1}{2}} \int x^{-\frac{3}{2}}s dx = \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

Calculation of this sum y .

$$\begin{aligned} \frac{dy}{dx} &= 1 + \frac{x}{1} + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} \\ &= 1 + y - \frac{x^n}{n!}. \end{aligned}$$

$$(1-2x) \int x^{-\frac{3}{2}}s dx = 4x^{\frac{1}{2}} - \frac{2s}{x^{\frac{1}{2}}} - \frac{4x^{n+\frac{1}{2}}}{n!}.$$

For large n , one obtains the infinite sum

$$s := x + \frac{3}{2!}x^2 + \frac{5}{3!}x^3 + \cdots + \frac{2n-1}{n!}x^n + \cdots$$

as a solution of the linear differential equation

$$\int x^{-\frac{3}{2}}s dx = \frac{4x-2s}{(1-2x)\sqrt{x}}.$$

Differentiate and obtain

$$\frac{ds}{dx} + \frac{(1+2x)}{1-2x}s = \frac{1+2x}{1-2x}.$$

Multiply by $\frac{1}{1-2x}e^{-x}$ and integrate [integrating factor] to obtain

$$s = 1 - (1-2x)e^x.$$

In particular, putting $x = \frac{1}{2}$,

$$1 = \frac{1}{2} + \frac{3}{2! \cdot 4} + \frac{5}{3! \cdot 8} + \cdots + \frac{2n-1}{n! \cdot 2^n} + \cdots$$

At the end of this paper, Euler mentioned the following two series:

$$1 + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \frac{1}{2^n - 1} \quad \text{finite}$$

and

$$\frac{1}{3} + \frac{1}{7} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{26} + \cdots$$

whose general term is the reciprocal of $a^\alpha - 1$, a , α greater than 1. The sum of the series is 1, demonstrated by Goldbach. This series is the subject of paper 72.

VB. An Example from Euler's earliest paper on infinite series**VB. Paper 20:** De summatione innumerabilium progressionum (1730/31) 1738.Euler evaluates, for an arbitrary α , the integral

$$\int -y^{\alpha-2} \log(1-y) dy$$

in two ways.

(1) Using the series expansion for $\log(1-y)$, he obtains the integral as

$$\frac{y^\alpha}{\alpha} + \frac{y^{\alpha+1}}{2(\alpha+1)} + \frac{y^{\alpha+2}}{3(\alpha+2)} + \dots$$

(2) On the other hand, writing $y = 1 - z$, and using

$$\int z^n \log z = C - \frac{z^{n+1}}{(n+1)^2} + \frac{z^{n+1}}{n+1} \log z,$$

the integral can be written as a "trium sequentium serierum":

$$\begin{aligned} & 1 - \frac{\alpha-2}{1 \cdot 4} + \frac{(\alpha-2)(\alpha-3)}{2! \cdot 9} - \frac{(\alpha-2)(\alpha-3)(\alpha-4)}{3! \cdot 16} + \dots \\ & - z + \frac{\alpha-2}{1 \cdot 4} z^2 - \frac{(\alpha-2)(\alpha-3)}{2! \cdot 9} z^3 + \frac{(\alpha-2)(\alpha-3)(\alpha-4)}{3! \cdot 16} z^4 - \dots \\ & + z \log z - \frac{\alpha-2}{2!} z^2 \log z + \frac{(\alpha-2)(\alpha-3)}{3!} z^3 \log z - \frac{(\alpha-2)(\alpha-3)(\alpha-4)}{4!} z^4 \log z + \dots \end{aligned}$$

In §22, Euler puts $\alpha = 1$, and obtains, for the trium sequentium serierum,

$$\begin{aligned} & +1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots \\ & - z - \frac{1}{4} z^2 - \frac{1}{9} z^2 - \frac{1}{16} z^2 - \dots \\ & + z \log z + \frac{1}{2} z^2 \log z + \frac{1}{3} z^3 \log z + \frac{1}{4} z^4 \log z + \dots \\ & = y + \frac{1}{4} y^2 + \frac{1}{9} y^3 + \frac{1}{16} y^4 + \dots \end{aligned}$$

Note that

$$z + \frac{1}{2} z^2 + \frac{1}{3} z^3 + \dots = -\log(1-z) = -\log y.$$

He was indeed aiming at:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots = \frac{y+z}{1} + \frac{y^2+z^2}{4} + \frac{y^3+z^3}{9} + \frac{y^4+z^4}{16} + \dots + \log y \log z$$

where $y + z = 1$. In particular, if $y = z = \frac{1}{2}$,

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = 1 + \frac{1}{8} + \frac{1}{36} + \frac{1}{128} + \cdots + (\log 2)^2.$$

Euler was trying to evaluate the sum

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

He did some numerical calculations, making use of

$$\log 2 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 8} + \frac{1}{4 \cdot 16} + \cdots$$

with

$$(\log 2)^2 \approx 0.480453,$$

and

$$1 + \frac{1}{8} + \frac{1}{36} + \frac{1}{128} + \cdots \approx 1.164481.$$

From this, he obtained

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots \approx 1.644934.$$

Si quis autem huius seriei summam addendis aliquot terminis initialibus determinare voluerit, plus quam mille terminos addere deberet, quo nostrum inventum numerum reperieret.

VC. Euler's Paper 72: Variæ Observationes Circa Series Infinitas (1737) 1744.

Theorem 1.

The sum of the series⁹

$$\frac{1}{3} + \frac{1}{7} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{26} + \frac{1}{31} + \frac{1}{35} + \cdots$$

where the denominators are numbers of the form $m^n - 1$, m, n both greater than 1, is unity.

Proof. Write

$$x = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots.$$

[But this is a divergent series!]

From

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$

⁹Communicated by Goldbach.

we have

$$x - 1 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{9} + \frac{1}{10} + \dots$$

Here the denominators of the fractions on the right sides exclude powers of 2. From

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \dots$$

we have

$$x - 1 - \frac{1}{2} = 1 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{10} + \frac{1}{11} + \dots$$

Now, the powers of 3 are excluded from the denominators on the right hand side. From

$$\frac{1}{4} = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \dots$$

we have

$$x - 1 - \frac{1}{2} - \frac{1}{4} = 1 + \frac{1}{6} + \frac{1}{7} + \frac{1}{10} + \dots$$

Proceeding, we have

$$x - 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{9} - \dots = 1,$$

or

$$x - 1 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{9} + \frac{1}{10} + \dots$$

Comparing with

$$x = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \dots$$

we have

$$1 = \frac{1}{3} + \frac{1}{7} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{26} + \dots$$

Theorem 2

$$\frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \frac{1}{31} + \frac{1}{35} + \frac{1}{63} + \dots = \log 2.$$

Here, the denominators are $m^n - 1$ for m even.

$$\frac{1}{4} + \frac{1}{24} + \frac{1}{26} + \frac{1}{48} + \frac{1}{80} + \dots = 1 - \log 2.$$

Theorem 3.

$$\frac{\pi}{4} = 1 - \frac{1}{8} - \frac{1}{24} + \frac{1}{28} - \frac{1}{48} - \frac{1}{80} - \frac{1}{120} - \frac{1}{124} - \frac{1}{168} - \frac{1}{224} + \frac{1}{244} - \frac{1}{288} - \dots$$

Theorem 4.

$$\frac{\pi}{4} - \frac{3}{4} = \frac{1}{28} - \frac{1}{124} + \frac{1}{244} + \frac{1}{344} + \dots$$

Corollary 5.

$$\pi = 3 + \frac{1}{7} - \frac{1}{31} + \frac{1}{61} + \frac{1}{86} + \frac{1}{333} + \frac{1}{547} - \frac{1}{549} - \frac{1}{781} + \frac{1}{844} - \dots$$

Theorem 5.

$$\frac{\pi}{4} - \log 2 = \left(\frac{1}{26} + \frac{1}{28}\right) + \left(\frac{1}{242} + \frac{1}{244}\right) + \left(\frac{1}{342} + \frac{1}{344}\right) + \dots$$

where

$$27 = 3^3, \quad 243 = 3^5, \quad 343 = 7^3,$$

are odd powers of an odd number of the form $4k - 1$.

Now, Euler made use of the sensational formula he found earlier in Paper 41 *De Summis Serierum Reciprocarum* (1734):

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots = \frac{\pi^2}{6}$$

to obtain

Theorem 6.

$$\frac{1}{15} + \frac{1}{63} + \frac{1}{80} + \frac{1}{255} + \frac{1}{624} + \dots = \frac{7}{4} - \frac{\pi^2}{6}.$$

Theorem 7.

$$\frac{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \dots}{1 \cdot 2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 16 \cdot 18 \dots} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

$$\prod_{p \text{ prime}} \frac{p}{p-1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

In a somewhat more familiar form, this reads

$$\prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p}} = \text{harmonic series}$$

Theorem 8.

$$\prod_{p \text{ prime}} \frac{p^n}{p^n - 1} = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \dots$$

Corollaries

$$\frac{\pi^2}{6} = \frac{2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 11 \cdot 11 \cdots}{1 \cdot 3 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdots}$$

Theorem 9.

$$\frac{5 \cdot 13 \cdot 25 \cdot 61 \cdot 85 \cdot 145 \cdots}{4 \cdot 12 \cdot 24 \cdot 60 \cdot 84 \cdot 144 \cdots} = \frac{3}{2}$$

Theorema 19. Summa seriei reciprocae numerorum primorum

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \cdots$$

ist infinite magna, infinites tamen minor quam summa seriei harmonicae

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

Atque illius summa est huius summae quasi logarithmus.

VIA. Euler's first calculation of

$$1 + \frac{1}{2^{2k}} + \frac{1}{3^{2k}} + \frac{1}{4^{2k}} + \frac{1}{5^{2k}} + \dots$$

Euler's papers on these series:

- Volume 14: 41(1734/5), 61(1743), 63(1743), 130(1740).
- Volume 15: 393(1769), 597(1785).
- Volume 16 (part 1): 664(1790),
- Volume 16 (part 2): 736(1809), 746(1812).

Paper 41: De Summis Serierum Reciprocarum (1734)

In this paper, Euler used p to denote π , and q for $\frac{\pi}{2}$.^{10 11}
 §§3,4 begin with the series for sine and cosine.

$$y = s - \frac{s^3}{3!} + \frac{s^5}{5!} - \frac{s^7}{7!} + \dots,$$

$$x = 1 - \frac{s^2}{2!} + \frac{s^4}{4!} - \frac{s^6}{6!} + \dots$$

§5. If, for a fixed y , the roots of the equation

$$0 = 1 - \frac{s}{y} + \frac{s^3}{3!y} - \frac{s^5}{5!y} + \dots$$

are A, B, C, D, E etc., then the "infinite polynomial" factors into the product of

$$1 - \frac{s}{A}, 1 - \frac{s}{B}, 1 - \frac{s}{C}, 1 - \frac{s}{D}, \dots$$

¹⁰First occurrence of the sign π . According to Cajori, *History of Mathematical Notations*, Art. 396, "[t]he modern notation for 3.14159... was introduced in 1706. It was in that year that William Jones made himself noted, without being aware that he was doing anything noteworthy, through his designation of the ratio of the length of the circle to its diameter by the letter π . He took this step with ostentation".

¹¹Euler's use of π . *Ibid.*, Art. 397: In 1734 Euler employed p instead of π and g instead of $\frac{\pi}{2}$. In a letter of April 16, 1738, from Stirling to Euler, as well as in Euler's reply, the letter p is used. But in 1736 he designated the ratio by the sign $1 : \pi$ and thus either consciously adopted the notation of Jones or independently fell upon it. ... But the letter is not restricted to this use in his *mechanica*, and the definition of π is repeated when it is taken for 3.14159... He represented 3.1415... again by π in 1737 (in a paper printed in 1744), in 1743, in 1746, and in 1748. Euler and Goldbach used $\pi = 3.1415...$ repeatedly in their correspondence in 1739. Johann Bernoulli used in 1739, in his correspondence with Euler, the letter c , (circumferentia), but in a letter of 1740 he began to use π . Likewise, Nikolaus Bernoulli employed π in his letters to Euler of 1742. Particularly favorable for wider adoption was the appearance of π for 3.1415... in Euler's *Introductio in analysin infinitorum* (1748). In most of his later publications, Euler clung to π as his designation of 3.1415....

By comparing coefficients,

$$\frac{1}{y} = \frac{1}{A} + \frac{1}{B} + \frac{1}{C} + \frac{1}{D} + \dots$$

§7. If A is an acute angle (smallest arc) with $\sin A = y$, then all the angles with sine equal to y are

$$A, \pm\pi - A, \pm 2\pi + A, \pm 3\pi - A, \pm 4\pi + A, \dots$$

The sums of these reciprocals is $\frac{1}{y}$. Sum of these reciprocals taking two at a time is 0, taking three at a time is $\frac{-1}{3!y}$. etc.

§8. In general, if

$$\begin{aligned} a + b + c + d + e + f + \dots &= \alpha, \\ ab + ac + bc + \dots &= \beta, \\ abc + abd + bcd + \dots &= \gamma, \end{aligned}$$

then

$$\begin{aligned} a^2 + b^2 + c^2 + \dots &= \alpha^2 - 2\beta, \\ a^3 + b^3 + c^3 + \dots &= \alpha^3 - 3\alpha\beta + 3\gamma, \end{aligned}$$

and

$$a^4 + b^4 + c^4 + \dots = \alpha^4 - 4\alpha^2\beta + 4\alpha\gamma + 2\beta^2 - 4\delta.$$

Euler denotes by Q the sum of the squares, R the sum of cubes, S the sum of fourth powers etc. and wrote

$$\begin{aligned} P &= \alpha, \\ Q &= P\alpha - 2\beta, \\ R &= Q\alpha - P\beta + 3\gamma, \\ S &= R\alpha - Q\beta + P\gamma - 4\delta, \\ T &= S\alpha - R\beta + Q\gamma - P\delta + 5\epsilon, \\ &\vdots \end{aligned}$$

Now he applies these to

$$\alpha = \frac{1}{y}, \quad \beta = 0, \quad \gamma = \frac{-1}{3!y}, \quad \delta = 0, \quad \epsilon = \frac{1}{5!y}, \dots$$

and obtains

$$P = \frac{1}{y}, \quad Q = \frac{1}{y^2}, \quad R = \frac{Q}{y} - \frac{1}{2!y}, \quad S = \frac{R}{y} - \frac{P}{3!y},$$

and

$$\begin{aligned} T &= \frac{S}{y} - \frac{Q}{3!y} + \frac{1}{4!y}, \\ V &= \frac{T}{y} - \frac{R}{3!y} + \frac{P}{5!y}, \\ W &= \frac{V}{y} - \frac{S}{3!y} + \frac{Q}{5!y} - \frac{1}{6!y}. \end{aligned}$$

§10. Now Euler takes $y = 1$. All the angles with sines equal to 1 are

$$\frac{\pi}{2}, \frac{\pi}{2}, \frac{-3\pi}{2}, \frac{-3\pi}{2}, \frac{5\pi}{2}, \frac{5\pi}{2}, \frac{-7\pi}{2}, \frac{-7\pi}{2}, \frac{9\pi}{2}, \frac{9\pi}{2}, \dots$$

From these,

$$\frac{4}{\pi} \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots \right) = 1.$$

This gives Leibniz's famous series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots = \frac{\pi}{4}.$$

§11. Noting that $P = \alpha = 1$, $\beta = 0$, so that $Q = P = 1$, Euler proceeded to obtain

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{8}.$$

From this, he deduced that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{6}.$$

Continuing with

$$R = \frac{1}{2}, S = \frac{1}{3}, T = \frac{5}{24}, V = \frac{2}{15}, W = \frac{6}{720}, X = \frac{17}{315}, \dots$$

Euler obtained

$$\begin{aligned} 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \dots &= \frac{\pi^3}{32}, \\ 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \dots &= \frac{\pi^4}{96}, \\ 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \dots &= \frac{\pi^4}{90}, \end{aligned}$$

$$\begin{aligned}
1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \cdots &= \frac{5\pi^5}{1536}, \\
1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \cdots &= \frac{\pi^6}{960}, \\
1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \cdots &= \frac{\pi^6}{945}, \\
1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \frac{1}{9^7} - \cdots &= \frac{61\pi^7}{184320}, \\
1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{9^8} + \cdots &= \frac{17\pi^8}{161280}, \\
1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \cdots &= \frac{\pi^8}{9450}.
\end{aligned}$$

Then Euler remarked that the general values of

$$1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots$$

can be determined from the sequence

$$1, 1, \frac{1}{2}, \frac{1}{3}, \frac{5}{24}, \frac{2}{15}, \frac{61}{720}, \frac{17}{315}, \dots$$

In the remaining articles, §16 – 18, Euler gave an alternative method of determining these sums. This time, he considered $y = 0$ and made use of

$$0 = s - \frac{s^3}{3!} + \frac{s^5}{5!} - \frac{s^7}{7!} + \cdots.$$

Removing the obvious factor s , he noted that the roots of

$$0 = 1 - \frac{s^2}{3!} + \frac{s^4}{5!} - \frac{s^6}{7!} + \cdots$$

are

$$\pm\pi, \pm 2\pi, \pm 3\pi, \dots$$

so that

$$1 - \frac{s^2}{3!} + \frac{s^4}{5!} - \frac{s^6}{7!} + \cdots = \left(1 - \frac{s^2}{\pi^2}\right) \left(1 - \frac{s^2}{4\pi^2}\right) \left(1 - \frac{s^2}{9\pi^2}\right) \left(1 - \frac{s^2}{16\pi^2}\right) \cdots$$

§17. With

$$\alpha = \frac{1}{3!}, \beta = \frac{1}{5!}, \gamma = \frac{1}{7!}, \dots$$

and

$$P = \frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \frac{1}{16\pi^2} + \cdots,$$

Q the sum of the squares of these terms, R the sum of the cubes, etc., Euler now obtained

$$\begin{aligned} P &= \alpha = \frac{1}{3!} = \frac{1}{6}, \\ Q &= P\alpha - 2\beta = \frac{1}{90}, \\ R &= Q\alpha - P\beta + 3\gamma = \frac{1}{945}, \\ S &= R\alpha - Q\beta + P\gamma - 4\delta = \frac{1}{9450}, \\ T &= S\alpha - R\beta + Q\gamma - P\delta + 5\epsilon = \frac{1}{93555}, \\ &\vdots \\ V &= T\alpha - S\beta + R\gamma - Q\delta + P\epsilon - 6\zeta = \frac{691}{6825 \cdot 93555}. \end{aligned}$$

In §18, he summarized the formulas which made him famous throughout Europe (for the first time):

$$\begin{aligned} 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots &= \frac{\pi^2}{6}, \\ 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \cdots &= \frac{\pi^4}{90}, \\ 1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \cdots &= \frac{\pi^6}{945}, \\ 1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \cdots &= \frac{\pi^8}{9450}, \\ 1 + \frac{1}{2^{10}} + \frac{1}{3^{10}} + \frac{1}{4^{10}} + \frac{1}{5^{10}} + \cdots &= \frac{\pi^{10}}{93555}, \\ 1 + \frac{1}{2^{12}} + \frac{1}{3^{12}} + \frac{1}{4^{12}} + \frac{1}{5^{12}} + \cdots &= \frac{691\pi^{12}}{6825 \cdot 93555}, \\ &\vdots \end{aligned}$$

VIB. Chapter X of Introductio ad Analysin Infinitorum, I

§165. Euler started with

$$1 + Az + Bz^2 + Cz^3 + Dz^4 + \cdots = (1 + \alpha z)(1 + \beta z)(1 + \gamma z)(1 + \delta z) \cdots$$

and write

$$\begin{aligned} P &= \alpha + \beta + \gamma + \delta + \epsilon + \cdots, \\ Q &= \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \epsilon^2 + \cdots, \\ R &= \alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \epsilon^3 + \cdots, \\ S &= \alpha^4 + \beta^4 + \gamma^4 + \delta^4 + \epsilon^4 + \cdots, \\ T &= \alpha^5 + \beta^5 + \gamma^5 + \delta^5 + \epsilon^5 + \cdots, \\ V &= \alpha^6 + \beta^6 + \gamma^6 + \delta^6 + \epsilon^6 + \cdots. \end{aligned}$$

He stated the relations

$$\begin{aligned} P &= A, \\ Q &= AP - 2B, \\ R &= AQ - BP + 3C, \\ S &= AR - BQ + CP - 4D, \\ T &= AS - BR + CQ - DP + 5E, \\ V &= AT - BS + CR - DQ + EP - 6F. \end{aligned}$$

The truth of these formulas is intuitively clear, but a rigorous proof will be given in the differential calculus.

§167. Then he applied to

$$\begin{aligned} \frac{e^z - e^{-z}}{2} &= x \left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} + \cdots \right) \\ &= x \left(1 + \frac{x^2}{\pi^2} \right) \left(1 + \frac{x^2}{4\pi^2} \right) \left(1 + \frac{x^2}{9\pi^2} \right) \left(1 + \frac{x^2}{16\pi^2} \right) \left(1 + \frac{x^2}{25\pi^2} \right) \cdots \end{aligned}$$

$$1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} + \cdots = \left(1 + \frac{x^2}{\pi^2} \right) \left(1 + \frac{x^2}{4\pi^2} \right) \left(1 + \frac{x^2}{9\pi^2} \right) \left(1 + \frac{x^2}{16\pi^2} \right) \left(1 + \frac{x^2}{25\pi^2} \right) \cdots$$

Write $x^2 = \pi^2 z$.

$$1 + \frac{\pi^2}{3!} z + \frac{\pi^4}{5!} z^2 + \frac{\pi^6}{7!} z^3 + \cdots = (1 + z) \left(1 + \frac{z}{4} \right) \left(1 + \frac{z}{9} \right) \left(1 + \frac{z}{16} \right) \left(1 + \frac{z}{25} \right) \cdots$$

From these

$$\begin{aligned}
 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots &= \frac{2^0}{3!} \cdot \frac{1}{1} \cdot \pi^2, \\
 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \cdots &= \frac{2^2}{5!} \cdot \frac{1}{3} \cdot \pi^4, \\
 1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \cdots &= \frac{2^4}{7!} \cdot \frac{1}{3} \cdot \pi^6, \\
 1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \cdots &= \frac{2^6}{9!} \cdot \frac{1}{5} \cdot \pi^8, \\
 1 + \frac{1}{2^{10}} + \frac{1}{3^{10}} + \frac{1}{4^{10}} + \frac{1}{5^{10}} + \cdots &= \frac{2^8}{11!} \cdot \frac{5}{3} \cdot \pi^{10}, \\
 1 + \frac{1}{2^{12}} + \frac{1}{3^{12}} + \frac{1}{4^{12}} + \frac{1}{5^{12}} + \cdots &= \frac{2^{10}}{13!} \cdot \frac{691}{105} \cdot \pi^{12}, \\
 &\vdots
 \end{aligned}$$

Not only these, he continued up to the 26th powers. Writing for $k = 1, 2, \dots, 13$,

$$1 + \frac{1}{2^{2k}} + \frac{1}{3^{2k}} + \frac{1}{4^{2k}} + \frac{1}{5^{2k}} + \cdots = \frac{2^{2k-2}}{(2k+1)!} \cdot \frac{X}{Y} \cdot \pi^{2k}.$$

We could continue with more of these, but we have gone far enough to see a sequence which at first seems quite irregular,

$$1, \frac{1}{3}, \frac{1}{3}, \frac{3}{5}, \frac{5}{3}, \frac{691}{105}, \frac{35}{1},$$

but it is of extraordinary usefulness in several places.

The remaining numbers are

$$\frac{3617}{15}, \frac{43867}{21}, \frac{122277}{55}, \frac{854513}{3}, \frac{1181820455}{273}, \frac{76977927}{1}.$$

VIC. Euler's "forgotten" Paper 63: *Demonstration de la somme de cette suite*

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \text{etc}$$

Euler begins by integrating the series

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{1 \cdot 3}{2 \cdot 4}x^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^6 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}x^8 + \cdots$$

to obtain

$$s = x + \frac{1}{2 \cdot 3} x^3 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} x^5 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7} x^7 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 9} x^9 + \dots$$

The function s is, of course, $\arcsin x$. But Euler continues to write s and considers

$$sds = \frac{xdx}{\sqrt{1-x^2}} + \frac{1}{2 \cdot 3} \cdot \frac{x^3 dx}{\sqrt{1-x^2}} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} \cdot \frac{x^5 dx}{\sqrt{1-x^2}} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7} \cdot \frac{x^7 dx}{\sqrt{1-x^2}} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 9} \cdot \frac{x^9 dx}{\sqrt{1-x^2}} + \dots$$

An easy integration by parts gives

$$\int \frac{x^{n+2} dx}{\sqrt{1-x^2}} = \frac{n+1}{n+2} \int \frac{x^n dx}{\sqrt{1-x^2}} - \frac{x^{n+1}}{n+2} \sqrt{1-x^2}.$$

Consequently,

$$\int_0^1 \frac{x^{n+2} dx}{\sqrt{1-x^2}} = \frac{n+1}{n+2} \int_0^1 \frac{x^n dx}{\sqrt{1-x^2}}.$$

Since

$$\int_0^1 \frac{xdx}{\sqrt{1-x^2}} = \left[1 - \sqrt{1-x^2} \right]_0^1 = 1,$$

we have

$$\begin{aligned} \int_0^1 \frac{x^3 dx}{\sqrt{1-x^2}} &= \frac{2}{3}, \\ \int_0^1 \frac{x^5 dx}{\sqrt{1-x^2}} &= \frac{2 \cdot 4}{3 \cdot 5}, \\ \int_0^1 \frac{x^7 dx}{\sqrt{1-x^2}} &= \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7}, \\ \int_0^1 \frac{x^9 dx}{\sqrt{1-x^2}} &= \frac{2 \cdot 4 \cdot 6 \cdot 8}{3 \cdot 5 \cdot 7 \cdot 9}, \\ &\vdots \end{aligned}$$

Therefore,

$$\int_{s=0}^{s=\frac{\pi}{2}} sds = \frac{\pi^2}{8}$$

is equal to

$$\begin{aligned} &1 + \frac{1}{2 \cdot 3} \cdot \frac{2}{3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} \cdot \frac{2 \cdot 4}{3 \cdot 5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7} \cdot \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 9} \cdot \frac{2 \cdot 4 \cdot 6 \cdot 8}{3 \cdot 5 \cdot 7 \cdot 9} + \dots \\ &= 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \dots \end{aligned}$$

From this, it is easy to obtain

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6}.$$

Euler next proceeds to sum the series

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \cdots$$

by considering the series expansion of $\frac{1}{2}(\arcsin x)^2$. This he writes as $y = \frac{1}{2}s^2$, and assumes

$$y = \alpha x^2 + \beta x^4 + \gamma x^6 + \delta x^8 + \cdots$$

By differentiation, $\frac{dy}{dx} = s \frac{ds}{dx}$. This gives

$$\begin{aligned} y' &= s \cdot s' = \frac{s}{\sqrt{1-x^2}}, \\ y'' &= \frac{s' \sqrt{1-x^2} + \frac{xs}{\sqrt{1-x^2}}}{1-x^2} = \frac{1+xy'}{1-x^2}. \end{aligned}$$

Therefore,

$$(1-x^2)y'' - xy' - 1 = 0.$$

By differentiating the series, we obtain

$$\begin{aligned} y' &= 2\alpha x + 4\beta x^3 + 6\gamma x^5 + 8\delta x^7 + \cdots \\ y'' &= 2\alpha + 3 \cdot 4\beta x^2 + 5 \cdot 6\gamma x^4 + 7 \cdot 8\delta x^6 + \cdots \\ x^2 y'' &= 2\alpha x^2 + 3 \cdot 4\beta x^4 + 5 \cdot 6\gamma x^6 + 7 \cdot 8\delta x^8 + \cdots. \end{aligned}$$

From these,

$$\begin{array}{r} 1 = 2\alpha + 3 \cdot 4\beta x^2 + 5 \cdot 6\gamma x^4 + 7 \cdot 8\delta x^6 + \cdots \\ \quad - 2\alpha x^2 - 3 \cdot 4\beta x^4 - 5 \cdot 6\gamma x^6 - \cdots \\ \quad - 2\alpha x^2 - 4\beta x^4 - 6\gamma x^6 - \cdots \end{array}$$

Therefore,

$$\begin{aligned} \alpha &= \frac{1}{2}, \\ \beta &= \frac{2 \cdot 2}{3 \cdot 4} \alpha = \frac{2 \cdot 2}{2 \cdot 3 \cdot 4}, \\ \gamma &= \frac{4 \cdot 4}{5 \cdot 6} \beta = \frac{2 \cdot 2 \cdot 4 \cdot 4}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}, \end{aligned}$$

$$\begin{aligned}\delta &= \frac{6 \cdot 6}{7 \cdot 8} \gamma = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}, \\ \epsilon &= \frac{8 \cdot 8}{9 \cdot 10} \delta = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10}, \\ &\vdots\end{aligned}$$

Simplifying these fractions, Euler obtains

$$y = \frac{1}{2}s^2 = \frac{1}{2}x^2 + \frac{2}{3} \cdot \frac{x^4}{4} + \frac{2 \cdot 4}{3 \cdot 5} \cdot \frac{x^6}{6} + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7} \cdot \frac{x^8}{8} + \frac{2 \cdot 4 \cdot 6 \cdot 8}{3 \cdot 5 \cdot 7 \cdot 9} \cdot \frac{x^{10}}{10} + \dots$$

Therefore,

$$\int_{s=0}^{s=\frac{\pi}{2}} y ds = \int_{s=0}^{s=\frac{\pi}{2}} \frac{1}{2} s^2 ds = \left[\frac{1}{6} s^3 \right]_0^{\frac{\pi}{2}} = \frac{1}{48} \pi^3$$

can also be obtained as

$$\begin{aligned}\frac{1}{2} \int_0^1 \frac{x^2 dx}{\sqrt{1-x^2}} + \frac{2}{3} \cdot \frac{1}{4} \int_0^1 \frac{x^4 dx}{\sqrt{1-x^2}} + \frac{2 \cdot 4}{3 \cdot 5} \cdot \frac{1}{6} \int_0^1 \frac{x^6 dx}{\sqrt{1-x^2}} + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7} \cdot \frac{1}{8} \int_0^1 \frac{x^8 dx}{\sqrt{1-x^2}} \\ + \frac{2 \cdot 4 \cdot 6 \cdot 8}{3 \cdot 5 \cdot 7 \cdot 9} \cdot \frac{1}{10} \int_0^1 \frac{x^{10} dx}{\sqrt{1-x^2}} + \dots\end{aligned}$$

Now,

$$\begin{aligned}\int_0^1 \frac{dx}{\sqrt{1-x^2}} &= \frac{\pi}{2}, \\ \int_0^1 \frac{x^2 dx}{\sqrt{1-x^2}} &= \frac{1}{2} \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{1}{2} \cdot \frac{\pi}{2}, \\ \int_0^1 \frac{x^4 dx}{\sqrt{1-x^2}} &= \frac{3}{4} \int_0^1 \frac{x^2 dx}{\sqrt{1-x^2}} = \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{\pi}{2}, \\ \int_0^1 \frac{x^6 dx}{\sqrt{1-x^2}} &= \frac{5}{6} \int_0^1 \frac{x^4 dx}{\sqrt{1-x^2}} = \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{\pi}{2}, \\ &\vdots\end{aligned}$$

It follows that

$$\frac{\pi^3}{48} = \frac{1}{2 \cdot 2} \cdot \frac{\pi}{2} + \frac{1}{4 \cdot 4} \cdot \frac{\pi}{2} + \frac{1}{6 \cdot 6} \cdot \frac{\pi}{2} + \frac{1}{8 \cdot 8} \cdot \frac{\pi}{2} + \dots$$

From this, we conclude, once again,

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots.$$

Ces deux méthodes, toutes faciles qu'elles méritoient une plus grande attention, si elles se pouvoient employer également pour trouver les sommes des plus hautes puissances paires, qui sont toutes comprises dans mon autre méthode générale tirée de la considération des racines d'une équation infinie. Mais malgré toute la peine que je me suis donnée pour trouver seulement la somme des biarrés

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \cdots$$

je n'ai pas encore pu réussir dans cette recherche, quoique la somme par l'autre méthodes me soit connue; laquelle est = $\frac{\pi^4}{90}$.

$$\begin{aligned} 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots &= \frac{2}{3!} \cdot \frac{1}{2} \cdot \pi^2, \\ 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \cdots &= \frac{2^3}{5!} \cdot \frac{1}{6} \cdot \pi^4, \\ 1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \cdots &= \frac{2^5}{7!} \cdot \frac{1}{6} \cdot \pi^6, \\ 1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \cdots &= \frac{2^7}{9!} \cdot \frac{3}{10} \cdot \pi^8, \\ 1 + \frac{1}{2^{10}} + \frac{1}{3^{10}} + \frac{1}{4^{10}} + \frac{1}{5^{10}} + \cdots &= \frac{2^9}{11!} \cdot \frac{5}{6} \cdot \pi^{10}, \\ 1 + \frac{1}{2^{12}} + \frac{1}{3^{12}} + \frac{1}{4^{12}} + \frac{1}{5^{12}} + \cdots &= \frac{2^{11}}{13!} \cdot \frac{691}{210} \cdot \pi^{12}, \\ &\vdots \\ 1 + \frac{1}{2^{26}} + \frac{1}{3^{26}} + \frac{1}{4^{26}} + \frac{1}{5^{26}} + \cdots &= \frac{2^{25}}{27!} \cdot \frac{76977927}{2} \cdot \pi^{26}, \end{aligned}$$

Thus,

$$1 + \frac{1}{2^{2k}} + \frac{1}{3^{2k}} + \frac{1}{4^{2k}} + \frac{1}{5^{2k}} + \cdots = \frac{2^{2k-1}}{(2k+1)!} \cdot ? \cdot \pi^{2k}$$

For $2k = 14, 16, 18, 20, 22, 24$, this factor ? is

$$\frac{35}{2}, \frac{3617}{30}, \frac{43867}{42}, \frac{1222277}{110}, \frac{854513}{6}, \frac{1181820455}{546}.$$

VIIA. Jacob Bernoulli's summation of the powers of natural numbers

Ars Conjectandi (1713)

[Bernoulli arranged the binomial coefficients in the table below and made use of them to sum the powers of natural numbers.]

1	0	0	0	0	0	0
1	1	0	0	0	0	0
1	2	1	0	0	0	0
1	3	3	1	0	0	0
1	4	6	4	1	0	0
1	5	10	10	5	1	0
1	6	15	20	15	6	1
1	7	21	35	35	21	7

Let the series of natural numbers 1, 2, 3, 4, 5, etc. up to n be given, and let it be required to find their sum, the sum of the squares, cubes, etc. [Bernoulli then gave a simple derivation of the formula

$$\int n = \frac{1}{2}n^2 + n,$$

for the sum of the first n natural numbers. He then continued]. A term in the third column is generally taken to be

$$\frac{(n-1)(n-2)}{1 \cdot 2} = \frac{n^2 - 3n + 2}{2},$$

and the sum of all terms (that is, of all $\frac{n^2-3n+2}{2}$) is

$$\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} = \frac{n^3 - 3n^2 + 2n}{6},$$

and

$$\int \frac{1}{2}n^2 = \frac{n^3 - 3n^2 + 2n}{6} + \int \frac{3}{2}n - \int 1;$$

but $\int \frac{3}{2}n = \frac{3}{2} \int n = \frac{3}{4}n^2 + \frac{3}{4}n$ and $\int 1 = n$. Substituting, we have

$$\frac{1}{2}n^2 = \frac{n^3 - 3n^2 + 2n}{6} = \frac{3n^2 + 3n}{4} - n = \frac{1}{6}n^3 + \frac{1}{4}n^2 + \frac{1}{12}n,$$

of which the double $\int n^2$ (the sum of the squares of all n)¹²

$$= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n.$$

¹²More generally, $\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$. This identity can be established by considering the number of $(k+1)$ -element subsets of $\{1, 2, 3, \dots, n+1\}$, noting that the greatest element m of each subset must be one of $k+1, k+2, \dots, n+1$, and that there are exactly $\binom{m-1}{k}$ subsets with greatest element m .

A term of the fourth column is generally

$$\frac{(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3} = \frac{n^3 - 6n^2 + 11n - 6}{6},$$

and the sum of all terms is

$$\frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{n^4 - 6n^3 + 11n^2 - 6n}{24}.$$

it must certainly be that

$$\int \frac{1}{6}n^3 - \int n^2 + \int \frac{11}{6}n - \int 1 = \frac{n^4 - 6n^3 + 11n^2 - 6n}{24}.$$

Hence,

$$\int \frac{1}{6}n^3 = \frac{n^4 - 6n^3 + 11n^2 - 6n}{24} + \int n^2 - \int \frac{11}{6}n + \int 1.$$

And before it was found that ... When all substitutions are made, the following results:

$$\int n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{24}n^2.$$

Thus, we can step by step reach higher and higher powers and with slight effort form the following table:

$$\begin{array}{l} \int n^1 = \frac{1}{2}n^2 + \frac{1}{2}n, \\ \int n^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n, \\ \int n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 \\ \int n^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n \\ \int n^5 = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{1}{12}n^4 - \frac{1}{12}n^2 \\ \int n^6 = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n \\ \int n^7 = \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{1}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}n^2 \\ \int n^8 = \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{1}{3}n^7 - \frac{7}{15}n^5 + \frac{2}{9}n^3 - \frac{1}{30}n \\ \int n^9 = \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 - \frac{7}{10}n^6 + \frac{1}{2}n^4 - \frac{3}{20}n^2 \\ \int n^{10} = \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 - n^7 + n^5 - \frac{1}{2}n^3 + \frac{5}{66}n. \end{array}$$

Whoever will examine the series as to their regularity may be able to continue the table. Taking c to be the power of any exponent, the sum of all n^c or

$$\begin{aligned} \int n^c &= \frac{1}{c+1}n^{c+1} + \frac{1}{2}n^c + \frac{c}{2}An^{c-1} + \frac{c(c-1)(c-2)}{2 \cdot 3 \cdot 4}Bn^{c-3} \\ &\quad + \frac{c(c-1)(c-2)(c-3)(c-4)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}Cn^{c-5} \end{aligned}$$

$$+ \frac{c(c-1)(c-2)(c-3)(c-4)(c-5)(c-6)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} D n^{c-7} + \dots,$$

the exponents of n continually decreasing by 2 until n or n^2 is reached. The capital letters A, B, C, D denote in order the coefficients of the last terms in the expressions for $f n^2, f n^4, f n^6, f n^8$ etc., namely,

$$A = \frac{1}{6}, \quad B = -\frac{1}{30}, \quad C = \frac{1}{42}, \quad D = -\frac{1}{30}.$$

The coefficients are such that each one completes the others in the same expression to unity. Thus, D must have the value $-\frac{1}{30}$, because

$$\frac{1}{9} + \frac{1}{2} + \frac{2}{3} - \frac{7}{15} + \frac{2}{9} - \frac{1}{30} = 1.$$

Euler on the Bernoulli numbers

Paper 642: De Singulari Ratione Differentiandi et Integrandi quae in Summis Serierum Occurrit.

§1. $\sum x^n := 1^n + 2^n + \dots + x^n.$

$\sum x^0 = x$	
Mult. $\frac{1}{2}x;$ $\sum x^1 = \frac{1}{2}x^2 + \frac{1}{2}x,$	
Mult. $\frac{2}{3}x, \frac{2}{2}x;$ $\sum x^2 = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x,$	
Mult. $\frac{3}{4}x, \frac{3}{3}x, \frac{3}{2}x;$ $\sum x^3 = \frac{1}{4}x^4 + \frac{1}{2}x^3 + \frac{1}{4}x^2 + 0x$	
Mult. $\frac{4}{5}x, \frac{4}{4}x, \frac{4}{3}x, \frac{4}{2}x;$ $\sum x^4 = \frac{1}{5}x^5 + \frac{1}{2}x^4 + \frac{1}{3}x^3 + 0x^2 - \frac{1}{30}x$	
Mult. $\frac{5}{6}x, \frac{5}{5}x, \frac{5}{4}x, \frac{5}{3}x, \frac{5}{2}x;$ $\sum x^5 = \frac{1}{6}x^6 + \frac{1}{2}x^5 + \frac{5}{12}x^4 + 0x^3 - \frac{1}{12}x^2 + 0x$	
Mult. $\frac{6}{7}x, \frac{6}{6}x, \frac{6}{5}x, \frac{6}{4}x, \frac{6}{3}x, \frac{6}{2}x;$ $\sum x^6 = \frac{1}{7}x^7 + \frac{1}{2}x^6 + \frac{1}{2}x^5 + 0x^4 - \frac{1}{6}x^3 + 0x^2 + \frac{1}{42}x$	
Mult. $\frac{7}{8}x, \frac{7}{7}x, \frac{7}{6}x, \frac{7}{5}x, \frac{7}{4}x, \frac{7}{3}x, \frac{7}{2}x;$ $\sum x^7 = \frac{1}{8}x^8 + \frac{1}{2}x^7 + \frac{7}{12}x^6 + 0x^5 - \frac{7}{24}x^4 + 0x^3 + \frac{1}{12}x^2 + 0x$	
Mult. $\frac{8}{9}x, \frac{8}{8}x, \frac{8}{7}x, \frac{8}{6}x, \frac{8}{5}x, \frac{8}{4}x, \frac{8}{3}x, \frac{8}{2}x;$ $\sum x^8 = \frac{1}{9}x^9 + \frac{1}{2}x^8 + \frac{2}{3}x^7 + 0x^6 - \frac{7}{15}x^5 + 0x^4 + \frac{2}{9}x^3 + 0x - \frac{1}{30}x$	

Here, each rightmost coefficient is determined by the condition that the sum of the coefficients on the same line should be equal to 1.

VIII. Euler's summation formula¹³

Institutiones Calculi Differentialis, Part II, Chapter V: Investigatio summae serierum ex termino generali

Euler establishes his famous summation formula.

§§105 – 106. Let y be a function of x . Then

$$v = y(x - 1) = y - \frac{dy}{dx} + \frac{1}{2} \cdot \frac{d^2y}{dx^2} - \frac{1}{6} \cdot \frac{d^3y}{dx^3} + \frac{1}{24} \cdot \frac{d^4y}{dx^4} - \frac{1}{120} \cdot \frac{d^5y}{dx^5} + \dots$$

Summation gives

$$Sv = Sy - S\frac{dy}{dx} + \frac{1}{2} \cdot S\frac{d^2y}{dx^2} - \frac{1}{6} \cdot S\frac{d^3y}{dx^3} + \frac{1}{24} \cdot S\frac{d^4y}{dx^4} - \frac{1}{120} \cdot S\frac{d^5y}{dx^5} + \dots$$

Since $Sv = Sy - y + A$ for some constant A , this can be rewritten as

$$y - A = S\frac{dy}{dx} - \frac{1}{2} \cdot S\frac{d^2y}{dx^2} + \frac{1}{6} \cdot S\frac{d^3y}{dx^3} - \frac{1}{24} \cdot S\frac{d^4y}{dx^4} + \frac{1}{120} \cdot S\frac{d^5y}{dx^5} - \dots$$

or

$$S\frac{dy}{dx} = y - A + \frac{1}{2} \cdot S\frac{d^2y}{dx^2} - \frac{1}{6} \cdot S\frac{d^3y}{dx^3} + \frac{1}{24} \cdot S\frac{d^4y}{dx^4} - \frac{1}{120} \cdot S\frac{d^5y}{dx^5} + \dots$$

Putting $z = \frac{dy}{dx}$, we have

$$Sz = \int z dz - A + \frac{1}{2} \cdot S\frac{dz}{dx} - \frac{1}{6} \cdot S\frac{d^2z}{dz^2} + \frac{1}{24} \cdot S\frac{d^3z}{dz^3} - \frac{1}{120} \cdot S\frac{d^4z}{dz^4} + \dots$$

In §§107 – 108, Euler illustrated with the example $y = x^{n+1}$, leading to formulas for the sums of powers of consecutive integers.

$$Sx^n = \frac{1}{n+1}x^{n+1} + \frac{n}{2}Sx^{n-1} - \frac{n(n-1)}{2 \cdot 3}Sx^{n-2} + \frac{n(n-1)(n-2)}{2 \cdot 3 \cdot 4}Sx^{n-3} - \dots$$

Specifically, since $Sx^0 = x$,

$$Sx^1 = \frac{1}{2}x^2 + \frac{1}{2}Sx^0 = \frac{1}{2}x^2 + \frac{1}{2}x;$$

$$Sx^2 = \frac{1}{3}x^3 + Sx - \frac{1}{3}Sx^0 = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x;$$

$$Sx^3 = \frac{1}{4}x^4 + \frac{3}{2}Sx^2 - Sx + \frac{1}{4}Sx^0 = \frac{1}{4}x^4 + \frac{1}{2}x^3 + \frac{1}{4}x^2;$$

¹³See also G. Ferraro, Some aspects of Euler's theory of series: inexplicable functions and the Euler - MacLaurin summation formula, *Hist. Math.*, 25 (1998) 290 – 317.

$$Sx^4 = \frac{1}{5}x^5 + \frac{4}{2}Sx^3 - \frac{4}{2}Sx^2 + Sx - \frac{1}{5}Sx^0 = \frac{1}{5}x^5 + \frac{1}{2}x^4 + \frac{1}{3}x^3 - \frac{1}{30}x.$$

§§110 – 111. Euler seeks to rewrite the formula

$$Sz = \int z dz - A + \frac{1}{2} \cdot S \frac{dz}{dx} - \frac{1}{6} \cdot S \frac{d^2 z}{dx^2} + \frac{1}{24} \cdot S \frac{d^3 z}{dx^3} - \frac{1}{120} \cdot S \frac{d^4 z}{dx^4} + \dots$$

in the form of

$$Sz = \int z dz - A + \alpha z + \beta \frac{dz}{dx} + \gamma \frac{d^2 z}{dx^2} + \delta \frac{d^3 z}{dx^3} + \delta \frac{d^4 z}{dx^4} + \epsilon \frac{d^5 z}{dx^5} + \dots$$

Applying the first formula in §109 repeatedly,

$$\begin{aligned} S \frac{dz}{dx} &= x + \frac{1}{2} \cdot S \frac{d^2 z}{dx^2} - \frac{1}{6} \cdot S \frac{d^3 z}{dx^3} + \frac{1}{24} \cdot S \frac{d^4 z}{dx^4} - \dots \\ S \frac{d^2 z}{dx^2} &= \frac{dz}{dx} + \frac{1}{2} \cdot S \frac{d^3 z}{dx^3} - \frac{1}{6} \cdot S \frac{d^4 z}{dx^4} + \frac{1}{24} \cdot S \frac{d^5 z}{dx^5} - \dots \\ S \frac{d^3 z}{dx^3} &= \frac{d^2 z}{dx^2} + \frac{1}{2} \cdot S \frac{d^4 z}{dx^4} - \frac{1}{6} \cdot S \frac{d^5 z}{dx^5} + \frac{1}{24} \cdot S \frac{d^6 z}{dx^6} - \dots \\ S \frac{d^4 z}{dx^4} &= \frac{d^3 z}{dx^3} + \frac{1}{2} \cdot S \frac{d^5 z}{dx^5} - \frac{1}{6} \cdot S \frac{d^6 z}{dx^6} + \frac{1}{24} \cdot S \frac{d^7 z}{dx^7} - \dots \\ &\vdots \end{aligned}$$

§111. By comparison,

$$\begin{aligned} \int z dz &= Sz - \frac{1}{2} \cdot S \frac{dz}{dx} + \frac{1}{6} \cdot S \frac{d^2 z}{dx^2} - \frac{1}{24} \cdot S \frac{d^3 z}{dx^3} + \frac{1}{120} S \frac{d^4 z}{dx^4} - \dots \\ \alpha z &= \quad + \quad \alpha S \frac{dz}{dx} - \frac{\alpha}{2} \cdot S \frac{d^2 z}{dx^2} + \frac{\alpha}{6} \cdot S \frac{d^3 z}{dx^3} - \frac{\alpha}{24} S \frac{d^4 z}{dx^4} + \dots \\ \beta \frac{dz}{dx} &= \quad \quad \quad \beta \cdot S \frac{d^2 z}{dx^2} - \frac{\beta}{2} \cdot S \frac{d^3 z}{dx^3} + \frac{\beta}{6} \cdot S \frac{d^4 z}{dx^4} - \dots \\ \gamma \frac{d^2 z}{dx^2} &= \quad \quad \quad \quad \quad \gamma \cdot S \frac{d^3 z}{dx^3} - \frac{\gamma}{2} \cdot S \frac{d^4 z}{dx^4} + \dots \\ \delta \frac{d^3 z}{dx^3} &= \quad \quad \quad \quad \quad \quad \quad \delta \cdot S \frac{d^4 z}{dx^4} - \dots \\ &\vdots \end{aligned}$$

he obtains the following relations

$$\begin{array}{ll}
 \alpha - \frac{1}{2} = 0, & \alpha = \frac{1}{2}, \\
 \beta - \frac{\alpha}{2} + \frac{1}{6} = 0, & \beta = \frac{\alpha}{2} - \frac{1}{6} = \frac{1}{12}, \\
 \gamma - \frac{\beta}{2} + \frac{\alpha}{6} - \frac{1}{24} = 0, & \gamma = \frac{\beta}{2} - \frac{\alpha}{6} + \frac{1}{24} = 0, \\
 \delta - \frac{\gamma}{2} + \frac{\beta}{6} - \frac{\alpha}{24} + \frac{1}{120} = 0, & \delta = \frac{\gamma}{2} - \frac{\beta}{6} + \frac{\alpha}{24} - \frac{1}{120} = -\frac{1}{720}, \\
 \epsilon - \frac{\delta}{2} + \frac{\gamma}{6} - \frac{\beta}{24} + \frac{\alpha}{120} - \frac{1}{720} = 0, & \epsilon = \frac{\delta}{2} - \frac{\gamma}{6} + \frac{\beta}{24} - \frac{\alpha}{120} + \frac{1}{720} = 0, \\
 \zeta - \frac{\epsilon}{2} + \frac{\delta}{6} - \frac{\gamma}{24} + \frac{\beta}{120} - \frac{\alpha}{720} + \frac{1}{5040} = 0, & \dots
 \end{array}$$

§113. From the recurrence relations defining $\alpha, \beta, \gamma, \delta$, Euler observes that the series

$$1 + \alpha u + \beta u^2 + \gamma u^3 + \delta u^4 + \epsilon u^5 + \zeta u^6 + \dots$$

is actually the series of

$$V = \frac{1}{1 - \frac{1}{2}u + \frac{1}{6}u^3 - \frac{1}{24}u^3 + \frac{1}{120}u^4 - \dots} = \frac{u}{1 - e^{-u}}.$$

VIII. Euler's summation formula (continued)

§114. Euler observes that the function V is almost an even function. More precisely, $V - \frac{1}{2}u$ contains only even powers of u , since

$$V - \frac{1}{2}u = \frac{\frac{1}{2}u(1 + e^{-u})}{1 - e^{-u}} = \frac{e^{\frac{1}{2}u} + e^{-\frac{1}{2}u}}{\frac{2}{u}(e^{\frac{1}{2}u} - e^{-\frac{1}{2}u})},$$

and each of the numerator and the denominator clearly contains only even powers of u .

This means that

$$\gamma = 0, \quad \epsilon = 0, \dots,$$

and

$$V = 1 + \frac{1}{2}u + \beta u^2 + \delta u^4 + \zeta u^6 + \theta u^8 + \chi u^{10} + \dots$$

§116. Now, $1 + \beta u^2 + \delta u^4 + \zeta u^6 + \theta u^8 + \dots$ is the quotient

$$\frac{1 + \frac{u^2}{2 \cdot 4} + \frac{u^4}{2 \cdot 4 \cdot 6 \cdot 8} + \frac{u^6}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12} + \frac{u^8}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14 \cdot 16} + \dots}{1 + \frac{u^2}{4 \cdot 6} + \frac{u^4}{4 \cdot 6 \cdot 8 \cdot 10} + \frac{u^6}{4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14} + \frac{u^8}{4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14 \cdot 16 \cdot 18} + \dots}.$$

From these, he obtains

$$\beta = \frac{1}{2 \cdot 4} - \frac{1}{4 \cdot 6},$$

$$\begin{aligned}\delta &= \frac{1}{2 \cdot 4 \cdot 6 \cdot 8} - \frac{\beta}{4 \cdot 6} - \frac{1}{4 \cdot 6 \cdot 8 \cdot 10}, \\ \zeta &= \frac{1}{2 \cdot 4 \cdot 6 \cdots 12} - \frac{\delta}{4 \cdot 6} - \frac{\beta}{4 \cdot 6 \cdot 8 \cdot 10} - \frac{1}{4 \cdot 6 \cdots 14}, \\ \theta &= \frac{1}{2 \cdot 4 \cdot 6 \cdots 16} - \frac{\zeta}{4 \cdot 6} - \frac{\delta}{4 \cdot 6 \cdot 8 \cdot 10} - \frac{\beta}{4 \cdot 6 \cdots 14} - \frac{1}{4 \cdot 6 \cdots 18}, \\ &\vdots\end{aligned}$$

These give the coefficients in Euler's summation formula

$$Sz = \int z dz - A + \frac{1}{2}z + \beta \frac{dz}{dx} + \delta \frac{d^3 z}{dx^3} + \zeta \frac{d^5 z}{dx^5} + \theta \frac{d^7 z}{dx^7} + \cdots$$

§117. To identify these with the Bernoulli numbers, Euler now changes the signs in this formula alternately. In other words, he replaces the above formula by

$$Sz = \int z dz - A + \frac{1}{2}z - \beta \frac{dz}{dx} + \delta \frac{d^3 z}{dx^3} - \zeta \frac{d^5 z}{dx^5} + \theta \frac{d^7 z}{dx^7} - \cdots$$

Now, these new coefficients are given by

$$\begin{aligned}&1 + \beta u^2 + \delta u^4 + \zeta u^6 + \theta u^8 + \cdots \\ &= \frac{1 - \frac{u^2}{2 \cdot 4} + \frac{u^4}{2 \cdot 4 \cdot 6 \cdot 8} - \frac{u^6}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12} + \frac{u^8}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14 \cdot 16} - \cdots}{1 - \frac{u^2}{4 \cdot 6} + \frac{u^4}{4 \cdot 6 \cdot 8 \cdot 10} - \frac{u^6}{4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14} + \frac{u^8}{4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14 \cdot 16 \cdot 18} - \cdots}.\end{aligned}$$

Specifically,

$$\begin{aligned}\beta &= \frac{1}{4 \cdot 6} - \frac{1}{2 \cdot 4}, \\ \delta &= \frac{\beta}{4 \cdot 6} - \frac{1}{4 \cdot 6 \cdot 8 \cdot 10} + \frac{1}{2 \cdot 4 \cdot 6 \cdot 8}, \\ \zeta &= \frac{\delta}{4 \cdot 6} - \frac{\beta}{4 \cdot 6 \cdot 8 \cdot 10} + \frac{1}{4 \cdot 6 \cdots 14} - \frac{1}{2 \cdot 4 \cdot 6 \cdots 12}, \\ &\vdots\end{aligned}$$

§118. Euler notes that the rational function in the preceding section is actually

$$\frac{\cos \frac{u}{2}}{2 \sin \frac{u}{2}} = \frac{u}{2} \cot \frac{u}{2}.$$

§124. To identify the coefficients in the series expansion of $\frac{u}{2} \cot \frac{u}{2}$, Euler invokes a formula established earlier (§43):

$$\frac{1}{1-z^2} + \frac{1}{4-z^2} + \frac{1}{9-z^2} + \frac{1}{16-z^2} + \cdots = \frac{1}{2z^2} - \frac{\pi}{2z} \cot \pi z.$$

Resolving each of these into series,

$$\begin{aligned} \frac{1}{1-z^2} &= 1 + z^2 + z^4 + z^6 + z^8 + \cdots \\ \frac{1}{4-z^2} &= \frac{1}{2^2} + \frac{z^2}{2^4} + \frac{z^4}{2^6} + \frac{z^6}{2^8} + \frac{z^8}{2^{10}} + \cdots \\ \frac{1}{9-z^2} &= \frac{1}{3^2} + \frac{z^2}{3^4} + \frac{z^4}{3^6} + \frac{z^6}{3^8} + \frac{z^8}{3^{10}} + \cdots \\ \frac{1}{16-z^2} &= \frac{1}{4^2} + \frac{z^2}{4^4} + \frac{z^4}{4^6} + \frac{z^6}{4^8} + \frac{z^8}{4^{10}} + \cdots \\ &\vdots \end{aligned}$$

§135. Writing

$$\begin{aligned} 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots &= a, \\ 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots &= b, \\ 1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \cdots &= c, \\ 1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \cdots &= d, \\ 1 + \frac{1}{2^{10}} + \frac{1}{3^{10}} + \frac{1}{4^{10}} + \cdots &= e, \\ 1 + \frac{1}{2^{12}} + \frac{1}{3^{12}} + \frac{1}{4^{12}} + \cdots &= f, \\ &\vdots \end{aligned}$$

we have

$$a + bz^2 + cz^4 + dz^6 + ez^8 + fz^{10} + \cdots = \frac{1}{2z^2} - \frac{\pi}{2z} \cot \pi z,$$

which, upon putting $z = \frac{u}{2\pi}$, becomes

$$a + \frac{b}{2^2\pi^2}u^2 + \frac{c}{2^4\pi^4}u^4 + \frac{d}{2^6\pi^6}u^6 + \frac{e}{2^8\pi^8}u^8 + \frac{f}{2^{10}\pi^{10}}u^{10} + \cdots$$

$$\begin{aligned}
&= \frac{2\pi^2}{u^2} - \frac{2\pi^2}{u^2} \cdot \frac{u}{2} \cot \frac{u}{2} \\
&= -\frac{2\pi^2}{u^2} (\beta u^2 + \delta u^4 + \zeta u^6 + \theta u^8 + \dots) \\
&= -2\pi^2 (\beta + \delta u^2 + \zeta u^4 + \theta u^6 + \dots).
\end{aligned}$$

By comparison,

$$a = -2\pi^2\beta, \quad b = -2^3\pi^4\delta, \quad c = -2^5\pi^6\zeta, \quad d = -2^7\pi^8\theta, \dots$$

Recalling the relation among $\beta, \delta, \zeta, \theta, \dots$, and $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \dots$, we have

$$\begin{aligned}
1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots &= a = -2\pi^2\beta = \frac{2\mathcal{A}}{2!}\pi^2 \\
1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots &= b = -2^3\pi^4\delta = -\frac{2^3\mathcal{B}}{4!}\pi^4, \\
1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \dots &= c = -2^5\pi^6\zeta = \frac{2^5\mathcal{C}}{6!}\pi^6, \\
1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \dots &= d = -2^7\pi^8\theta = -\frac{2^7\mathcal{D}}{8!}\pi^8,
\end{aligned}$$

and more generally,

$$1 + \frac{1}{2^{2k}} + \frac{1}{3^{2k}} + \frac{1}{4^{2k}} + \dots = \frac{2^{2k-1} \cdot |k\text{-th Bernoulli number}|}{(2k)!} \cdot \pi^{2k}.$$

VIII.A. The series expansions of cotangent, tangent, and secant

In Chapter 2 of *Institutes*, *De investigatione serierum summabilium*, Euler actually explains in several ways the expansion of the cotangent function. In §42, Euler recalls from Chapter X of *Introductio*, I the expression

$$\sin \pi x = \pi x \cdot \frac{1-x^2}{1} \cdot \frac{4-x^2}{4} \cdot \frac{9-x^2}{9} \cdot \frac{16-x^2}{16} \dots$$

taking logarithms,

$$\log \frac{\sin \pi x}{\pi} = \log x + \log \frac{1-x^2}{1} + \log \frac{4-x^2}{4} + \log \frac{9-x^2}{9} + \log \frac{16-x^2}{16} + \dots$$

and differentiating,

$$\frac{\pi \cos \pi x}{\sin \pi x} = \frac{1}{x} - \frac{2x}{1-x^2} - \frac{2x}{4-x^2} - \frac{2x}{9-x^2} - \frac{2x}{16-x^2} - \dots$$

From this,

$$\frac{1}{1-x^2} + \frac{1}{4-x^2} + \frac{1}{9-x^2} + \frac{1}{16-x^2} + \cdots = \frac{1}{2x^2} - \frac{\pi}{2x} \cot \pi x.$$

Institutiones Calculi Differentialis

Part II, Chapter VIII: De usu calculi differentialis in formandis seriebus

§221. Ex his seriebus pro sinu et cosinu notissimis deducuntur series pro tangente, cotangente, secante et cosecante cuiusvis angulis. Tangens enim prodit, si sinus per cosinum, cotangens si cosinus per sinum, secans, si radius 1 per cosinum, et cosecans, si radius per sinum dividatur. Series autem ex his divisionibus ortae maxime videntur irregulares; verum excepta serie secantem exhibente reliquae per numeros Bernoullianos supra definitos \mathcal{A} , \mathcal{B} , \mathcal{C} , \mathcal{D} etc. ad facilem progressionis legem reduci possunt. Quoniam enim supra (§127) invenimus esse

$$\frac{\mathcal{A}u^2}{2!} + \frac{\mathcal{B}u^4}{4!} + \frac{\mathcal{C}u^6}{6!} + \frac{\mathcal{D}u^8}{8!} + \cdots = 1 - \frac{u}{2} \cot \frac{1}{2}u,$$

erit posito $\frac{1}{2}u = x$

$$\cot x = \frac{1}{x} - \frac{2^2 \mathcal{A}x}{2!} - \frac{2^4 \mathcal{B}x^3}{4!} - \frac{2^6 \mathcal{C}x^5}{6!} - \frac{2^8 \mathcal{D}x^7}{8!} - \cdots,$$

atque si ponatur $\frac{1}{2}x$ pro x , erit

$$\cot \frac{1}{2}x = \frac{2}{x} - \frac{2\mathcal{A}x}{2!} - \frac{2\mathcal{B}x^3}{4!} - \frac{2\mathcal{C}x^5}{6!} - \frac{2\mathcal{D}x^7}{8!} - \cdots.$$

§222. Hinc autem tangens cuiusvis arcus sequenti modo per seriem exprimetur. Cum sit¹⁴

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x},$$

erit

$$\cot 2x = \frac{1}{2 \tan x} - \frac{\tan x}{2} = \frac{1}{2} \cot x - \frac{1}{2} \tan x$$

ideoque

$$\tan x = \cot x - 2 \cot 2x.$$

Cum igitur sit

$$\cot x = \frac{1}{x} - \frac{2^2 \mathcal{A}x}{2!} - \frac{2^4 \mathcal{B}x^3}{4!} - \frac{2^6 \mathcal{C}x^5}{6!} - \frac{2^8 \mathcal{D}x^7}{8!} - \cdots,$$

¹⁴Euler wrote $\text{tang } x$ for $\tan x$ and $\text{tang } x^2$ for $\tan^2 x$.

$$2 \cot 2x = \frac{1}{x} - \frac{2^4 Ax}{2!} - \frac{2^8 Bx^3}{4!} - \frac{2^{12} Cx^5}{6!} - \frac{2^{16} Dx^7}{8!} - \dots,$$

erit hanc seriem ab illa subtrahendo

$$\tan x = \frac{2^2(2^2 - 1)Ax}{2!} + \frac{2^4(2^4 - 1)Bx^3}{4!} + \frac{2^6(2^6 - 1)Cx^5}{6!} + \frac{2^8(2^8 - 1)Dx^7}{8!} + \dots.$$

§224. Per hos autem numeros Bernoullianos secans exprimi non potest, sed requirit alios numeros, qui in summas potestatum reciprocarum imparium ingrediuntur. Si enim ponatur

$$\begin{aligned} 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots &= \alpha \cdot \frac{\pi}{2^2}, \\ 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \dots &= \frac{\beta}{2!} \cdot \frac{\pi^3}{2^4}, \\ 1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \dots &= \frac{\gamma}{4!} \cdot \frac{\pi^5}{2^6}, \\ 1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \frac{1}{9^7} - \dots &= \frac{\delta}{6!} \cdot \frac{\pi^7}{2^8}, \\ 1 - \frac{1}{3^9} + \frac{1}{5^9} - \frac{1}{7^9} + \frac{1}{9^9} - \dots &= \frac{\epsilon}{8!} \cdot \frac{\pi^9}{2^{10}}, \\ 1 - \frac{1}{3^{11}} + \frac{1}{5^{11}} - \frac{1}{7^{11}} + \frac{1}{9^{11}} - \dots &= \frac{\zeta}{10!} \cdot \frac{\pi^{11}}{2^{12}}, \\ &\vdots \end{aligned}$$

erit

$$\begin{aligned} \alpha &= 1, \\ \beta &= 1, \\ \gamma &= 5, \\ \delta &= 61, \\ \epsilon &= 1385, \\ \zeta &= 50521, \\ \eta &= 2702765, \\ \theta &= 199360981, \\ \iota &= 19391512145, \\ \kappa &= 2404879661671, \end{aligned}$$

ex hisque valoribus obtienbitur

$$\sec x = \alpha + \frac{\beta}{2!}x^2 + \frac{\gamma}{4!}x^4 + \frac{\delta}{6!}x^6 + \frac{\epsilon}{8!}x^8 + \dots$$

§226. Euler obtained the series expansion of $\sec x$ by forming the reciprocal of

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

By equating the product of the series for $\cos x$ and $\sec x$ to 1, namely,

$$\begin{aligned} 1 &= \alpha + \frac{\beta}{2!}x^2 + \frac{\gamma}{4!}x^4 + \frac{\delta}{6!}x^6 + \frac{\epsilon}{8!}x^8 + \dots \\ &\quad - \frac{\alpha}{2!}x^2 - \frac{\beta}{2!2!}x^4 - \frac{\gamma}{2!4!}x^6 - \frac{\delta}{2!6!}x^8 - \dots \\ &\quad + \frac{\alpha}{4!}x^4 + \frac{\beta}{4!2!}x^6 + \frac{\gamma}{4!4!}x^8 + \dots \\ &\quad - \frac{\alpha}{6!}x^6 - \frac{\beta}{6!2!}x^8 - \dots \\ &\quad + \frac{\alpha}{8!}x^8 + \dots \\ &\quad \vdots \end{aligned}$$

Euler obtains a set of recurrence relations for the coefficients.

$$\begin{aligned} \alpha &= 1, \\ \beta &= \frac{2 \cdot 1}{1 \cdot 2} \alpha, \\ \gamma &= \frac{4 \cdot 3}{1 \cdot 2} \beta - \frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4} \alpha, \\ \delta &= \frac{6 \cdot 5}{1 \cdot 2} \gamma - \frac{6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4} \beta + \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \alpha, \\ \epsilon &= \frac{8 \cdot 7}{1 \cdot 2} \delta - \frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4} \gamma + \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \beta - \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} \alpha, \\ &\quad \vdots \end{aligned}$$

Euler actually explained in §225 the connection of the series of $\sec x$ and the sums ¹⁵

$$1 - \frac{1}{3^{2k+1}} + \frac{1}{5^{2k+1}} - \frac{1}{7^{2k+1}} + \dots$$

¹⁵Precisely, this sum is $(-1)^k \frac{E_{2k}}{2^{2k+2} (2k)!} \pi^{2k+1}$. This is actually positive since E_k is positive or negative according as k is even or odd.

VIIIB. Euler's constant γ

Paper 43: De Progressionibus Harmonicis Observationes (1734/5)

Euler begins by explaining the divergence of the harmonic series.

§8. Ex hac consideratione innumerabiles oriuntur series ad logarithmos quarumvis numerorum designandos. Summamus primo hanc progressionem harmonicam

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

... Differentia igitur inter hanc seriem

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{i}$$

ad terminum indicis i continuatam et eandem

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{ni}$$

ad terminum indicis ni continuatam erit $= \log n$. Quare ille series ab hac subtracta relinquit $\log n$. Quia autem huius seriei numerus terminorum est n vicibus maior quam illius, ab n terminis seriei

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{ni}$$

subtrahi oportet unicum alterius seriei

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{i}$$

quo subtractio in infinitum eodem modo possit perfici. Quare erit

$$\begin{aligned} \log n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n} + \frac{1}{2n+1} + \cdots + \frac{1}{3n} + \cdots \\ -1 \qquad \qquad \qquad -\frac{1}{2} \qquad \qquad \qquad -\frac{1}{3} \end{aligned}$$

Si igitur inferioris seriei singuli termini a suprascriptis terminis superioris seriei actu subtrahantur et pro n numeri integri scribantur 2, 3, 4, etc.

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \cdots$$

$$\log 3 = 1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \frac{1}{7} + \frac{1}{8} - \frac{2}{9} + \frac{1}{10} + \frac{1}{11} - \frac{2}{12} + \cdots$$

$$\log 4 = 1 + \frac{1}{2} + \frac{1}{3} - \frac{3}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} - \frac{3}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} - \frac{3}{12} + \cdots$$

$$\begin{aligned} \log 5 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{4}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{4}{10} + \frac{1}{11} + \frac{1}{12} + \dots \\ \log 6 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{5}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} - \frac{5}{12} + \dots \\ &\vdots \end{aligned}$$

§11.

$$\begin{aligned} 1 &= \log 2 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \dots \\ \frac{1}{2} &= \log \frac{3}{2} + \frac{1}{2 \cdot 4} - \frac{1}{3 \cdot 8} + \frac{1}{4 \cdot 16} - \frac{1}{5 \cdot 32} + \frac{1}{6 \cdot 64} - \frac{1}{7 \cdot 128} + \dots \\ \frac{1}{3} &= \log \frac{4}{3} + \frac{1}{2 \cdot 9} - \frac{1}{3 \cdot 27} + \frac{1}{4 \cdot 81} - \frac{1}{5 \cdot 243} + \frac{1}{6 \cdot 729} - \frac{1}{7 \cdot 2187} + \dots \\ \frac{1}{4} &= \log \frac{5}{4} + \frac{1}{2 \cdot 16} - \frac{1}{3 \cdot 64} + \frac{1}{4 \cdot 256} - \frac{1}{5 \cdot 1024} + \frac{1}{6 \cdot 4096} - \frac{1}{7 \cdot 16384} + \dots \\ &\vdots \\ \frac{1}{i} &= \log \frac{i+1}{i} + \frac{1}{2 \cdot i^2} - \frac{1}{3 \cdot i^3} + \frac{1}{4 \cdot i^4} - \frac{1}{5 \cdot i^5} + \frac{1}{6 \cdot i^6} - \frac{1}{7 \cdot i^7} + \dots \end{aligned}$$

Adding these series, Euler obtains

$$\begin{aligned} 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{i} &= \log(i+1) + \frac{1}{2}(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots) \\ &\quad - \frac{1}{3}(1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \dots) \\ &\quad + \frac{1}{4}(1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \dots) \\ &\quad - \dots \end{aligned}$$

From this, Euler obtains

$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{i} = \log(i+1) + 0.577218.$$

This is the first appearance of the Euler constant γ .¹⁶

¹⁶ $\gamma := \lim_{n \rightarrow \infty} (1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log n)$. It is still unknown if γ is irrational.

VIIIC. Summary on Bernoulli and Euler numbers

The **Bernoulli numbers** B_n are the coefficients in the series expansion of

$$\frac{x}{e^x - 1} = 1 + B_1x + \frac{B_2}{2!}x^2 + \frac{B_3}{3!}x^3 + \cdots + \frac{B_n}{n!}x^n + \cdots$$

These can be generated recursively by

$$(1 + B)^{n+1} - B^{n+1} = 0$$

interpreting each B^j as B_j .

$$B_1 = -\frac{1}{2}, \quad B_{2j+1} = 0 \text{ for } k = 1, 2, 3, \dots,$$

$$B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730}, \dots$$

These give the series expansion of

$$\tan x = \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k} - 1)B_{2k}}{(2k)!} x^{2k-1},$$

and other functions.

The Bernoulli numbers also appear in the **Euler summation formula**

$$\begin{aligned} f(1) + f(2) + \cdots + f(n) &= \int_0^n f(x)dx + \text{Euler constant} \\ &+ \frac{1}{2}f(n) - \frac{B_2}{2!}f'(n) + \frac{B_4}{4!}f'''(n) - \frac{B_6}{6!}f^{(5)}(n) + \cdots \end{aligned}$$

The **Euler number** E_{2n} are defined recursively by

$$(E + 1)^n + (E - 1)^n = 0, \quad E_0 = 1,$$

interpreting each E^j as E_j . These are

$$E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad E_6 = -61, \quad E_8 = 1385,$$

These give the series expansion of $\sec x$:

$$\sec x = 1 - \frac{E_2}{2!}x^2 + \frac{E_4}{4!}x^4 - \frac{E_6}{6!}x^6 + \cdots$$

In terms of these numbers, Euler gave

$$1 - \frac{1}{3^{2k+1}} + \frac{1}{5^{2k+1}} - \frac{1}{7^{2k+1}} + \frac{1}{9^{2k+1}} - \cdots = (-1)^k \frac{E_{2k}}{2^{2k+2}(2k)!} \pi^{2k+1}.$$

IXA. Continued fractions

Paper 71: De fractionibus continuis dissertio¹⁷ (1737)

Euler begins with a general continued fraction

$$a + \frac{\alpha}{b + \frac{\beta}{c + \frac{\gamma}{d + \frac{\delta}{e + \dots}}}}$$

and computes their convergents. For typographical reasons, we shall also write this continued fraction in the form

$$a + \frac{\alpha}{b} + \frac{\beta}{c} + \frac{\gamma}{d} + \frac{\delta}{e} + \frac{\epsilon}{f} + \dots$$

The convergents are

$$\begin{array}{cccccc} a & b & c & d & e & \\ \frac{1}{0}, & \frac{a}{1}, & \frac{ab+\alpha}{b}, & \frac{abc+\alpha c+\beta\alpha}{bc+\beta}, & \frac{abcd+\alpha cd+\beta ad+\gamma ab+\alpha\gamma}{bcd+\beta d+\gamma b} & \dots \end{array}$$

$$\begin{array}{cccccc} & & & & & \\ \alpha & \beta & \gamma & \delta & \epsilon & \end{array}$$

From this Euler observes (§7) a general rule for the formation of these convergents.

Introductio, I, §361. In the above scheme each fraction has a superscript and a subscript. Again the first fraction is $\frac{1}{0}$, and the second is $\frac{a}{1}$. Thereafter, any fraction is formed by multiplying the numerator of the preceding fraction by the superscript and multiplying the second predecessor by the subscript. The new numerator is the sum of these two products. The new denominators likewise is the sum of the product of the denominator of the predecessor by the subscript and the product of the denominator of the second predecessor by its subscript.

superscript	a	b	c	d	e	\dots
fraction	$\frac{1}{0},$	$\frac{a}{1},$	$\frac{a \cdot b + 1 \cdot \alpha}{1 \cdot b + 0 \cdot \alpha},$	$\frac{(ab + \alpha)c + \beta\alpha}{b \cdot c + 1 \cdot \beta},$	$\frac{(abc + \alpha c + \beta\alpha)d + (ab + \alpha)\gamma}{(bc + \beta)d + b \cdot \gamma}$	\dots
subscript	α	β	γ	δ	ϵ	\dots

§12. Thus, if it is proposed to change the fraction $\frac{A}{B}$ into a continued fraction all of whose numerators are one, I divide A by B with quotient a and remainder C ; the preceding divisor B is divided by this remainder C with quotient b and remainder D , by which D is divided, and so on until a zero remainder an an infinitely large quotient is obtained. Moreover this operation is represented in the following manner:

¹⁷This paper has been translated into English by Wyman and Wyman, and appeared in *Math. Systems Theory*, 18 (1985) 295 – 328.

$$\begin{array}{r}
 B \mid A \mid a \\
 \hline
 C \mid B \mid b \\
 \hline
 D \mid C \mid c \\
 \hline
 E \mid D \mid d \\
 \hline
 F \mid E \mid e
 \end{array}$$

Therefore, the quotients a, b, c, d, e , etc., will be found by this operation, and it follows that

$$\frac{A}{B} = a + \frac{1}{b + \frac{1}{c + \frac{1}{d + \frac{1}{e + \dots}}}$$

Introductio, I, §382.¹⁸ Since fractions arise from this operation which very quickly approximate the value of the expression, this method can be used to express decimal fractions by ordinary fractions which approximate them. Indeed, if the given fraction has a very large numerator and denominator, then a fraction expressed by smaller numbers can be found which does not give the exact value, but is a very close approximation. This is the problem discussed by Wallis and has an easy solution in that we find fractions, expressed by smaller numbers, which almost equal the given fraction expressed in large numbers. Our fractions, obtained by this methods, have a value so close to the continued fraction from which they come, that there are no other numbers, unless they be larger, which give a closer approximation.

Example I. We would like to find a fraction which expresses the ratio of the circumference of a circle to the diameter such that no more accurate fraction can be found unless large numbers are used. If the decimal equivalent $3.1415926535 \dots$ is expressed by our method of continued division, the sequence of quotients is

$$3, 7, 15, 1, 292, 1, 1, \dots$$

From this sequence we form the fractions

$$\frac{1}{0}, \frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}, \dots$$

The second fraction already shows that the ratio of the diameter to circumference to be $1 : 3$, and is certainly the most accurate approximation unless larger numbers are used. The third fraction gives the Archimedean ratio of $7 : 22$, and the fourth fraction give the Metian ratio which is so close to the true value that the error is less than $\frac{1}{113 \cdot 33102}$.¹⁹ In addition, these fractions are alternately greater and less than the true value.

¹⁸Concluding sections of Chapter XIII, and of the whole book.

¹⁹The fraction $\frac{355}{113}$ as an approximation of π was used by the Chinese mathematician Zu Congzhi (430 – 501) in the fifth century. Zu gave this as an “accurate” value (*mi lii*), and $\frac{22}{7}$ as a “crude” value (*yue lii*).

Example II. We would like to express the approximate ratio of one day to one solar year in smallest possible numbers. This year is 365 days, 5 hours, 48 minutes, and 55 seconds. That means that one year is $365\frac{20935}{86400}$ days. We need be concerned only with the fraction, which gives the sequence of quotients

$$4, 71, 1, 6, 1, 2, 2, 4,$$

and the sequence of fractions

$$\frac{0}{1}, \frac{1}{4}, \frac{7}{29}, \frac{8}{33}, \frac{55}{227}, \frac{63}{280}, \frac{181}{747}.$$

The hours, minutes, and seconds which exceed 365 days make about one day in four years, and this is the origin of the **Julian** calendar. More exact, however, is eight days in 33 years, or 181 days in 747 years. For this reason, in 400 years there are 97 extra days, while the **Julian** calendar gives 100 extra days. This is the reason that the **Gregorian** calendar in 400 years converts three years, which would be leap years, into ordinary years.

Paper 71: §§18 – 19. We seek now the fractions which approximate $\sqrt{2}$ so closely that no fractions with smaller denominators approach more closely. In fact,

$$\sqrt{2} = 1.41421356 = \frac{141421356}{1000000}.$$

If continuing division is carried out in the prescribed manner, this fraction gives the quotients

$$1, 2, 2, 2, 2, 2, 2, 2, \dots$$

from which the following fractions are formed

$$\frac{1}{0}, \frac{2}{1}, \frac{2}{2}, \frac{2}{5}, \frac{2}{12}, \frac{2}{29}, \frac{2}{70}, \frac{2}{169}$$

§19. This description of $\sqrt{2}$ has been suitably presented, since all the quotients except the first have the value 2, so that

$$\sqrt{2} = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$$

Similarly, if $\sqrt{3}$ is analyzed, the quotients

$$1, 1, 2, 1, 2, 1, 2, 1, 2, 1, \dots$$

are found, so that

$$\sqrt{3} = 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{2} + \dots$$

§19[a]. Then he considers the continued fraction

$$a + \frac{1}{b} + \frac{1}{b} + \frac{1}{b} + \frac{1}{b} + \dots$$

which is set equal to x . We have

$$x - a = \frac{1}{b} + \frac{1}{b} + \frac{1}{b} + \frac{1}{b} + \dots = \frac{1}{b + x - a}$$

from which

$$x^2 - 2ax + bx + a^2 - ab = 1$$

and

$$x = a - \frac{b}{2} + \sqrt{1 + \frac{b^2}{4}}.$$

Substituting $b = 2$ and $a = 1$, this becomes

$$x = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots = \sqrt{2}.$$

If $b = 2a$, we have

$$\sqrt{a^2 + 1} = a + \frac{1}{2a} + \frac{1}{2a} + \frac{1}{2a} + \frac{1}{2a} + \dots.$$

He continued to investigate continued fractions of the types

$$\begin{aligned} a + \frac{1}{b} + \frac{1}{c} + \frac{1}{b} + \frac{1}{c} + \frac{1}{b} + \frac{1}{c} + \dots, \\ a + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \dots, \end{aligned}$$

and find that “every such value is the root of a quadratic equation”.

Appendix. §11. Every simple continued fraction

$$a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \dots + \frac{1}{a_n} + \dots$$

in which the a_n are fractions, say $a_n = \frac{b_n}{B_n}$ can be converted into a generic one involving only “integers”.

$$a_0 + \frac{B_1}{b_1} + \frac{B_1 B_2}{b_2} + \frac{B_2 B_3}{b_3} + \frac{B_3 B_4}{b_4} + \dots + \frac{B_{n-1} B_n}{b_n} + \dots$$

On the other hand, every (generic) continued fraction can be converted into a simple continued fraction “whose denominators may be fractions”:

$$\begin{aligned}
& a_0 + \frac{\alpha_1}{a_1} + \frac{\alpha_2}{a_2} + \frac{\alpha_3}{a_3} + \frac{\alpha_4}{a_4} + \cdots + \frac{\alpha_n}{a_n} + \frac{\alpha_{n+1}}{a_{n+1}} + \cdots \\
= & a_0 + \frac{1}{\frac{a_1}{\alpha_1}} + \frac{1}{\frac{\alpha_1 a_2}{\alpha_2}} + \frac{1}{\frac{\alpha_2 a_3}{\alpha_1 \alpha_3}} + \frac{1}{\frac{\alpha_1 \alpha_3 a_4}{\alpha_2 \alpha_4}} + \frac{1}{\frac{\alpha_2 \alpha_4 a_5}{\alpha_1 \alpha_3 \alpha_5}} + \cdots
\end{aligned}$$

IXB. Continued fractions and infinite series

§8. Euler converts the sequence of convergents into (the sequence of partial sums of) an infinite series. More generally, every given sequence can be regarded as the sequence of partial sums of the sequence of successive differences: given a sequence

$$B_0, B_1, B_2, \cdots, B_n, B_{n+1}, \cdots,$$

writing

$$\begin{aligned}
A_0 &= B_0, \\
A_1 &= B_1 - B_0, \\
A_2 &= B_2 - B_1, \\
&\vdots \\
A_n &= B_n - B_{n-1}, \\
&\vdots
\end{aligned}$$

we easily see that

$$A_0 + A_1 + \cdots + A_n = B_n.$$

Applying this simple idea to the sequence of convergents of a continued fraction, Euler converts the continued fraction into an infinite series. The successive differences of the convergents are

$$\frac{\alpha}{b}, -\frac{\alpha\beta}{b(bc+\beta)}, +\frac{\alpha\beta\gamma}{(bc+\beta)(bcd+\beta d+\gamma b)}, \cdots$$

Therefore,²⁰

$$a_0 + \frac{\alpha}{b} + \frac{\beta}{c} + \frac{\gamma}{d} + \frac{\delta}{e} + \cdots$$

²⁰

$$a_0 + \frac{\alpha_1}{a_1} + \frac{\alpha_2}{a_2} + \frac{\alpha_3}{a_3} + \frac{\alpha_4}{a_4} + \cdots + \frac{\alpha_n}{a_n} + \frac{\alpha_{n+1}}{a_{n+1}} + \cdots = a_0 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\alpha_1 \alpha_2 \cdots \alpha_n}{Q_{n-1} Q_n}.$$

$$= a_0 + \frac{\alpha}{1 \cdot b} - \frac{\alpha\beta}{b(bc + \beta)} + \frac{\alpha\beta\gamma}{(bc + \beta)(bcd + \beta d + \gamma b)} - \frac{\alpha\beta\gamma\delta}{(bcd + \beta d + \gamma b)(bcde + \dots)} + \dots$$

Euler did not give any illustrative examples in this paper. Applied to the golden ratio

$$\phi = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}}}}$$

this procedure gives the series

$$F_0 + \frac{1}{F_0 F_1} - \frac{1}{F_1 F_2} + \frac{1}{F_2 F_3} - \frac{1}{F_3 F_4} + \dots$$

since the convergents are

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \dots, \frac{F_{n+1}}{F_n}, \dots$$

where F_n is the n -Fibonacci number.

In the *Introductio*, I, Chapter 18, Euler reverts this process to write an infinite series as a continued fraction. This is also the main topics of Paper 593, *De transformatione serierum in fractiones continuas, ubi simul haec theoria non mediocriter amplificatur* (1785).

Theorem 1 If

$$s = \frac{1}{\alpha} - \frac{1}{\beta} + \frac{1}{\gamma} - \frac{1}{\delta} + \frac{1}{\epsilon} - \dots,$$

then

$$\frac{1}{s} = \alpha + \frac{\alpha^2}{\beta - \alpha + \frac{\beta^2}{\gamma - \beta + \frac{\gamma^2}{\delta - \gamma + \dots}}}$$

Applying this to the series

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots,$$

he obtained

$$\log 2 = \frac{1}{1 + \frac{1^2}{1 + \frac{2^2}{1 + \frac{3^2}{1 + \frac{4^2}{1 + \dots}}}}}$$

and to the Gregory series for $\frac{\pi}{4}$,

$$\frac{\pi}{4} = \frac{1}{1 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \dots}}}}$$

Euler's proof of Theorem 1: Write

$$\begin{aligned} s &= \frac{1}{\alpha} - \frac{1}{\beta} + \frac{1}{\gamma} - \frac{1}{\delta} + \frac{1}{\epsilon} - \dots, \\ t &= \frac{1}{\beta} - \frac{1}{\gamma} + \frac{1}{\delta} - \frac{1}{\epsilon} + \dots, \\ u &= \frac{1}{\gamma} - \frac{1}{\delta} + \frac{1}{\epsilon} - \dots, \\ &\vdots \end{aligned}$$

Now, $s = \frac{1}{\alpha} - t = \frac{1-at}{\alpha}$, so that $\frac{1}{s} = \frac{\alpha}{1-at} = \alpha + \frac{\alpha^2 t}{1-at}$. Also, $\frac{\alpha^2 t}{1-at} = \frac{\alpha^2}{-\alpha + \frac{1}{t}}$, so that

$$\frac{1}{s} = \alpha + \frac{\alpha^2}{-\alpha + \frac{1}{t}}.$$

Similarly,

$$\begin{aligned} \frac{1}{t} &= \beta + \frac{\beta^2}{-\beta + \frac{1}{u}}, \\ \frac{1}{u} &= \gamma + \frac{\gamma^2}{-\gamma + \frac{1}{v}}, \\ &\vdots \end{aligned}$$

Combining these, we obtain the continued fraction expansion of $\frac{1}{s}$.

§369, Example IV. Euler recalls from §178 that

$$\cot \frac{m\pi}{n} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \dots$$

and obtains

$$\cot \frac{m\pi}{n} = \frac{1}{m + \frac{m^2}{n - 2m + \frac{(n-m)^2}{m + \frac{(n+m)^2}{n - 2m + \frac{(2n-m)^2}{2m + \dots}}}}}$$

Theorem 2. If

$$s = \frac{1}{ab} - \frac{1}{bc} + \frac{1}{cd} - \frac{1}{de} + \frac{1}{ef} - \dots,$$

then

$$\frac{1}{as} = b + \frac{ab}{c - a + \frac{bc}{d - b + \frac{cd}{e - c + \frac{de}{f - d + \dots}}}}$$

In §§24,25, Euler applies this to

$$2 \log 2 - 1 = \frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} - \frac{1}{4 \cdot 5} + \frac{1}{5 \cdot 6} - \dots$$

to obtain

$$2 \log 2 - 1 = \frac{1}{2 + \frac{1 \cdot 2}{2 + \frac{2 \cdot 3}{2 + \frac{3 \cdot 4}{2 + \dots}}}}$$

and to

$$\frac{\pi}{4} - \frac{1}{2} = \frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - \frac{1}{7 \cdot 9} + \dots$$

to obtain

$$\frac{\pi + 2}{\pi - 2} = 4 + \frac{1 \cdot 3}{4 + \frac{3 \cdot 5}{4 + \frac{5 \cdot 7}{4 + \frac{7 \cdot 9}{4 + \dots}}}}$$

Theorem 6. If

$$s = \frac{1}{\alpha} - \frac{1}{\alpha\beta} + \frac{1}{\alpha\beta\gamma} - \frac{1}{\alpha\beta\gamma\delta} + \dots,$$

then

$$\frac{1}{s} = \alpha + \frac{\alpha}{\beta - 1 + \frac{\beta}{\gamma - 1 + \frac{\gamma}{\delta - 1 + \dots}}}}$$

§41. From

$$\frac{e - 1}{e} = \frac{1}{1} - \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} - \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \dots,$$

we have

$$\frac{e}{e - 1} = 1 + \frac{1}{1 + \frac{2}{2 + \frac{3}{3 + \dots}}}}$$

$$\frac{1}{e - 1} = \frac{1}{1 + \frac{2}{2 + \frac{3}{3 + \dots}}}}$$

XC. Continued fraction expansion of functions

In Introductio, I, §381, Example III, Euler computed the continued fraction expansion of $\frac{e-1}{2}$ using the approximation 0.8591409142295, and found the quotients 1, 6, 10, 14, 18, 22. He commented that “[i]f the value for e at the beginning had been more exact, then the sequence of quotients would have been

$$1, 6, 10, 14, 18, 22, 26, 30, 34, \dots,$$

which form the terms of an arithmetic²¹ progression. It follows that

$$\frac{e-1}{2} = \frac{1}{1 + \frac{1}{6 + \frac{1}{10 + \frac{1}{14 + \frac{1}{18 + \frac{1}{22 + \dots}}}}}}$$

This result can be confirmed by infinitesimal calculus”. The *simple* continued fraction expansion of e is given in §21 of Paper 71:

$$e = 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{4 + \frac{1}{1 + \frac{1}{1 + \frac{1}{6 + \frac{1}{1 + \dots}}}}}}}}}}$$

More generally, by solving differential equations (§§28 – 30),

$$\sqrt[s]{e} = 1 + \frac{1}{s-1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{3s-1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{5s-1 + \frac{1}{1 + \frac{1}{7s-1 + \frac{1}{1 + \dots}}}}}}}}}}}}$$

In Paper 593, §§34 – 36, Euler makes use of the following theorem to express some familiar functions as continued fractions.

Theorem 4. If

$$s = \frac{x}{\alpha y} - \frac{x^2}{\beta y^2} + \frac{x^3}{\gamma y^3} - \frac{x^4}{\delta y^4} + \dots$$

²¹The English translation erroneously renders *progressionem arithmetica* “geometric progression”.

then

$$\frac{x}{s} = \alpha y + \frac{\alpha^2 xy}{\beta y - \alpha x + \frac{\beta^2 xy}{\gamma y - \beta x + \frac{\gamma^2 xy}{\delta y - \gamma x + \frac{\delta^2 xy}{\epsilon y - \delta x + \dots}}}$$

§35. From

$$\log\left(1 + \frac{x}{y}\right) = \frac{x}{y} - \frac{x^2}{2y^2} + \frac{x^3}{3y^3} - \frac{x^4}{4y^4} + \dots$$

with $s = \log\left(1 + \frac{x}{y}\right)$,

$$\alpha = 1, \quad \beta = 2, \quad \gamma = 3, \quad \delta = 4, \dots,$$

we have

$$\frac{x}{\log\left(1 + \frac{x}{y}\right)} = y + \frac{xy}{2y - x + \frac{4xy}{3y - 2x + \frac{9xy}{4y - 3x + \frac{16xy}{5y - 4x + \dots}}}}$$

§36. From

$$\arctan t = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} - \dots,$$

putting $t^2 = \frac{x}{y}$, we have

$$\sqrt{\frac{x}{y}} \arctan \frac{x}{y} = \frac{x}{y} - \frac{x^2}{3y^2} + \frac{x^3}{5y^3} - \frac{x^4}{7y^4} + \dots,$$

and with $s = \sqrt{\frac{x}{y}} \arctan \frac{x}{y}$,

$$\alpha = 1, \quad \beta = 3, \quad \gamma = 5, \quad \delta = 7, \dots,$$

we have

$$\frac{\sqrt{xy}}{\arctan \sqrt{\frac{x}{y}}} = y + \frac{xy}{3y - x + \frac{9xy}{5y - 3x + \frac{25xy}{7y - 5x + \frac{49xy}{9y - 7x + \dots}}}}$$

In particular, with $x = 1, y = 3, \arctan \frac{1}{\sqrt{3}} = \frac{\pi}{6}$,

$$\frac{6\sqrt{3}}{\pi} = 3 + \frac{1 \cdot 3}{8 + \frac{3 \cdot 9}{12 + \frac{3 \cdot 25}{16 + \frac{3 \cdot 49}{20 + \dots}}}}$$

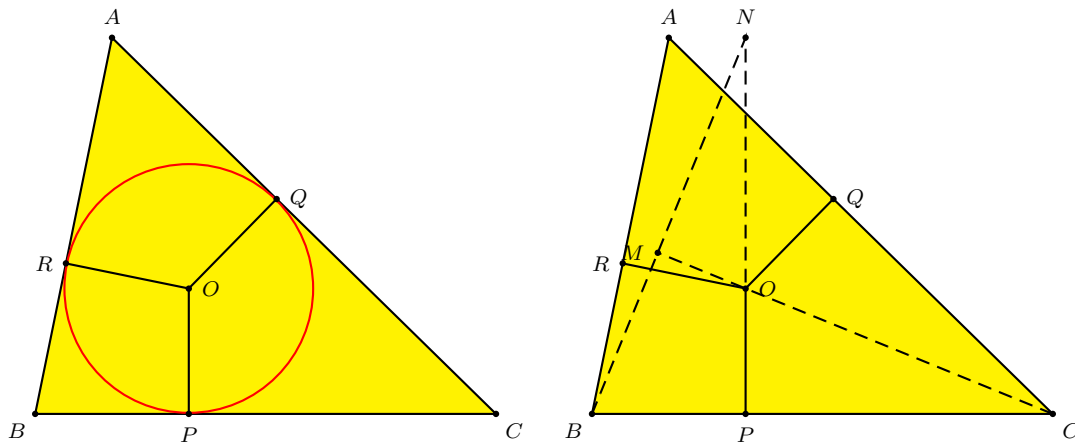
XA. Euler’s proof of Heron’s formula

Standard notation. For triangle ABC ,

- a, b, c lengths of BC, CA, AB
- s semiperimeter $\frac{1}{2}(a + b + c)$
- R circumradius
- ρ inradius
- Δ area

Paper 135: *Variae Demonstrationes Geometriae* (1747/48)

§6. Theorema. Area cuiusque trianguli ABC aequatur rectangulo ex seimsumma laterum in radium circuli inscripti, seu area ΔABC est $\frac{1}{2}(AB + BC + CA)OP$.



§6. Theorema. $\Delta = \rho s$.

§7. Theorema. $AR = AQ = s - a, BR = BP = s - b,$ and $CP = CQ = s - c$.

§9. $\Delta = \sqrt{s(s - a)(s - b)(s - c)}$.

§8. Let ABC be a triangle with incenter O , P , Q , and R the projections of O on the sides BC , CA , AB respectively. Euler first established the relation

$$AR \cdot BP \cdot CQ = s \cdot OP^2,$$

where s is the semiperimeter of the triangle.

Let the perpendicular from B to the line CO meet the line CO produced at a point M , and the line PO produced at another point N . Then $\angle OBM = \frac{A}{2}$, and the right triangles BOM and AOR are similar, so that

$$\begin{aligned} AR : RO &= BM : MO, \\ AR : OP &= BM : MO. \end{aligned}$$

Now, also the right triangles CBM , NBP and NOM are similar, and

$$\begin{aligned} BM : BC &= MO : ON, \\ BM : MO &= BC : ON. \end{aligned}$$

It follows that $AR : OP = BC : ON$, and $AR \cdot ON = OP \cdot BC$. Now, since $ON = PN - OP$, we have

$$AR \cdot PN - AR \cdot OP = BC \cdot OP$$

and

$$AR \cdot PN = AR \cdot OP + BC \cdot OP = (AR + BC)OP = s \cdot OP,$$

since Euler had previously established that $AR + BC = s$. Now, from the similar right triangles COP and NBP , one has

$$PN : BP = CP : OP,$$

and $OP \cdot PN = BP \cdot CP$,

$$AR \cdot BP \cdot CP = AR \cdot OP \cdot PN = s \cdot OP^2.$$

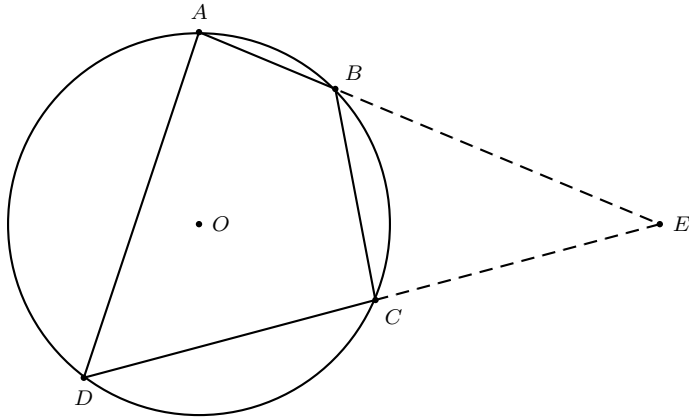
XB. Area of a cyclic quadrilateral

§13. Theorema. Si quadrilateri circulo inscripti $ABCD$ duo latera sibi opposita AB , DC ad occursum usque in E producantur, erit area quadrilateri $ABCD$ ad aream trianguli BCE ut $AD^2 - BC^2$ ad BC^2 .

Notation: $Q = \text{area } ABCD$; $\Delta = \triangle BEC$

Corollary 4. $16Q^2$ is the product of the following 4 factors

$$\text{I. } \frac{(AD - BC)(BE + CE + BC)}{BC},$$



$$\text{II. } \frac{(AD - BC)(BE + CE - BC)}{BC},$$

$$\text{III. } \frac{(AD + BC)(BC + BE - CE)}{BC},$$

$$\text{IV. } \frac{(AD + BC)(BC - BE + CE)}{BC}.$$

§§18,21. Theorema.

$$\begin{aligned} BE + CE : BC &= AB + CD : AD - BC, \\ CE - BE : AB - DC &= BC : AD + BC. \end{aligned}$$

§§19,20,22,23. Corollaria:

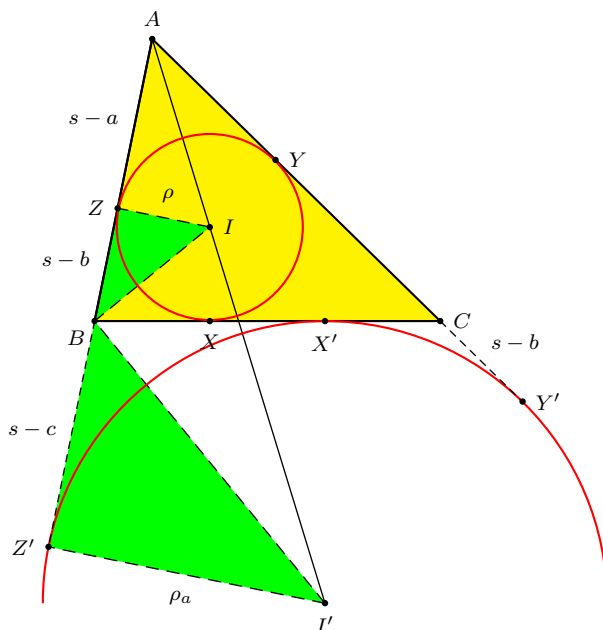
- I. $BE + CE + BC : BC = AB + CD + AD - BC : AD - BC,$
- II. $BE + CE - BC : BC = AB + CD - AD - BC : AD - BC,$
- III. $BC + CE - BE : BC = AD + BC + AB - CD : AD + BC,$
- IV. $BC - CE + BE : BC = AD + BC - AB + CD : AD + BC.$

§24. Theorema. Quadrilateri circulo inscripti $ABCD$ area invenitur, si a semisumma omnium eius laterum singula latera seorsim subtrahantur, haec quatuor residua in se invicem multiplicentur atque ex producto radix quadrata extrahatur.

$$Q = \sqrt{(s - a)(s - b)(s - c)(s - d)}.$$

XC. The excircles and Heron's formula

Euler's proof of Heron's formula, like those given by Heron²² and Newton,²³ made use of the incircle and an ingenious construction of similar triangles. In modern geometry textbooks, the Heron formula is proved elegantly by considering the incircle together with an excircle.²⁴ Suppose the excircle on the side BC has radius ρ_a .



From the similarity of triangles AIZ and $AI'Z'$,

$$\frac{\rho}{\rho_a} = \frac{s-a}{s}.$$

Also, from the similarity of triangles BIZ and $I'BZ'$,

$$\rho \cdot \rho_a = (s-b)(s-c).$$

It is easy to see that

$$\rho = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \quad \text{and} \quad \rho_a = \sqrt{\frac{s(s-b)(s-c)}{s-a}}.$$

²²I Thomas, *Greek Mathematical Works*, II, *Aristarchus to Pappus of Alexandria*, Loeb Classical Library, number 362, Harvard University Press, 1941; pp. 472 – 477.

²³D.T. Whiteside, *The Mathematical Papers of Issac Newton*, V, 1683 – 1684, pp. 50 – 53. This is part of Newton's preliminary notes and drafts for his *Arithmetica*. See also Problem 23 of his *Lectures on Algebra*, *ibid.* pp.224 – 227.

²⁴See, for example, John Casey, *A sequel to the first six books of the Elements of Euclid*, (Part I), 1904 edition.

The Heron formula now follows easily from $\Delta = \rho s$.

Was Euler aware of the excircles of a triangle? When did the excircles first appear? In 1822, K.W. Feuerbach²⁵, in the monograph²⁶ proving his celebrated theorem on the nine-point circle, began with a description of the *four* circles tangent to the sides of a triangle, marking their centers as the points of intersection of the bisectors of the angles, and gave the radii of these circles as²⁷

$$\rho = \frac{\Delta}{a+b+c}, \quad \rho' = \frac{2\Delta}{-a+b+c}, \quad \rho'' = \frac{2\Delta}{a-b+c}, \quad \rho''' = \frac{2\Delta}{a+b-c}.$$

The excircles were not mentioned in some of the important popular geometry textbooks in the 18th and 19th centuries, including Robert Simson's *The Elements of Euclid*, (9th edition, 1793), John Playfair's *Elements of Geometry* (1840 edition), and A.M.Legendre's *Elements of Geometry and Trigonometry*, (English translation by Charles Davis, 1855 edition).

In Heath's *Euclid, the Thirteen Books of the Elements*, the escribed circles are mentioned only in the remarks following Book IV, Propositions 3, 4. *Euclid* IV.4 constructs the inscribed circle of a given triangle. Heath states "this problem in the more general form: to describe a circle touching three given straight lines which do not all meet in one point, and of which not more than two are parallel". Then he proceeds to describe the construction of the *escribed* circles.²⁸

More interesting is Heath's remark on *Euclid* IV.3: "about a given circle to circumscribed a triangle equiangular with a given triangle":

Peletarius²⁹ and Borelli³⁰ gave an alternative solution, first inscribing a triangle equiangular to the given triangle, by IV.2, and then drawing tangents to the circle parallel to the sides of the inscribed triangle respectively. This method will of course give two solutions, since two tangents can be drawn parallel to each of the sides of the inscribed triangle.

If the three pairs of parallel tangents be drawn and produced far enough, they will form *eight* triangles, two of which are the triangles circumscribed to the circle in the manner required in the proposition. The other six triangles are so related to the circle that the circle touches two of the sides in each *produced*, i.e., the circle is an *escribed* circle to each of the six triangles.

A proof of Heron's formula strikingly similar to the proof given above can be found in MEI Wending

²⁵Karl Wilhelm Feuerbach (1800 – 1834).

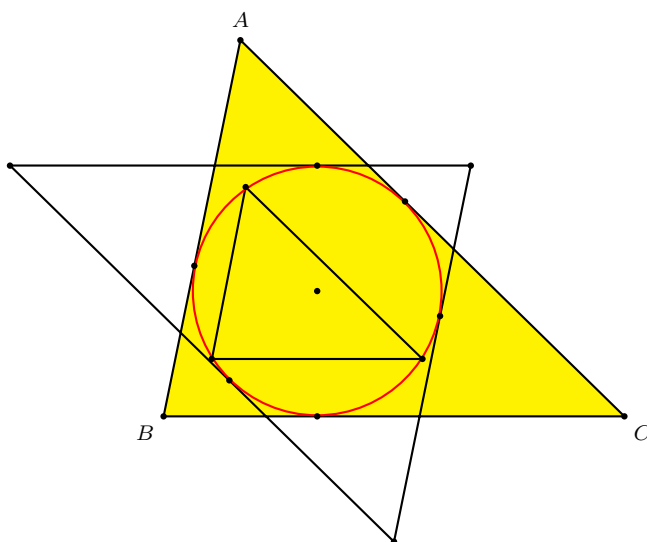
²⁶*Eigenschaften einiger merkwürdigen Punkte des geradlinigen Dreiecks, und mehrerer durch Sie bestimmten Linien und Figuren.*

²⁷D.E.Smith, *A Source Book of Mathematics*, p. 339.

²⁸Following this, Heath gives Heron's proof of his formula.

²⁹Jacques Peletier published "Demonstrations to the geometrical elements of Euclid, six books" in 1557.

³⁰Giovanni Alfonso Borelli (1608 – 1679) published *Euclides restitutus* in 1658.



³¹ *Elements of Plane Trigonometry*, ³² but without the excircle. MEI first used the similarity of triangles ADF and AIN to write down

$$\frac{AF}{AN} = \frac{DF^2}{IN \cdot DF}.$$

Then he asserted that the quadrilaterals $JINC$ and $HCFD$ are similar. From this,

$$\frac{DF}{FC} = \frac{CN}{IN},$$

so that $DF \cdot IN = CN \cdot FC$, and

$$\frac{AF}{AN} = \frac{DF^2}{FC \cdot CN}.$$

From the lengths of these segments, we have

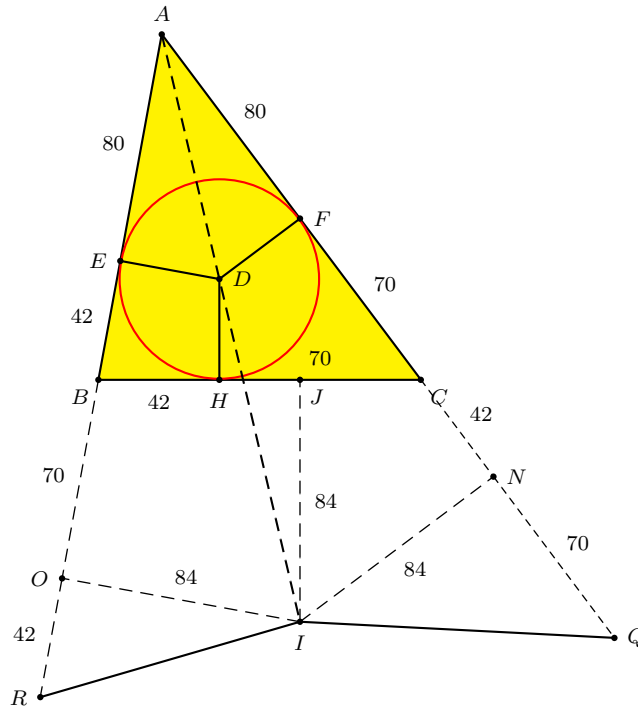
$$\frac{s-a}{s} = \frac{\rho^2}{(s-a)(s-b)}.$$

This determines the inradius ρ , and hence the area.

³¹ 1633 – 1721. On MEI, see J.C. Martzloff, *A History of Chinese Mathematics*, Springer, 1997.

³² *Pin san jiao ju yao*, Book IV, pp.12 – 13. In earlier Chinese literature, (QIN Jiushao, *Shushu jiuzhang* (13th century), the area formula of a triangle appeared in the form

$$\Delta = \sqrt{\frac{1}{4} \left\{ a^2 b^2 - \left(\frac{a^2 + b^2 - c^2}{2} \right)^2 \right\}}$$



XIA. Triangle centers

Paper 325: *Solutio facilis problematum quorundam geometricorum difficillimorum* (1765)

In this interesting paper, Euler analyzed and solved the construction problem a triangle with given orthocenter, circumcenter, and incenter. The Euler line and the famous Euler formula $OP^2 = R^2 - 2Rr$ emerged from this analysis.

§1. Review of the collinearity of the following triples of lines associated with a triangle ABC :

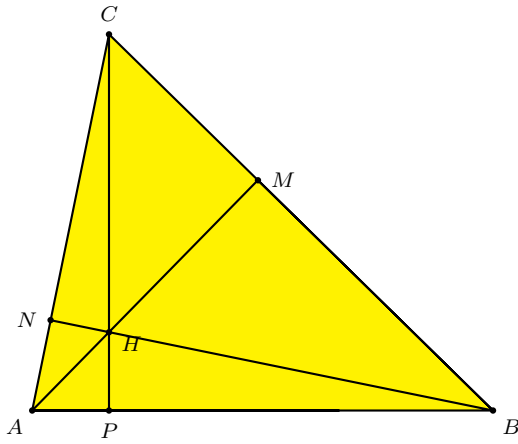
1. the altitudes intersecting at the orthocenter H ,
2. the medians intersecting at the centroid G ,
3. the angle bisectors intersecting at the incenter I ,
4. the perpendicular bisectors intersecting at the circumcenter O .³³

§6. Orthocenter [Figure 1]. $HP = \frac{(b^2+c^2-a^2)(c^2+a^2-b^2)}{8c \cdot \Delta}$.

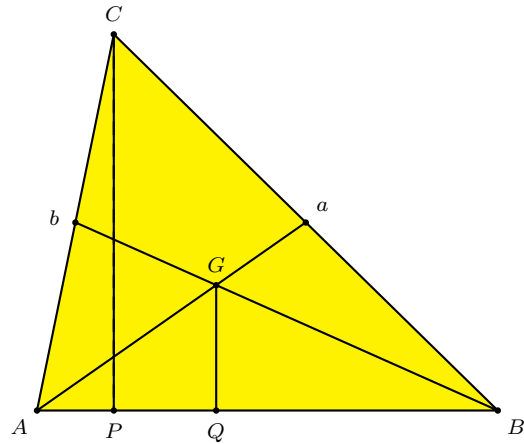
Proof. $AM = \frac{2\Delta}{a}$.

$$AP = \frac{b^2 + c^2 - a^2}{2c}, \quad BM = \frac{c^2 + a^2 - b^2}{2a}.$$

³³Euler used E, F, G, H for the orthocenter, the centroid, the incenter, and the circumcenter respectively.



(c) Figure 1



(d) Figure 2

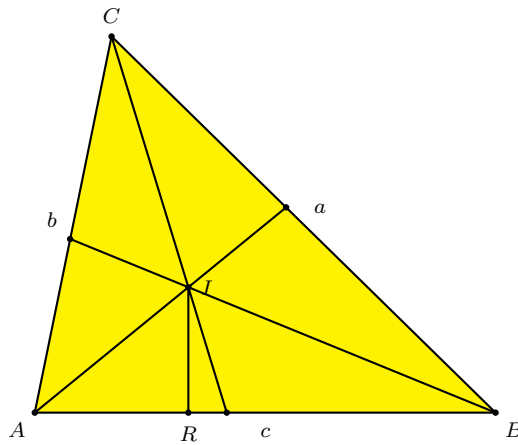
Similarity of right triangles ABM and AHP gives

$$AM : BM = AP : HP$$

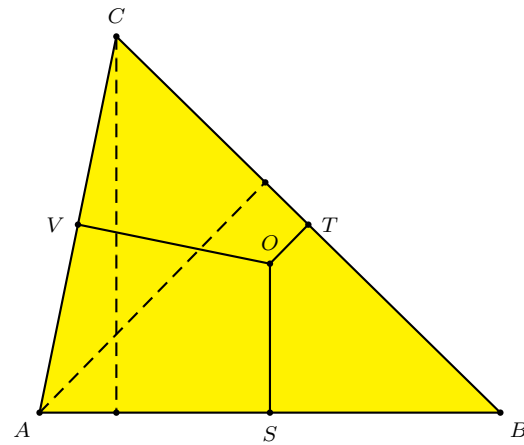
from which HP can be found.

§7. Centroid [Figure 2]. $AQ = \frac{3c^2 + b^2 - a^2}{6c}$, $GQ = \frac{2\Delta}{3c}$.

§8. Incenter [Figure 3]. $IR = \frac{2\Delta}{a+b+c}$.



(e) Figure 3

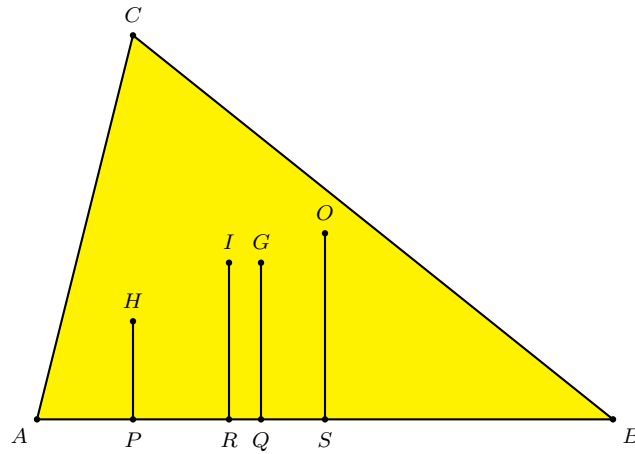


(f) Figure 4

§9. Circumcenter [Figure 4]. $OS = \frac{c(a^2 + b^2 - c^2)}{8\Delta}$.

XIB. Distances between various centers

§10.



$$\begin{aligned}
 HG^2 &= (AP - AQ)^2 + (PH - QG)^2, \\
 HI^2 &= (AP - AR)^2 + (PH - RI)^2, \\
 HO^2 &= (AP - AS)^2 + (PH - SO)^2, \\
 GI^2 &= (AQ - AR)^2 + (QG - RI)^2, \\
 GO^2 &= (AQ - AS)^2 + (QG - SO)^2, \\
 IO^2 &= (AR - AS)^2 + (RO - SO)^2.
 \end{aligned}$$

§11. Distances in terms of elementary symmetric functions of side lengths. Write

$$a + b + c = p, \quad ab + bc + ca = q, \quad abc = r.$$

The side lengths are the roots of the cubic equation

$$z^3 - pz^2 + qz - r = 0.$$

$$\begin{aligned}
 a^2 + b^2 + c^2 &= p^2 - 2q, \\
 a^2b^2 + b^2c^2 + c^2a^2 &= q^2 - 2pr, \\
 a^4 + b^4 + c^4 &= p^4 - 4p^2q + 2q^2 + 4pr.
 \end{aligned}$$

$$\Delta^2 = \frac{1}{16}(2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4)$$

$$= \frac{1}{16}(-p^4 + 4p^2q - 8pr).$$

§18. Summary of square distances.

$$\begin{aligned} HG^2 &= \frac{r^2}{4\Delta^2} - \frac{4}{9}(p^2 - 2q), \\ HI^2 &= \frac{r^2}{4\Delta^2} - \frac{4r}{p} - p^2 + 3q, \\ HO^2 &= \frac{9r^2}{16\Delta^2} - p^2 + 2q, \\ GI^2 &= \frac{1}{9p}(-p^3 + 5pq - 18r), \\ GO^2 &= \frac{r^2}{16\Delta^2} - \frac{1}{9}(p^2 - 2q), \\ IO^2 &= \frac{r^2}{16\Delta^2} - \frac{r}{p}. \end{aligned}$$

The Euler line

§18. ubi evidens est esse $HO = \frac{3}{2}HG$ et $GO = \frac{1}{2}HG$, sicque punctum O per puncta H , G sponte determinatur, scilicet si tria puncta H , G , I forment triangulum HGI , tum quartum punctum O ita in rect HG producta erit situm, ut sit

$$GO = \frac{1}{2}HG \quad \text{ideoque} \quad HO = \frac{3}{2}HG.$$

Hinc vero deducitur

$$4IO^2 + 2HI^2 = 3HG^2 + 6GI^2,$$

quod cum valoribus inventis apprime congruit.

§19. Quo nunc has formulas ad maiorem simplicitatem recocemus, ponamus

$$4pq - p^3 - 8r = 4S,$$

ut sit

$$4\Delta^2 = pS, \quad \text{et} \quad 4q = p^2 + \frac{8r}{p} + \frac{4S}{p};$$

tum vero faciamus:

$$\begin{aligned} \frac{r^2}{pS} &= \mathcal{R}, & \frac{r}{p} &= \mathcal{Q}, & p^2 &= \mathcal{P}, \\ HG^2 &= \mathcal{R} - \frac{2}{9}\mathcal{P} + \frac{16}{9}\mathcal{Q} + \frac{8\mathcal{Q}^2}{9\mathcal{R}}, \\ HI^2 &= \mathcal{R} - \frac{1}{4}\mathcal{P} + 2\mathcal{Q} + \frac{3\mathcal{Q}^2}{\mathcal{R}}, \\ HO^2 &= \frac{9}{4}\mathcal{R} - \frac{1}{2}\mathcal{P} + 4\mathcal{Q} + \frac{2\mathcal{Q}^2}{\mathcal{R}}, \\ GI^2 &= + \frac{1}{36}\mathcal{P} - \frac{8}{9}\mathcal{Q} + \frac{5\mathcal{Q}^2}{9\mathcal{R}}, \\ GO^2 &= \frac{1}{4}\mathcal{R} - \frac{1}{18}\mathcal{P} + \frac{4}{9}\mathcal{Q} + \frac{2\mathcal{Q}^2}{9\mathcal{R}}, \\ IO^2 &= \frac{1}{4}\mathcal{R} - \mathcal{Q}. \end{aligned}$$

Euler's formula

Note that

$$\frac{1}{4}\mathcal{R} = \frac{\rho^2}{4pS} = \frac{(abc)^2}{16\Delta^2} = \left(\frac{abc}{4\Delta}\right)^2 = R^2.$$

Also,

$$Q = \frac{r}{p} = \frac{abc}{2s} = 2 \cdot \frac{abc}{4\Delta} \cdot \frac{\Delta}{s} = 2R\rho.$$

This is the famous Euler's formula:

$$OI^2 = R^2 - 2R\rho = R(R - 2\rho).$$

Note: $R \geq 2\rho$, with equality for equilateral triangles.

XIC. The nine - point circle and Feuerbach's theorem

The nine - point circle of a triangle is the circle passing through the midpoints of the three sides, the perpendicular feet of the vertices on their opposite sides, and the midpoints between the orthocenter and each of the vertices. It was discovered by C.J. Brianchon³⁴ and J.V. Poncelet³⁵, one year before³⁶ K.W. Feuerbach proved³⁷ that this nine - point circle is tangent internally to the incircle, and externally to each of the excircles. He begins by showing that the center N of the nine - point circle is the midpoint of the segment joining the circumcenter O to the orthocenter H . The proof given by Feuerbach is computational in nature, similar in spirit to Euler's Paper 325.

The nine-point circle is indeed simply the circumcircle of the triangle DEF , D, E, F being the midpoints of the sides of triangle ABC . The two triangles ABC and DEF are similar, with corresponding sides parallel and similarity ratio 2 : 1, and sharing a common centroid. The nine-point circle, therefore, has radius $\frac{1}{2}R$.

Triangle	Circumcenter	Centroid	Orthocenter
ABC	O	G	H
DEF	N	G	O

From this table, it is clear that the two triangles have the same Euler line. Since the distance from the centroid to the orthocenter is always twice that from the centroid to the circumcenter, $OG = 2GN$. It follows that

$$ON = OG + GN = \frac{3}{2} \cdot OG = \frac{3}{2} \cdot \frac{1}{3} \cdot OH = \frac{1}{2} \cdot OH,$$

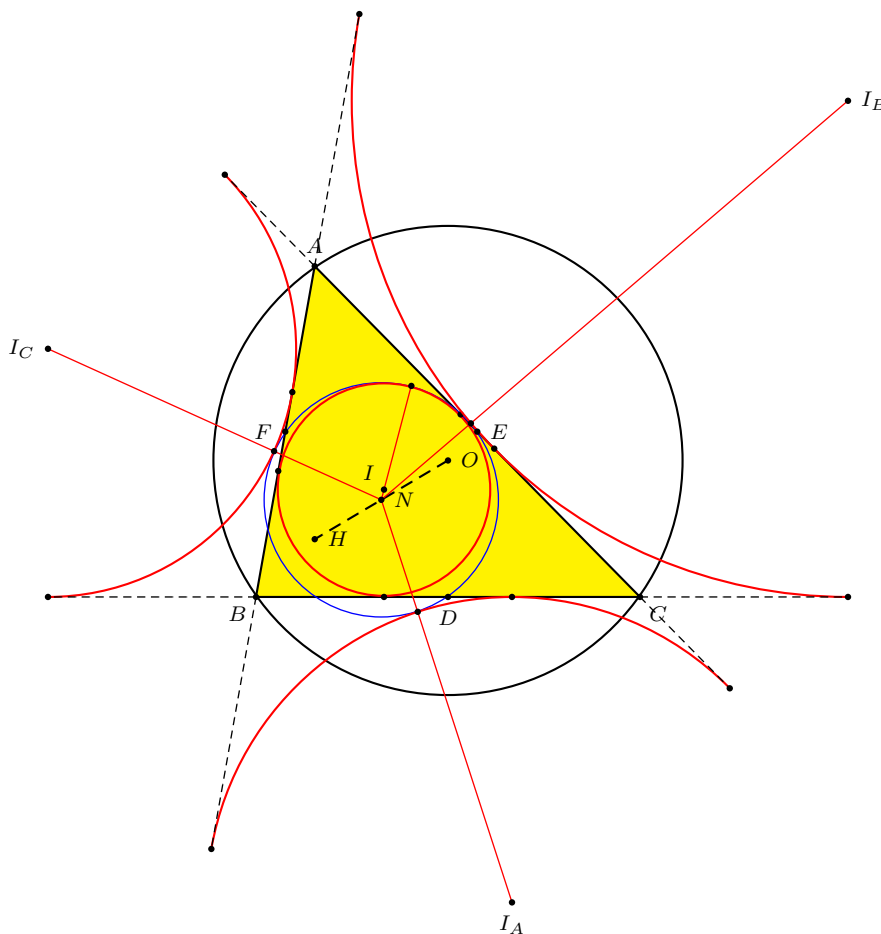
and N is the midpoint of OH .

³⁴Charles Jules Brianchon (1785 – 1864).

³⁵Jean Victor Poncelet (1788 – 1867).

³⁶*Recherches sur la détermination d'une hyperbole équilatère, au moyen de quatre conditions données*, Gergonne's Annales, 11 (1820-21) 205 – 220. See D.E.Smith, *A Source Book of Mathematics*, pp. 337 – 338.

³⁷See D.E.Smith, *A Source Book in Mathematics*, pp. 339 – 345.



Once it is known that N is the midpoint of OH , Feuerbach's theorem can be proved by computing the length of the median IN in triangle IOH , making use of Apollonius' theorem,^{38 39} and the distances

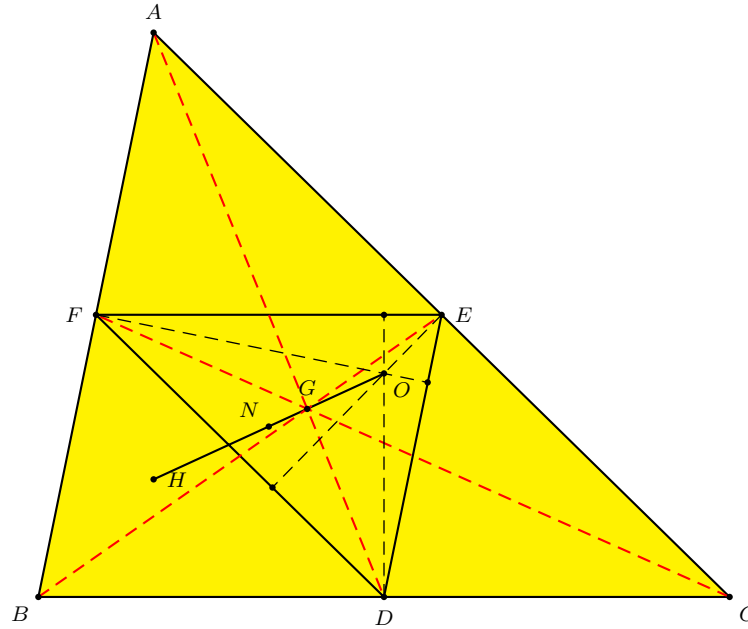
³⁸The sum of the squares on two sides of a triangle is equal to twice the square on half the base, together with twice the square on the median on this base. This is an easy consequence of *Eucl.* II.9, 10. Euler makes use of this in his Paper 135 *Variae demonstrationes geometriae* to prove that for a quadrilateral $ABCD$ with midpoints P, Q of the diagonals,

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4PQ^2.$$

³⁹More generally, if D is a point outside a line ABC , then

$$AD^2 \cdot BC + BD^2 \cdot CA + CD^2 \cdot AB + BC \cdot CA \cdot AB = 0.$$

This is usually referred to as Stewart's theorem. Howard Eve, *An Introduction to the History of Mathematics*, (6th ed.) p. 183, states that Robert Simson (1687 – 1768) anticipated M. Stewart (1717 – 1785) in the discovery of the theorem.. Euler stated this as a lemma in his 1780 Paper 543, *Problematis cuiusdam Pappi Alexandrini constructio*.



that Euler had determined. In his monograph, Feuerbach re-established the distance formulae that Euler obtained, expressing each of them in terms of R , ρ , and a third quantity $r = 2R \cos A \cos B \cos C$. Here, we make use of Euler's formulae.

$$\begin{aligned} IN^2 &= \frac{1}{2} \cdot HI^2 + \frac{1}{2} \cdot IO^2 - \left(\frac{1}{2} \cdot OH\right)^2 \\ &= \frac{1}{2} \left(\mathcal{R} - \frac{1}{4}\mathcal{P} + 2\mathcal{Q} + \frac{3\mathcal{Q}^2}{\mathcal{R}}\right) + \frac{1}{2} \left(\frac{1}{4}\mathcal{R} - \mathcal{Q}\right) - \frac{1}{4} \left(\frac{9}{4}\mathcal{R} - \frac{1}{2}\mathcal{P} + 4\mathcal{Q} + \frac{2\mathcal{Q}^2}{\mathcal{R}}\right) \\ &= \frac{1}{16}\mathcal{R} - \frac{1}{2}\mathcal{Q} + \frac{\mathcal{Q}^2}{\mathcal{R}} = \frac{1}{4}R^2 - R\rho + \rho^2 = \left(\frac{1}{2}R - \rho\right)^2 \end{aligned}$$

since $\mathcal{R} = 4R^2$ and $\mathcal{Q} = 2R\rho$. The distance between the centers of the nine-point circle and the incircle is equal to the difference between their radii. The two circles are therefore tangent internally. This is the first part of Feuerbach's theorem.

The proof of the tangency of the nine-point circle with each of the excircles is along the same line. Feuerbach obtained the distance between O and the excenter I_A :

$$OI_A^2 = R^2 + 2R\rho_a,$$

and in a historical note attributed this to Euler. However, the excircles do not seem to have appeared in Euler's work.

Referring to the Feuerbach theorem, Uta Merzbach, in her revision of Boyer's *A History of Mathematics*,⁴⁰ wrote:

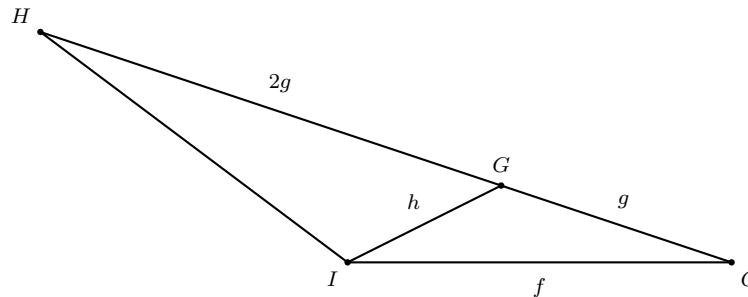
One enthusiast, the American geometer Julian Lowell Coolidge (1873 – 1954), called this “the most beautiful theorem in elementary geometry that has been discovered since the time of Euclid”.⁴¹ It should be noted that the charm of such theorems supported considerable investigation in the geometry of triangles and circles throughout the nineteenth century.

XIIA. Euler's construction of a triangle with given circumcenter, incenter, and orthocenter

§20. Problema: Datis positione his quatuor punctis in quolibet triangulo assignabilibus

1. Intersectione perpendicularium ex singulis angulis in latera opposita ductarum H ,
2. Centro gravitatis G ,
3. Centro circuli inscripti I et
4. Centro circuli circumscripti O .

Quod problema ex hactenus erutis horum punctorum affectionibus satisconcinne resolvere licebit.



Solutio: Cum positio horum quatuor punctorum per eorum distantias detur, vocemus:

$$IO = f, \quad GO = g \quad \text{et} \quad GI = h.$$

novimusque fore

$$HG = 2g, \quad HO = 3g \quad \text{itemque} \quad HI = \sqrt{6g^2 + 3h^2 - 2f^2}.$$

Nunc igitur statim habemus has tres aequationes

$$f^2 = \frac{1}{4}\mathcal{R} - \mathcal{Q},$$

⁴⁰2nd ed. p.537.

⁴¹J.L. Coolidge, The Heroic Age of Geometry, *Bull. Amer. Math. Soc.*, 35 (1929) 19 – 37.

$$\begin{aligned} g^2 &= \frac{1}{4}\mathcal{R} - \frac{1}{18}\mathcal{P} + \frac{4}{9}\mathcal{Q} + \frac{2\mathcal{Q}^2}{9\mathcal{R}}, \\ h^2 &= \frac{1}{36}\mathcal{P} - \frac{8}{9}\mathcal{Q} + \frac{5\mathcal{Q}^2}{9\mathcal{R}}, \end{aligned}$$

ex quarum resolutione colligimus:

$$\begin{aligned} \mathcal{R} &= \frac{4f^4}{3g^2 + 6h^2 - 2f^2}, \\ \mathcal{Q} &= \frac{3f^2(f^2 - g^2 - 2h^2)}{3g^2 + 6h^2 - 2f^2}, \\ \mathcal{P} &= \frac{27f^4}{3g^2 + 6h^2 - 2f^2} - 12f^2 - 15g^2 + 6h^2, \end{aligned}$$

unde fit

$$\frac{\mathcal{Q}^2}{\mathcal{R}} = \frac{9(f^2 - g^2 - 2h^2)^2}{4(3g^2 + 6h^2 - 2f^2)}.$$

§22. His valoris inventis investigentur tre sequentes expressiones:

$$p = \sqrt{\mathcal{P}}, \quad q = \frac{1}{4}\mathcal{P} + 2\mathcal{Q} + \frac{\mathcal{Q}^2}{\mathcal{R}}, \quad r = \mathcal{Q}\sqrt{\mathcal{P}}$$

indeque formetur haec aequatio cubica:

$$z^3 - pz^2 + qz - r = 0,$$

cuius tres radices debunt tria latera trianguli quaesiti, quo pacto eius constructio facillima habetur.

§§22, 23: Example. $a = 5, b = 6, c = 7: \Delta = 6\sqrt{6}$.

$$HG^2 = \frac{155}{72}, \quad HI^2 = \frac{11}{8}, \quad HO^2 = \frac{155}{32},$$

$$IG^2 = \frac{1}{9}, \quad GO^2 = \frac{155}{288}, \quad IO^2 = \frac{35}{32}.$$

$$f^2 = \frac{35}{32}, \quad g^2 = \frac{155}{288}, \quad h^2 = \frac{1}{9}.$$

From these,

$$3g^2 + 6h^2 - 2f^2 = \frac{3}{32}, \quad f^2 - g^2 - 2h^2 = \frac{1}{3}, \quad 4f^2 + 5g^2 - 2h^2 = \frac{219}{32};$$

and

$$\mathcal{R} = \frac{1225}{24}, \quad \mathcal{Q} = \frac{35}{3}, \quad \mathcal{P} = 324$$

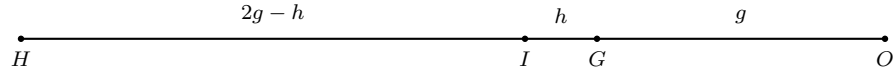
and $\frac{Q^2}{\mathcal{R}} = \frac{8}{3}$.

$$p = \sqrt{\mathcal{P}} = 18, \quad q = 107, \quad r = \frac{35}{3} \cdot 18 = 210$$

leading to the cubic equation $z^3 - 18z^2 + 107z - 210 = 0$, whose roots are 5, 6, 7.

XIIB. Special cases

§25. Triangles with the incenter on the Euler line.



$$\begin{aligned} GO &= g, & GI &= h, & IO &= f, \\ HG &= 2g, & HO &= 3g, & HI &= 2g - h. \end{aligned}$$

Putting $g = f - h$, we have

$$\mathcal{R} = \frac{4f^4}{(f-3h)^2}, \quad \mathcal{Q} = \frac{3f^2h(2f-3h)}{(f-3h)^2}, \quad \mathcal{P} = \frac{3h(4f-3h)^2}{(f-3h)^2}.$$

$$\frac{\mathcal{Q}^2}{\mathcal{R}} = \frac{9h^2(2f-3h)^2}{4(f-3h)^2}.$$

$$p = \frac{(4f-3h)\sqrt{3h(4f-3h)}}{f-3h},$$

$$q = \frac{3fh(4f-3h)(5f-6h)}{(f-3h)^2},$$

$$r = \frac{3f^2h(2f-3h)(4f-3h)\sqrt{3h(4f-3h)}}{(f-3h)^2}.$$

§26. Solution of the cubic equation $z^3 - pz^2 + qz - r = 0$.

Euler puts $z = \frac{\sqrt{3h(4f-3h)}}{f-3h} \cdot y$ and transforms the equation into

$$y^3 - (4f-3h)y^2 + f(5f-6h)y - f^2(2f-3h) = 0.$$

The roots of this equation are f , f , and $2f-3h$. The triangle is isosceles, with sides

$$a = b = \frac{f\sqrt{3h(4f-3h)}}{f-3h}, \quad c = \frac{(2f-3h)\sqrt{3h(4f-3h)}}{f-3h}.$$

§30. Euler takes $HI = e$, $IO = f$, $HO = k$, and writes

$$GO = \frac{1}{3}k, \quad HG = \frac{2}{3}k, \quad GI = \frac{1}{3}\sqrt{3e^2 + 6f^2 - 2k^2}.$$

Here,

$$\begin{aligned} \mathcal{R} &= \frac{4f^4}{2f^2 + 2e^2 - k^2}, \\ \mathcal{Q} &= \frac{f^2(k^2 - f^2 - 2e^2)}{2e^2 + 2f^2 - k^2}, \\ \mathcal{P} &= \frac{4e^4 + 11f^4 + 3k^4 - 12e^2f^2 + 2f^2k^2 - 8e^2k^2}{2e^2 + 2f^2 - k^2}, \\ \frac{\mathcal{Q}^2}{\mathcal{R}} &= \frac{(k^2 - f^2 - 2e^2)^2}{4(2e^2 + 2f^2 - k^2)}; \\ p &= \sqrt{\mathcal{P}}, \\ q &= \frac{2e^4 + f^4 + k^4 - 6e^2f^2 - 3e^2k^2 + 2f^2k^2}{2e^2 + 2f^2 - k^2}, \\ r &= \mathcal{Q}\sqrt{\mathcal{P}}. \end{aligned}$$

The sides of the triangle have lengths $y\sqrt{\mathcal{P}}$, where y satisfies the cubic equation

$$y^3 - y^2 + \frac{(2e^4 + f^4 + k^4 - 6e^2f^2 - 3e^2k^2 + 2f^2k^2)y - f^2(k^2 - 2e^2 - f^2)}{4e^4 + 11f^4 + 3k^4 - 12e^2f^2 + 2f^2k^2 - 8e^2k^2} = 0$$

Then Euler works out the condition for the existence of such a triangle.

Example Triangle OIH isosceles. $e = f$

$$\begin{aligned} p &= \frac{\sqrt{3}(k^2 - f^2)}{\sqrt{4f^2 - k^2}}, \\ q &= \frac{k^4 - f^2k^2 - 3f^4}{4f^2 - k^2}, \\ r &= \frac{\sqrt{3}f^2(k^2 - 3f^2)(k^2 - f^2)}{(4f^2 - k^2)\sqrt{4f^2 - k^2}}. \end{aligned}$$

If $z = \frac{y}{\sqrt{3(4f^2 - k^2)}}$, y satisfies

$$y^3 - 3(k^2 - f^2)y^2 + 3(k^4 - f^2k^2 - 3f^4)y - 9f^2(k^2 - 3f^2)(k^2 - f^2) = 0.$$

Euler observes that one of the roots is $y = 3f^2$. The other two therefore satisfy the quadratic

$$y^2 - 3(k^2 - 2f^2)y + 3(k^2 - 3f^2)(k^2 - f^2) = 0$$

and are

$$y = \frac{3(k^2 - 2f^2) \pm k\sqrt{3(4f^2 - k^2)}}{2}.$$

The sides of the triangle have lengths

$$\frac{\sqrt{3}(k^2 - 2f^2)}{2\sqrt{4f^2 - k^2}} \pm \frac{1}{2}k, \quad \frac{\sqrt{3}f^2}{\sqrt{4f^2 - k^2}}.$$

§34. Example $e^2 = 3$, $f^2 = 2$, $k^2 = 9$. The sides of the triangle are the roots of

$$z^3 - \sqrt{71}z^2 + 22z - 2\sqrt{71} = 0.$$

Euler ends the paper with the remark that the roots of this equation can be expressed in terms of an angle

$$\alpha = \arccos \sqrt{\frac{71}{125}}.$$

The side lengths are

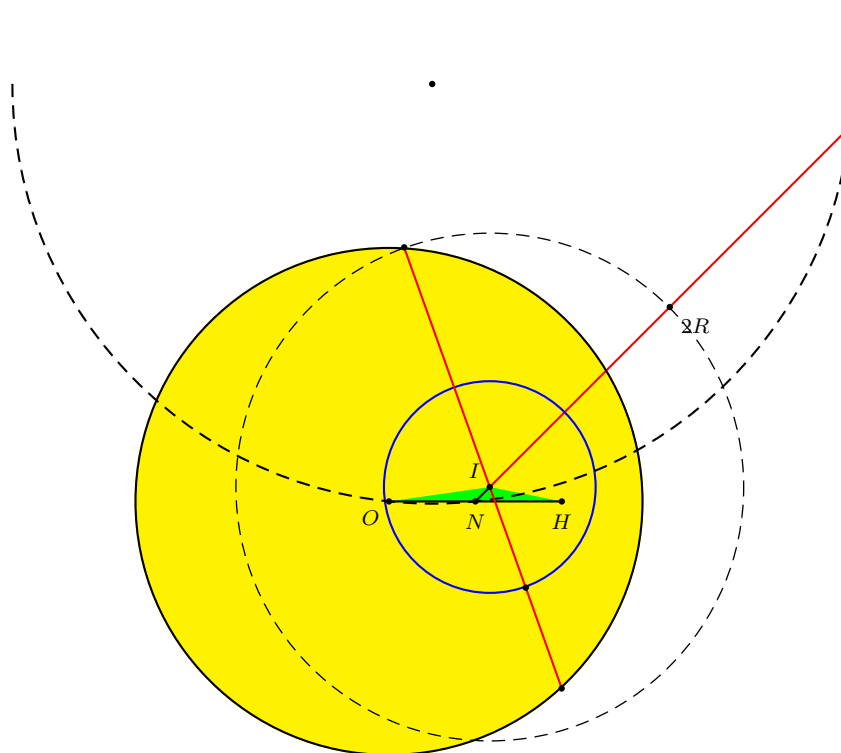
$$\begin{aligned} a &= \frac{1}{3}\sqrt{71} + \frac{2}{3}\sqrt{5} \cdot \cos(60^\circ - \frac{1}{3}\alpha), \\ b &= \frac{1}{3}\sqrt{71} + \frac{2}{3}\sqrt{5} \cdot \cos(60^\circ + \frac{1}{3}\alpha), \\ c &= \frac{1}{3}\sqrt{71} - \frac{2}{3}\sqrt{5} \cdot \cos \frac{1}{3}\alpha, \end{aligned}$$

ubi est proxime $\alpha = 41^\circ 5' 30''$ sicque per anguli trisectionem problema semper satis expedite resolvetur.

XIIC. Constructions

Given the triangle OIH , it is indeed possible to construct the circumcircle, and the incircle of the triangle ABC . This is possible by making use of Feuerbach's theorem.

The nine-point center N is the midpoint of the segment OH . According to Feuerbach's theorem, $IN = \frac{1}{2}(R - 2\rho)$. On the other hand, by Euler's formula, $OI^2 = R(R - 2\rho)$. From these, R and ρ can be determined. Since $OI^2 = 2R \cdot IN$, if we construct the circle through N , tangent to OI at O , and extend IN to intersect the circle again at M , then $IM = 2R$. (*Eucl.* III.36). From this, the circumcircle and the nine-point circle can be constructed.



The fact that the side lengths of triangle ABC satisfy a cubic equation shows that the triangle in general *cannot* be using ruler and compass. This was not confirmed in Euler's time. But Euler has reduced the problem to the trisection of an angle. Since it is always true that the lines joining a vertex to the circumcenter and the orthocenter are symmetric with respect to the bisector of the angle at that vertex,⁴² the line AI should bisect angle OAH . While in general such a point cannot be constructed (using ruler and compass only), the special case when IOH is isosceles does admit a euclidean construction. The intersection of the half - line NI with the circumcircle is one vertex A . If the half line AH intersects the nine-point circle at X , the perpendicular at X to AX would intersect the circumcircle at the other vertices of the required triangle.

Cubic equations and geometric constructions

A geometric solution of a cubic equation $x^3 = az + b$ can be reduced to one of the ancient problems of trisection of an angle or duplication of a cube. With the notation of Euler's paper 20, §3, in writing a root in the form $\sqrt[3]{A} + \sqrt[3]{B}$, A and B are the roots of the quadratic

$$z^2 = bz - \frac{a^3}{27}.$$

⁴²This is usually summarized by saying that the circumcenter and the orthocenter are *isogonal conjugates*.

If $\frac{b^2}{4} - \frac{a^3}{27} > 0$, A and B are real, and exactly one of the roots is real, and the other two are imaginary. The construction of these roots depends on the extraction of the cube roots of the real, constructible numbers A and B .

If, however, $\frac{b^2}{4} - \frac{a^3}{27} \leq 0$, A and B are imaginary, and the three roots of the cubic equation are indeed real. Note that a must be positive. These can be found by rewriting the cubic equation $x^3 = ax + b$ to bring to the form

$$4 \cos^3 \theta - 3 \cos \theta = \cos 3\theta = \text{constant}.$$

To this end, put $x = r \cos \theta$ and choose r such that

$$r^3 : ar = 4 : 3.$$

This means $r^2 = \frac{4a}{3}$. With $r = 2\sqrt{\frac{a}{3}}$, the cubic equation becomes

$$b = x^3 - az = 2\left(\frac{a}{3}\right)^{\frac{3}{2}} \cos 3\theta,$$

and

$$\cos 3\theta = \frac{b}{2} \cdot \left(\frac{3}{a}\right)^{\frac{3}{2}}$$

Since $\left(\frac{b}{2} \cdot \left(\frac{3}{a}\right)^{\frac{3}{2}}\right)^2 = \frac{b^2}{4} \cdot \frac{27}{a^3} < 1$, there is a unique angle α in the range $0 < \alpha < 180^\circ$ so that $\cos 3\theta = \cos(180^\circ - \alpha)$. Consequently,

$$\theta = 60^\circ - \frac{1}{3}\alpha, \quad -60^\circ - \frac{1}{3}\alpha, \quad 180^\circ - \frac{1}{3}\alpha.$$

The corresponding x are then

$$r \cos \theta = 2\sqrt{\frac{a}{3}} \cos(60^\circ - \frac{1}{3}\alpha), \quad 2\sqrt{\frac{a}{3}} \cos(60^\circ + \frac{1}{3}\alpha), \quad -2\sqrt{\frac{a}{3}} \cos \frac{1}{3}\alpha.$$

XIIIA. Euler and the Fermat numbers**References on Euler's number theory works**

1. Andre Weil, *Number Theory, An Approach Through History, from Hammurapi to Legendre*, Birkhäuser, 1983.
2. Winfred Scharlau and Hans Opolka, *From Fermat to Minkowski, Lectures on the theory of numbers and its historical development*, Springer, 1984.

The number theory works of Euler are contained in volumes 2 – 5 of the *Opera Omnia*. It started with the correspondences with Christian Goldbach in 1729. Goldbach remarked to Euler⁴³, “Is Fermat’s observation known to you, that all numbers $2^{2^n} + 1$ are primes? He said he could not prove it; nor has anyone else done so to my knowledge”.

Paper 26: Observationes de Theoremate quodam Fermatiano aliisque ad numeros primos spectantibus (1732/33)

In this short paper, Euler began by examining when a number of the form $d^n + 1$ can be prime. He quickly established that this must be of the form $2^{2^m} + 1$. Then he quoted Fermat:

Cum autem numeros a binario quadraticae in se ductos et unitate auctos esse semper numeros primos apud me constet et iam dudum Analystis illius theorematis veritas fuerit significata, nempe esse primos 3, 5, 17, 257, 65537 etc. in infinit., nullo negotio etc.

and Euler continued

Veritas istius theorematis elucet, ut iam dixi, si pro m ponatur 1, 2, 3 et 4; prodeunt enim hi numeri 5, 17, 257 et 65537, qui omnes inter numeros primos in tabula reperiuntur. Sed nescio, quo fato eveniat, ut statim sequens, nempe $2^{2^5} + 1$, cesset esse numerus primus; observavi enim his diebus longe alia agens posse hunc numerum dividi per 641, ut cuique tentanti statim patebit. Est enim

$$2^{2^5} + 1 = 2^{32} + 1 = 4294967297.$$

Ex quo intelligi potest theorema hoc etiam in aliis, qui sequuntur, casibus fallere et hanc ob rem problema de inveniendo numero primo quovis dato maiore etiam nunc non esse solutum.

How did Euler find the divisor 641? In his paper 134: *Theoremata circa divisores numerorum (1747/48)*, he studied in detail the factorization of numbers of the form $d^n \pm b^n$.

§29. Theorema 8. Summa duarum huiusmodi potestatum $a^{2^m} + b^{2^m}$, quarum exponens est dignitas binarii, alios divisores non admittit, nisi qui contineantur in hac forma $2^{m+1}n + 1$.

⁴³Weil, p.172.

§32. Scholion 1. ⁴⁴ Fermat affirmaverat, etiamsi id se demonstrare non posse ingenue esset confessus, omnes numeros ex hac forma $2^{2^m} + 1$ ortos esse primos; hincque problema alias difficillimum, quo quaerebatur numerus primus dato numero maior, resolvere est conatus. Ex ultimo theoremate autem perspicuum est, nisi numerus $2^{2^m} + 1$ sit primus, eum alios divisores haberes non posse praeter tales, qui in forma $2^{m+1}n + 1$ contineantur. Cum igitur veritatem huius effati Fermatiani pro casu $2^{32} + 1$ examinare voluissem, ingens hinc compendium sum nactus, dum divisionem aliis numeris primis praeter eos, quos formula $64n + 1$ suppeditat, tentare non opus habebam. Huc igitur inquisitione reducta mox deprehendi ponendo $n = 10$ numerum primum 641 esse divisorem numeri $2^{32} + 1$, unde problema memoratum, quo numerus primus dato numero maior requiritur, etiamnum manet insolutum.

Thus,

$$F_5 = 2^{32} + 1 = 641 \times 6700417.$$

Euler did not mention if the other divisor is prime or not. But using the same theorem, he could easily have decided that it is indeed prime. All he needed was to test for divisibility by primes less than the square root of 6700417, (between 641 and 2600), and of the form $64n + 1$. Such are

$$641, \quad 769, \quad 1153, \quad 1217, \quad 1409, \quad 1601, \quad \text{and } 2113.$$

It is routine to check that none of these divides 6700417.

In 1742, Euler wrote to Goldbach that $4m + 1$ is prime if and only if it can be written as $a^2 + b^2$ in one and only one way. He illustrated this by exhibiting

$$F_5 = 2^{32} + 1 = (2^{16})^2 + 1^2 = 62264^2 + 20449^2.$$

Appendix. Fermat on the primality of $F_m = 2^{2^m} + 1$ ⁴⁵

Fermat to Frenicle, 1640:

3. Mais voici ce que j'admire le plus : c'est que je suis quasi persuadé (1) que tous les nombres progressifs augmentés de l'unité, desquels les exposants sont des nombres de la progression double, sont nombres premiers, comme

$$3, \quad 5, \quad 17, \quad 257, \quad 65537, \quad 4\,294\,967\,297$$

et le suivant de 20 lettres

$$18\,446\,744\,073\,709\,551\,617, \quad \text{etc.}$$

⁴⁴See XIII below for more details of this paper.

⁴⁵M.S. Mahoney, *The Mathematical Career of Pierre de Fermat*, 2nd ed. Princeton University Press, 1994.

Je n'en ai pas la démonstration exacte, mais j'ai exclu si grande quantité de diviseurs par démonstrations infaillibles, et j'ai de si grandes lumières, qui établissent ma pensée, que j'aurois peine à me dédire. ⁴⁶

Mahoney comments that “[i]n the years that followed, Fermat’s conviction of the validity of his conjecture grew, while a proof continued to elude him. The primality of all numbers of the form $2^n + 1$ undelay the climax toward which he steered Part III of the *Tripartite Dissertation*, and it figured among the first propositions with which he tried to engage Blaise Pascal’s interest in number theory in 1654. At each mention of the conjecture, Fermat bemoaned his inability to find a proof, and his tone of growing exasperation suggests that he was continually trying to do so. In view of Euler’s disproof of the conjecture by counterexample in 1732, Fermat’s quandary is understandable in retrospect. So much more surprising, then, is his claim in the “Relation” to Carcavi ⁴⁷ in 1659 to have found the long - elusive demonstration. ⁴⁸

5. J'ai ensuite considéré certaines questions qui, bien que négatives, ne restent pas de recevoir très grande difficulté, la méthode pour y pratiquer la descente étant tout fait diverse des précédentes, comme il sera aisé d'éprouver. Telles sont les suivantes:

...

Toutes les puissances quarrées de 2, argumentées de l'unité, sont nombres premiers.

Cette dernière question est d'une très subtile et très ingénieuse recherche et, bien qu'elle soit connue affirmativement, elle est négative, puisque dire qu'un nombre est premier, c'est dire qu'il ne peut être divisé par aucun nombre. ⁴⁹

⁴⁶Mahoney, p.301: “But here is what I admire most of all: it is that I am just about convinced that all progressive numbers augmented by unity, of which the exponents are numbers of the double progression, are prime numbers, such as 3, 5, 17, 257, 65537, 4294967297, and the following of twenty digits: 18446744073709551617, etc. I do not have an exact proof of it, but I have excluded such a large quantity of divisors by infallible demonstrations, and my thoughts rest on such clear insights, that I can hardly be mistaken.”

⁴⁷Pierre de Carcavi, 1600? – 1684.

⁴⁸Mahoney, pp.301, 356.

⁴⁹“I have then considered some questions that, although negative, do not remain to receive very great difficulty, the method for there to practice the descent being entirely various precedents, as it will be well-off to feel. Such are following: ... All square powers of 2, argmented by the unit, are prime numbers. This last problem results from very subtle and very ingenious research and, even though it is conceived affirmatively, it is negative, since to say that a number is prime is to say that it cannot be divided by any number.”

XIIIB. Perfect numbers and Mersenne primes⁵⁰

Eucl. VII. Definition 22: A perfect number is that which is equal to its own parts.

Eucl. IX.36: If as many numbers as we please beginning from an unit be set out continuously into double proportion, until the sum of all becomes prime, and if the sum multiplied into the last make some number, the product will be perfect.

In response to Frenicle's question of finding a perfect number between 10^{20} and 10^{22} , Fermat discovered in 1640, that $10^{37} - 1$ is not a prime:

$$2^{37} - 1 = 137438953471 = 223 \cdot 616318177.$$

Fermat's factorization depended on three basic observations:

- (1) If n is not a prime, then $2^n - 1$ is not a prime.
 - (2) If n is a prime, then $2^n - 2$ is a multiple of $2n$.
 - (3) If n is a prime, and p is a prime divisor of $2^n - 1$, then $p - 1$ is a multiple of n .
- In the same paper 26, Euler claimed that $2^n - 1$ are prime for $n = 19, 31, 41, 47$:⁵¹

Dat autem $2^{n-1}(2^n - 1)$ numerum perfectum, quoties $2^n - 1$ est primus; debet ergo etiam n esse numerus primus. Operae igitur pretium for existimavi eos notare casus, quibus $2^n - 1$ non est numerus primus, quamvis n sit talis. Inveni autem hoc semper fieri, si sit $n = 4m - 1$ atque $8m - 1$ fuerit numerus primus; tum enim $2^n - 1$ semper poterit dividi per $8m - 1$. Hinc excludendi sunt casus sequentes: 11, 23, 83, 131, 179, 191, 239 etc., qui numeri pro n substituti reddunt $2^n - 1$ numerum compositum. Neque tamen reliqui numeri primi omnes loco n positi satisfaciunt, sed plures insuper excipiuntur; sic observavi $2^{37} - 1$ dividi posse per 223, $2^{43} - 1$ per 431, $2^{29} - 1$ per 1103, $2^{73} - 1$ per 439; omnes tamen excludere non est in primos minores quam 50 et forte quam 100 efficere $2^{n-1}(2^n - 1)$ esse numerum perfectum sequentibus numeris pro n positus 1, 2, 3, 5, 7, 13, 17, 19, 31, 41, 47, unde 11 proveniunt numeri perfecti. Deduxi has observationes ex theoremate quodam non ineleganti, cuius quidem demonstrationem quoque non habeo, varum tamen de eius veritate sum certissimus. Theorema hoc est: $a^n - b^n$ semper potest dividi per $n + 1$, si $n + 1$ fuerit numerus primus atque a et b non possint per eum dividi, . . .⁵²

According to Dickson,⁵³ Euler, in a letter to Goldbach on October 28, 1752, stated that he knew

⁵⁰Dunham, Chapter 1.

⁵¹ $2^{41} - 1 = 219902325551 = 13367 \cdot 164511353$; $2^{47} - 1 = 140737488355327 = 2351 \cdot 4513 \cdot 13264529$.

⁵²Translation in Dickson, *History of the theory of numbers*, vol. 1 (1919) p.17: "However, I venture to assert that aside from the cases noted, every prime less than 50, and indeed than 100, makes $2^{n-1}(2^n - 1)$ a perfect number, . . . I derived these results from the elegant theorem, of whose truth I am certain, although I have no proof: $a^n - b^n$ is divisible by the prime $n + 1$, if neither a nor b is".

⁵³*History*, vol.I, p.18.

only of seven perfect numbers, and was uncertain whether $2^{31} - 1$ is prime or not. Later that same year,⁵⁴ Euler wrote to Goldbach again, confirming the primality of $2^{31} - 1$:

Der folgende [Ausdruck, der eine vollkommene Zahl liefern könnte,] wäre $2^{30}(2^{21} - 1)$, wenn nur $2^{31} - 1$ ein numerus primus wäre, welches aber weder behauptet noch untersucht werden kann. So viel ist gewiß, daß diese Zahl $2^{31} - 1$ keine andere divisores haben kann, als welche in dieser Formel $62n + 1$ enthalten sind, woraus ich so viel gefunden, daß kein divisor unter 2000 Statt findet.

The primality test was more efficient in a later letter⁵⁵ to Jean III Bernoulli⁵⁶ in 1771, Euler remarked that he had verified that $2^{31} - 1 = 2147483647$ is prime by examining primes (up to 46339) of the form $248n + 1$ and $248n + 63$.^{57 58}

Euler's proof of Euclid's expression of even perfect numbers

Paper 798: De numeris amicabilibus (1849).

§8. Hinc inventio numerorum perfectorum nulla laborat difficultate; cum enim numerus perfectus vocetur, qui aequalis summae suarum partium aliquotarum, si numerus perfectus ponatur = a , oportebit esse $a = A - a$, ideoque $A = 2a$.⁵⁹ Iam numerus perfectus a vel est par vel impar; priori casu ergo factorem habebit 2 eiusque quampiam dignitatem. Sit igitur $a = 2^n b$, erit $A = (2^{n+1} - 1)B$, ideoque $(2^{n+1} - 1)B = 2^{n+1}b$, unde fit $\frac{B}{b} = \frac{2^{n+1}}{2^{n+1}-1}$. Cum igitur fractio $\frac{2^{n+1}}{2^{n+1}-1}$ ad minores numeros reduci nequeat, necesse est, ut sit vel $b = 2^{n+1} - 1$ vel $b = (2^{n+1} - 1)c$. Prius autem fieri nequit, nisi sit $2^{n+1} - 1$ numerus primus, quia summa divisorum esse debet = 2^{n+1} , ideoque summa partium aliquotarum = 1; quoties vero est $2^{n+1} - 1$ numerus primus, toties posito $b = 2^{n+1} - 1$, erit $B = 2^{n+1}$; hincque numerus perfectus erit $a = 2^n(2^{n+1} - 1)$. Sin autem pro b sumeretur multipulum ipsius $2^{n+1} - 1$, puta $(2^{n+1} - 1)c$, eius pars aliquota foret $2^{n+1} - 1$ et c ; unde omnium divisorum summa B certe non minor esset quam $2^{n+1} + c + b$; talis enim foret, si tam c quam $2^{n+1} - 1$ essent numeri primi. Fracto ergo $\frac{B}{b}$ non minor esset futura quam $\frac{2^{n+1}+c+b}{b}$, hoc est quam $\frac{2^{n+1}(1+c)}{(2^{n+1}-1)c}$ ob $b = (2^{n+1} - 1)c$. At fractio $\frac{2^{n+1}(1+c)}{(2^{n+1}-1)c}$ necessario maior est quam $\frac{2^{n+1}}{2^{n+1}-1}$, unde pro numero b multipulum ipsius $2^{n+1} - 1$ accipi nequit. Quamobrem alii numeri perfecti pares reperiri non possunt, nisi qui contineantur in formula prius inventa $a = 2^n(2^{n+1} - 1)$ existente $2^{n+1} - 1$ numero primo; haecque est ipsa regula ab Euclide praescripta.

⁵⁴16 Dezember.

⁵⁵Paper 461: *Extract d'une lettre ... a M. Bernoulli concernant le memoire imprime parmi ceux de 1771, p.318.*

⁵⁶Jean III Bernoulli (1744 – 1807), son of Jean II Bernoulli (710 – 1790) and grandson of Jean I Bernoulli (1667 – 1748).

⁵⁷There are 145 primes of the form $248n + 1$ and 143 primes of the form $248n + 63$.

⁵⁸At the end of the same letter, Euler remarked that *Cette progression 41, 43, 47, 53, 61, 71, 83, 97, 113, 131, ... dont le terme général est $41 - x + x^2$ est d'autant plus remarquable que les quarante premiers terms sont tous des nombres premiers.*

⁵⁹Editor's footnote: Si $a, b, c \dots$ denotant numeros quoscunque integros, litterae maiusculae A, B, C, \dots expriment summas omnium numerorum a, b, c, \dots . Eulerus in posterioribus commentationibus hanc notationem per $\int a, \int b, \int c, \dots$ supplevit.

Then Euler stated that an odd perfect number must be of the form⁶⁰ $(4m + 1)^{4n+1}x^2$, with x odd and $4m + 1$ prime:

Utrum autem praeter hos dentur numeri perfecti impare necne, difficillima est quaestio; neque quisquam adhuc talem numerum invenit, neque nullum omnino dari demonstravit. Sin autem huiusmodi numeri perfecti derentur, ii necessario in hac formula:
 $(4m + 1)^{4n+1}x^2$ continerentur, ubi $4m + 1$ denotat numerum primum et x numerum imparem.

In his long paper 792, *Tractatus de numerorum doctrina*, (1849), Euler gave another proof of the euclidean expression for even perfect numbers.

§106. Numerus perfectus est, cuius summa divisorum ipsi duplo est aequalis. Ita si fuerit $\int N = 2N$, erit N numerus perfectus. Qui si sit par, erit huiusmodi $2^n A$ existente A numero impari sive primo sive composito. Cum ergo sit $N = 2^n A$, erit

$$\int N = (2^{n+1} - 1) \int A = 2^{n+1} A, \quad \text{unde fit } \frac{\int A}{A} = \frac{2^{n+1}}{2^{n+1} - 1}.$$

§107. Quia huius fractionis $\frac{2^{n+1}}{2^{n+1}-1}$ numerator unitate tantum superat denominatorem, excedere nequit summam divisorum denominatoris; erit ergo vel aequalis vel minor. Posteriori casu nulla datur solutio, prior vero existere nequit, nisi sit $2^{n+1} - 1$ numerus primus. Quare quoties $2^{n+1} - 1$ fuerit numerus primus, ei A aequalis capi debet, eritque numerus perfectus $2^n(2^{n+1} - 1)$.

§108. Omnes ergo numeri perfecti pares in hac formula $2^n(2^{n+1} - 1)$ continentur, siquidem $2^{n+1} - 1$ fuerit numerus primus, quod quidem evenit nequit, nisi $n + 1$ sit numerus primus; etiamsi non omnes primi pro $n + 1$ assumpti praebeant $2^{n+1} - 1$ primum. Utrum vero praeter hos numeros perfecti pares dentur quoque impares necne, nemo adhuc demonstravit.

Dickson⁶¹ commented that this proof is not complete. It can be easily salvaged by writing, at the end of §106,

$$\int A = \frac{2^{n+1}}{2^{n+1} - 1} \cdot A = A + \frac{A}{2^{n+1} - 1}$$

instead. The latter summand must be an integer, so that $2^{n+1} - 1$ is a divisor of A , and $\int A$ is the sum of A and its divisor $\frac{A}{2^{n+1}-1}$. This means that A has exactly two divisors. It must be prime, and is equal to $2^{n+1} - 1$.

⁶⁰The existence of an odd perfect number is still an open problem.

⁶¹*History*, vol.I, p.19.

Appendix: Records of Mersenne primes. The primality of M_k for $k = 2, 3, 5, 7, 13$ has been known since antiquity. The following Mersenne numbers $M_k = 2^k - 1$ are known to be primes.⁶²

k	Digits	Year	Discoverer
17	6	1588	P.A.Cataldi
19	6	1588	P.A.Cataldi
31	10	1752	L.Euler
61	19	1883	I.M.Pervushin
89	27	1911	R.E.Powers
107	33	1913	E.Fauquembergue
127	39	1876	E.Lucas
521	157	1952	R.M.Robinson
607	183	1952	R.M.Robinson
1279	386	1952	R.M.Robinson
2203	664	1952	R.M.Robinson
2281	687	1952	R.M.Robinson
3217	969	1957	H.Riesel
4253	1281	1961	A.Hurwitz
4423	1332	1961	A.Hurwitz
9689	2917	1963	D.B.Gillies
9941	2993	1963	D.B.Gillies
11213	3376	1963	D.B.Gillies
19937	6002	1971	B.Tuckerman
21701	6533	1978	C.Noll, L.Nickel
23209	6987	1979	C.Noll
44497	13395	1979	H.Nelson, D.Slowinski
86243	25962	1982	D.Slowinski
110503	33265	1988	W.N.Colquitt, L.Welsch
132049	39751	1983	D.Slowinski
216091	65050	1985	D.Slowinski
756839	227832	1992	D.Slowinski, P.Gage
859433	258716	1993	D.Slowinski
1257787	378632	1996	D.Slowinski and Gage
1398269	420921	1996	Armengaud, Woltman et al.
2976221	895932	1997	Spence, Woltman et al.
3021377	909526	1998	Clarkson, Woltman, Kurowski et. al.
6972593	2098960	1999	Hajratwala, Woltman, Kurowski et. al.

This last entry was announced early this month (July, 1999).

⁶²See, for example, the webpage of Chris K. Caldwell, <http://www.utm.edu/research/primes/mersenne.shtml>

XIIIC. Euler's proof of Fermat's little theorem

Paper 54: Theorematum quorundam ad numeros primos spectantium demonstratio (1736)

§3. Significante p numerum primum formula $a^{p-1} - a$ semper per p dividi poterit, nisi a per p dividi queat.

§4. Significante p numerum primum imparem quemcunque formula $2^{p-1} - 1$ semper per p dividi poterit.

Demonstratio. Loca 2 ponatur 1+1 eritque

$$(1+1)^{p-1} = 1 + \frac{p-1}{1} + \frac{(p-1)(p-2)}{1 \cdot 2} + \frac{(p-1)(p-2)(p-3)}{1 \cdot 2 \cdot 3} + \frac{(p-1)(p-2)(p-3)(p-4)}{1 \cdot 2 \cdot 3 \cdot 4} + \dots$$

cuius seriei terminorum numerus est $-p$ et proinde impar. Pareteera quilibet terminus, quamvis habeat fractionis speciem, debet numerum integrum; quisque enim numerator, uti satis constat, per suum denominatorem dividi potest. Demto igitur seriei termino primo 1 erit

$$\begin{aligned} (1+1)^{p-1} - 1 &= 2^{p-1} - 1 \\ &= \frac{p-1}{1} + \frac{(p-1)(p-2)}{1 \cdot 2} + \frac{(p-1)(p-2)(p-3)}{1 \cdot 2 \cdot 3} + \dots \end{aligned}$$

quorum numerus est $= p - 1$ et propterea par. Colligantur igitur bini quique termini in unam summam, quo terminorum numerus fiat duplo minor; erit

$$2^{p-1} - 1 = \frac{p(p-1)}{1 \cdot 2} + \frac{p(p-1)(p-2)(p-3)}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{p(p-1)(p-2)(p-3)(p-4)(p-5)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \dots$$

cuius seriei ultermis terminus ob p numerum imparem erit

$$\frac{p(p-1)(p-2) \cdots 2}{1 \cdot 2 \cdot 3 \cdots (p-1)} = p.$$

Apparet autem singulos terminos per p esse divisibiles; nam cum p sit numerus primus et maior quam ullus denominatorum factor, nusquam divisione tolli poterit. Quamobrem si fuerit p numerus primus impar, per illum semper $2^{p-1} - 1$ dividi poterit.

Theorema Denotante p numerum primum si $a^p - a$ per p dividi potest, tum per idem p quoque formula

$$(a+1)^p - a - 1$$

dividi poterit.

Demonstratio: Resolvatur $(1+a)^p$ consueto more in seriem; erit

$$(1+a)^p = 1 + \frac{p}{1}a + \frac{p(p-1)}{2!}a^2 + \frac{p(p-1)(p-2)}{3!}a^3 + \dots + \frac{p}{1}a^{p-1} + a^p,$$

cuius seriei singuli termini per p dividi possunt praeter primum et ultimum, si quidem p fuerit numerus primus. Quamobrem $(1 + a)^p - a^p - 1$ divisionem per p admittet; haec autem formula congruit cum hac $(1 + a)^p - a - 1 - a^p + a$. At $a^p - a$ per hypothesin per p dividi potest, ergo et $(1 + a)^p - a - 1$. Q.E.D. ⁶³

Paper 134 Theoremata circa divisores numerorum (1747/48)

§1. Theorema 1. Si p fuerit numerus primus, omnis numerus in hac forma $(a + b)^p - a^p - b^p$ contentus divisibilis erit per p .

§4. Theorema 2. Si utraque harum formularum $a^p - a$ et $b^p - b$ fuerit divisibilis per numerum primus p , tum quoque ista formula $(a + b)^p - a - b$ divisibilis erit per eundem numerum primum p .

§7. Theorema 3. Si p fuerit numerus primus, omnis numerus huius formae $c^p - c$ per p erit divisibilis.

§10. Si ergo p fuerit numerus primus, omnes numeri hac forma contenti $a^{p-1} - 1$ erunt divisibiles per p exceptis iis casibus, quibus ipse numerus a per p est divisibilis.

§11. Theorema 4. Si neuter numerorum a et b divisibilis fuerit per numerum primum p , tum omnis numerus huius formae $a^{p-1} - b^{p-1}$ erit divisibilis per p .

§15. Corollarium 4. Si m sit numerus par, puta $m = 2n$, atque $a^m - b^m$ seu $a^{2n} - b^{2n}$ divisibilis per $2m + 1 = 4n + 1$, tum ob eandem rationem vel $a^n - b^n$ vel $a^n + b^n$ divisibilis erit per numerum primum $4n + 1$.

§16. Summa duorum quadratorum $a^2 + b^2$ per nullum numerum primum huius formae $4n - 1$ unquam dividi potest, nisi utriusque radix seorsim a et b sit divisibilis per $4n - 1$.

§21. Theorema 6. Omnes divisores summae duorum biquadratorum inter se primorum sunt vel 2 vel numeri huius formae $8n + 1$.

§26. Theorema 7. Omnes divisores huiusmodi numerorum $a^8 + b^8$, si quidem a et b sunt numeri inter se primorum, sunt vel 2 vel in hac formae $16n + 1$.

§29. Theorema 8. Summa duarum huiusmodi potestatum $a^{2^m} + b^{2^m}$, quarum exponens est dignitas binarii, alios divisores non admittit, nisi qui contineantur in hac formae $2^{m+1}n + 1$.

⁶³Quod erat demonstrandum

XIVA. Euler's Tractatus de Numerorum Doctrina**Caput 2: De divisoribus numerorum**

§77. Hinc patet regula facilis multitudinem divisorum cuiuscuno numeri definiendi: Sit enim $p^\lambda q^\mu r^\nu s^\zeta$ forma numeri propositi; et quia numeri multitudo divisorum est $\lambda + 1$, erit numeri $p^\lambda q^\mu$ multitudo divisorum $(\lambda + 1)(\mu + 1)$, huius vero numeri $p^\lambda q^\mu r^\nu$ erit $(\lambda + 1)(\mu + 1)(\nu + 1)$, porroque huius $p^\lambda q^\mu r^\nu s^\zeta$ erit $(\lambda + 1)(\mu + 1)(\nu + 1)(\zeta + 1)$. Classis autem ad quam hic numerus est referendus, indicatur numero $\lambda + \mu + \nu + \zeta$, est summa exponentium.

Caput 3: De summa divisorum cuiusque numeri

§82. Proposito quocunque numero n summam omnium eius divisorum hoc modo $\int n$ designemus, ita ut haec scriptura $\int n$ denotet summam divisorum numeri n .

§84. Pro numeris primis p , quia alios non agnoscunt divsores praeter se ipsos et unitatem, erit $\int p = p + 1$. Tum vero pro potestatibus numerorum primorum erit

$$\begin{aligned}\int p^1 &= p + 1 = \frac{p^2 - 1}{p - 1}, \\ \int p^2 &= p^2 + p + 1 = \frac{p^3 - 1}{p - 1}, \\ \int p^3 &= p^3 + p^2 + p + 1 = \frac{p^4 - 1}{p - 1},\end{aligned}$$

et in genere

$$\int p^n = p^n + p^{n-1} + p^{n-2} + \dots + 1 = \frac{p^{n+1} - 1}{p - 1}.$$

§90. Proposito ergo numero N , cuius summam divisorum assignari oporteat, resolvatur is in suos factores primos, sitque $N = p^\lambda q^\mu r^\nu s^\zeta$, quo facto erit

$$\int N = \int p^\lambda \cdot \int q^\mu \cdot \int r^\nu \cdot \int s^\zeta.$$

This chapter concludes with a proof of the euclidean expression for even perfect numbers.⁶⁴

Caput 4. De numeris inter se primis et compositis

§111. Duo numeri, qui praeter unitatem nullum alium habent factores seu divisorem communem, vacantur numeri primi inter se; qui autem praeter unitatem alium habent divisorum communem, vocantur compositi inter se. Iam 8 et 15 sunt numeri inter se primi, et 9 et 15 numeri inter se compositi.

§117. Si ergo a sit numerus primus $= p$, quia omnes numeri ipso minores ad eum sunt primi, horum multitudo est $p - 1$.

§121. Generalius, si sit $q = pq$ existente utroque factore p et q primo, ab unitate ad a dantur p numeri per q divisibiles scilicet $q, 2q, 3q, \dots, pq$; deinde dantur q numeri per p divisibiles, scilicet

⁶⁴See XIII B.

$p, 2p, 3p, \dots, qp$, quorum ultimus pq iam est numeratus. Multitudo ergo omnium numerorum a non superantium, qui ad a sunt compositi, erit $= p + q - 1$, unde reliqui, quorum multitudo est

$$= pq - p - q + 1 = (p - 1)(q - 1),$$

ad a erunt primi.

§122. Hic autem pro p et q numeros primos diversos sumsimus. Namque esset $a = p^2$. alii numeri ad a non essent compositi, nisi qui sunt per p divisibles, quorum multitudo cum sit $-p$, reliquorum, qui ad a sunt praeter multitudo erit $= p^2 - p = p(p - 1)$.

§124. Hinc in genere patet, si a fuerit potestas quaecunque p^n numerus primi p , multitudinem numerorum ad a primorum, qui quidem ipso a non maiores, fore $= p^{n-1}(p - 1)$.

§132. Cum ergo multitudo numerorum ad p^n primorum ipsoque minorum sit $= p^{n-1}(p - 1)$, ex praecedente propositione summo rigore concludin. Si numerus propositus sit $= p^\lambda q^\mu r^\nu s^\zeta \dots$, for multitudinem omnium numerorum ad eum primorum ipsoque minorum

$$= p^{\lambda-1}(p - 1)q^{\mu-1}(q - 1)r^{\nu-1}(r - 1)s^{\zeta-1}(s - 1) \dots$$

XIVB. Euler's extraordinary relation involving sums of divisors

Paper 175: *Decouverte d'une loi tout extraordinaire des nombres par rapport a la somme de leurs diviseurs*⁶⁵ (1751)

Euler begins by explaining the sum of divisors function⁶⁶ $\int n$, the multiplicative property, and tabulating $\int n$ for $n = 1, 2, \dots, 100$.

n	0	1	2	3	4	5	6	7	8	9
0	--	1	3	4	7	6	12	8	15	13
10	18	12	28	14	24	24	31	18	39	20
20	42	32	36	24	60	31	42	40	56	30
30	72	32	63	48	54	48	91	38	60	56
40	90	42	96	44	84	78	72	48	124	57
50	93	72	98	54	120	72	120	80	90	60
60	168	62	96	104	127	84	144	68	126	96
70	144	72	195	74	114	124	140	96	168	80
80	186	121	126	84	224	108	132	120	180	90
90	234	112	168	128	144	120	252	98	171	156

§5. *Néanmoins, j'ai remarqué que cette progression suit une loi bien régl'ee est même comprise dans l'ordre des progressions que les Geometres nomment recurrentes, de sorte qu'on peut toujours former*

⁶⁵English translation by G. Polya, *Mathematics and Plausible Reasoning*, volume I. Induction and Analogy in Mathematics, Princeton University Press, 1954.

⁶⁶Modern notation: $\sigma(n)$ = sum of all divisors of n , including 1 and n .

chacun de ces termes par quelques - uns des précédens, suivant une règle constante. Car si $\int n$ marque un terme quelconque de cette irrégulière progression, et $\int(n - 1)$, $\int(n - 2)$, $\int(n - 3)$, $\int(n - 4)$, $\int(n - 5)$, etc. des termes précédens, je dis que la valeur de $\int n$ est toujours composée de quelques - uns des précédens suivant cette formule:

$$\begin{aligned} \int n &= \int(n - 1) + \int(n - 2) - \int(n - 5) - \int(n - 7) + \int(n - 12) + \int(n - 15) \\ &\quad - \int(n - 22) - \int(n - 26) + \int(n - 35) + \int(n - 40) - \int(n - 51) - \int(n - 57) \\ &\quad + \int(n - 70) + \int(n - 77) - \int(n - 92) - \int(n - 100) + \dots \end{aligned}$$

Of this formula we must make the following remarks:

I. In the sequence of the signs + and -, each arises twice in succession.

II. The law of the numbers 1, 2, 5, 7, 12, 15, ... which we have to subtract from the proposed number n , will become clear if we take their differences:

Num.	1,	2,	5,	7,	12,	15,	22,	26,	35,	40,	51,	57,	70,	77,	92,	100,	...
Diff.	1,	3,	2,	5,	3,	7,	4,	9,	5,	11,	6,	13,	7,	15,	8,	...	

In fact, we have here, alternately, all the integers 1, 2, 3, 4, 5, 6, ... and the odd numbers 3, 5, 7, 9, 11 ..., and hence we can continue the sequence of these numbers as far as we please.

III. Although this sequence goes to infinity, we must take, in each case, only those terms for which the numbers under the sign \int are still positive and omit the \int for negative values.

IV. If the sign $\int 0$ turns up in the formula, we must, as its value in itself is indeterminate, substitute for $\int 0$ the number n proposed.

Euler then gave illustrative examples in §§6, 7.

diff	=	\int	300	+	\int	299	-	\int	296	-	294	+	\int	289	+	\int	286	-	\int	279	-	\int	275	+	

§9. In considering the partitions of numbers, I examined, a long time ago,⁶⁷ the expression

$$(1 - x)(1 - x^2)(1 - x^3)(1 - x^4)(1 - x^5)(1 - x^6)(1 - x^7)(1 - x^8) \dots$$

⁶⁷See also Paper 158 *Observationes analyticae variae de combinationibus* (1741/3), 191 *De partitione numerorum* (1750/1), 243 *Observatio de summis divisorum* (1754/5), 244 *Demonstratio theorematis circa ordinem in summis divisorum observatum* (1754/5).

in which the product is assumed to be infinite. In order to see what kind of series will result, I multiplied actually a great number of factors and found

$$1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + \dots$$

The exponents of x are the same which enter into the above formula; also the signs + and - arise twice in succession. It suffices to undertake this multiplication and to continue it as far as it is deemed proper to become convinced of the truth of this series. Yet I have not other evidence for this, except a long induction which I have carried out so far that I cannot in any way doubt the law governing the formation of these terms and their exponents. I have long searched in vain for a rigorous demonstration of the equation between the series and the above infinite product $(1 - x)(1 - x^2)(1 - x^3) \dots$, and I have proposed the same question to some of my friends with whose ability in these matters I am familiar, but all have agreed with me on the truth of this transformation of the product into a series, without being able to unearth any clue of a demonstration. Thus, it will be a known truth, but not yet demonstrated, that if we put

$$s = (1 - x)(1 - x^2)(1 - x^3)(1 - x^4)(1 - x^5)(1 - x^6) \dots$$

the same quantity s can also be expressed as follows:

$$s = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + \dots$$

For each of us can convince himself of this truth by performing the multiplication as far as he may wish; and it seems impossible that the law which has been discovered to hold for 20 terms, for example, would not be observed in the terms that follow.

§10. As we have thus discovered that those two infinite expressions are equal even though it has not been possible to demonstrate their equality,⁶⁸ all the conclusions which may be deduced from it will be of the same nature, that is, true but not demonstrated. Or, if one of these conclusions could be demonstrated, one could reciprocally obtain a clue to the demonstration of that equation and it was with this purpose in mind that I maneuvered those two expressions in many ways, and so I was led among other discoveries to that which I explained above; its truth, therefore, must be as certain as that of the equation between the two infinite expressions. I proceeded as follows. Being given the two expressions

$$\begin{aligned} s &= (1 - x)(1 - x^2)(1 - x^3)(1 - x^4)(1 - x^5)(1 - x^6)(1 - x^7) \dots \\ s &= 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + \dots \end{aligned}$$

are equal, I got rid of the factors in the first by taking logarithms

$$\log s = \log(1 - x) + \log(1 - x^2) + \log(1 - x^3) + \log(1 - x^4) + \dots$$

In order to get rid of the logarithms, I differentiate and obtain the equation

$$\frac{1}{s} \cdot \frac{ds}{dx} = -\frac{1}{1-x} - \frac{2x}{1-x^2} - \frac{3x^2}{1-x^3} - \frac{4x^3}{1-x^4} - \frac{5x^4}{1-x^5} - \dots$$

⁶⁸See XIC below; also G.E.Andrew, *Euler's pentagonal number theorem*, Math. Magazine, 56 (1983) pp. 279 – 284.

or

$$-\frac{x}{s} \cdot \frac{ds}{dx} = \frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \frac{4x^4}{1-x^4} + \frac{5x^5}{1-x^5} + \dots$$

From the second expression for s , as infinite series, we obtain another value for the same quantity

$$-\frac{x}{s} \cdot \frac{ds}{dx} = \frac{x + 2x^2 - 5x^5 - 7x^7 + 12x^{12} + 15x^{15} - 22x^{22} - 26x^{26} + \dots}{1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \dots}$$

§11. Let us put $-\frac{x}{s} \cdot \frac{ds}{dx} = t$. We have above two expressions for the quantity t . In the first expression, I expand each term into a geometric series and obtain

$$\begin{aligned} t = & x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + \dots \\ & + 2x^2 + 2x^4 + 2x^6 + 2x^8 + \dots \\ & + 3x^3 + 3x^6 + \dots \\ & + 4x^4 + 4x^8 + \dots \\ & + 5x^5 + \dots \\ & + 6x^6 + \dots \\ & + 7x^7 + \dots \\ & + 8x^8 + \dots \\ & \vdots \end{aligned}$$

Here, we see easily that each power of x arises as many times as its exponent has divisors, and that each divisor arises as a coefficient of the same power of x . Therefore, if we collect the terms with like powers, the coefficients of each power of x will be the sum of the divisors of its exponent. And, therefore, using the above notation $\int n$ for the sum of the divisors of n , I obtain

$$t = \int 1 \cdot x + \int 2 \cdot x^2 + \int 3 \cdot x^3 + \int 4 \cdot x^4 + \int 5 \cdot x^5 + \int 6 \cdot x^6 + \int 7 \cdot x^7 + \dots$$

The law of the series is manifest. And although it might appear that some induction was involved in the determination of the coefficients, we can easily satisfy ourselves that this law is a necessary consequence.

§12. By virtue of the definition of t , the last formula of §10 can be written as follows:

$$\begin{aligned} t(1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \dots) \\ -x - 2x^2 + 5x^5 + 7x^7 - 12x^{12} - 15x^{15} + 22x^{22} + 26x^{26} - \dots = 0. \end{aligned}$$

Substituting for t the value obtained at the end of §11, we find

$$\begin{aligned} 0 = & \int 1 \cdot x + \int 2 \cdot x^2 + \int 3 \cdot x^3 + \int 4 \cdot x^4 + \int 5 \cdot x^5 + \int 6 \cdot x^6 + \dots \\ & -x - \int 1 \cdot x^2 - \int 2 \cdot x^3 - \int 3 \cdot x^4 - \int 4 \cdot x^5 - \int 5 \cdot x^6 - \dots \\ & - 2x^2 - \int 1 \cdot x^3 - \int 2 \cdot x^4 - \int 3 \cdot x^5 - \int 4 \cdot x^6 - \dots \\ & + 5x^5 + \int 1 \cdot x^6 + \dots \end{aligned}$$

Collecting the terms, we find the coefficient for any given power of x . This coefficient consists of several terms. First comes the sum of the divisors of the exponent of x , and then sums of divisors of some preceding numbers, obtained from that exponent by subtracting successively 1, 2, 5, 7, 12, 15, 22, 26, Finally, if it belongs to this sequence, the exponent itself arises. We need not explain again the signs assigned to the terms just listed. Therefore, generally, the coefficient of x^n is

$$\int n - \int (n - 1) - \int (n - 2) + \int (n - 5) + \int (n - 7) - \int (n - 12) - \int (n - 15) + \dots$$

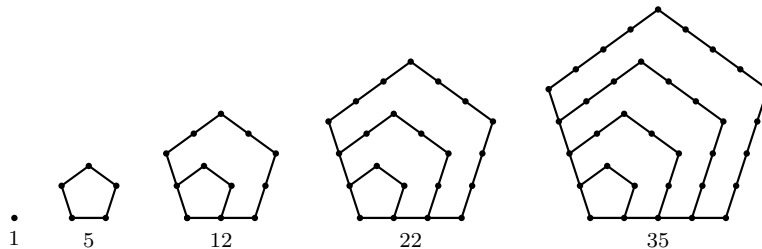
This is continued as long as the numbers under the sign \int are not negative. Yet, if the term $\int 0$ arises, we must substitute n for it.

XIVC. Euler’s pentagonal number theorem

The infinite series expansion that Euler made use of to establish his extraordinary relation involving the sum of divisors function can be written as

$$\prod_{n=1}^{\infty} (1 - x^n) = \sum_{n=-\infty}^{+\infty} (-1)^n x^{\frac{1}{2}n(3n+1)}.$$

This is usually called Euler’s pentagonal number theorem, since for negative values of n , upon writing $n = -m$, the exponent becomes the pentagonal number $\frac{1}{2}m(3m - 1)$.



Euler proved this theorem in Paper 244: *Demonstratio theorematis circa ordinem in summis divisorum observatum*, (1754). After a brief explanation of how to form the infinite series from the sequence of differences⁶⁹

$$1, 1, 3, 2, 5, 3, 7, 4, 9, 5, 11, 6, 13, 7, 15, 8, \dots$$

he gave the proof in three propositions.

Propositio 1 Si sit

$$s = (1 + \alpha)(1 + \beta)(1 + \gamma)(1 + \delta)(1 + \epsilon)(1 + \zeta)(1 + \eta) \dots$$

⁶⁹See §5, II of Paper 175, XIVB above.

productum hoc ex infinitis factoribus constans in seriem sequentum convertitur

$$s = (1 + \alpha) + \beta(1 + \alpha) + \gamma(1 + \alpha)(1 + \beta) + \delta(1 + \alpha)(1 + \beta)(1 + \gamma) \\ + \epsilon(1 + \alpha)(1 + \beta)(1 + \gamma)(1 + \delta) + \zeta(1 + \alpha)(1 + \beta)(1 + \gamma)(1 + \delta)(1 + \epsilon) + \dots$$

Propositio 2 Si fuerit

$$s = (1 - x)(1 - x^2)(1 - x^3)(1 - x^4)(1 - x^5)(1 - x^6) \dots$$

productum hoc ex infinitis factoribus constans reducetur ad hanc seriem

$$s = 1 - x - x^2(1 - x) - x^3(1 - x)(1 - x^2) - x^4(1 - x)(1 - x)(1 - x^3) - \dots$$

Propositio 3 Si fuerit

$$s = (1 - x)(1 - x^2)(1 - x^3)(1 - x^4)(1 - x^5)(1 - x^6)(1 - x^7) \dots$$

erit hoc productum infinitum per multiplicationem evolvendo terminoque secundum potestates ipsius x disponendo

$$s = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + x^{51} + x^{57} - \dots$$

cuius seriei ratio est ea ipsa, quae supra est exposita.

Demonstratio. Cum sit

$$s = (1 - x)(1 - x^2)(1 - x^3)(1 - x^4)(1 - x^5)(1 - x^6)(1 - x^7) \dots$$

erit

$$s = 1 - x - x^2(1 - x) - x^3(1 - x)(1 - x^2) - x^4(1 - x)(1 - x^2)(1 - x^3) - \dots$$

Ponatur

$$s = 1 - x - Ax^2;$$

erit

$$A = 1 - x + x(1 - x)(1 - x^2) + x^2(1 - x)(1 - x^2)(1 - x^3) + \dots$$

Evolvantur singuli termini tantum secundum factorem $1 - x$ ac sequenti modo disponantur

$$A = \begin{cases} -x & -x^2(1 - x^2) & -x^3(1 - x^2)(1 - x^3) & - \dots \\ 1 + x(1 - x^2) & +x^2(1 - x^2)(1 - x^3) & +x^3(1 - x^2)(1 - x^3)(1 - x^4) & + \dots \end{cases}$$

eritque terminis subscriptis colligendis

$$A = 1 - x^3 - x^5(1 - x^2) - x^7(1 - x^2)(1 - x^3) - x^9(1 - x^2)(1 - x^3)(1 - x^4) - \dots$$

Ponatur

$$A = 1 - x^3 - Bx^5;$$

erit

$$B = 1 - x^2 + x^2(1 - x^2)(1 - x^3) + x^4(1 - x^2)(1 - x^3)(1 - x^4) + \dots$$

in quibus terminis singulis $1 - x^2$ tantum evolvatur, ac fiet

$$B = \begin{cases} -x^2 & -x^4(1 - x^3) & -x^6(1 - x^3)(1 - x^4) & -\dots \\ 1 + x^2(1 - x^3) & +x^4(1 - x^3)(1 - x^4) & +x^6(1 - x^3)(1 - x^4)(1 - x^5) & +\dots \end{cases}$$

denuoque terminis subscriptis colligendis habetur

$$B = 1 - x^5 - x^8(1 - x^3) - x^{11}(1 - x^3)(1 - x^4) - x^{14}(1 - x^3)(1 - x^4)(1 - x^5) - \dots$$

Ponatur

$$B = 1 - x^5 - Cx^8;$$

erit

$$C = 1 - x^3 + x^3(1 - x^3)(1 - x^4) + x^6(1 - x^3)(1 - x^4)(1 - x^5) + \dots$$

ubi in singulis terminis factor $1 - x^3$ evolvatur, ut fiat scribendo ut supra

$$C = \begin{cases} -x^3 & -x^6(1 - x^4) & -x^9(1 - x^4)(1 - x^5) & -\dots \\ 1 + x^3(1 - x^4) & +x^6(1 - x^4)(1 - x^5) & +x^9(1 - x^4)(1 - x^5)(1 - x^6) & +\dots \end{cases}$$

unde colligetur

$$C = 1 - x^7 - x^{11}(1 - x^4) - x^{15}(1 - x^4)(1 - x^5) - x^{19}(1 - x^4)(1 - x^5)(1 - x^6) - \dots$$

Ponatur

$$C = 1 - x^7 - Dx^{11};$$

erit

$$D = 1 - x^4 + x^4(1 - x^4)(1 - x^5) + x^8(1 - x^4)(1 - x^5)(1 - x^6) + \dots$$

quae abit in hanc formam

$$D = \begin{cases} -x^4 & -x^8(1 - x^5) & -x^{12}(1 - x^5)(1 - x^6) & -\dots \\ 1 + x^4(1 - x^5) & +x^8(1 - x^5)(1 - x^6) & +x^{12}(1 - x^5)(1 - x^6)(1 - x^7) & +\dots \end{cases}$$

sicque erit

$$D = 1 - x^9 - x^{14}(1 - x^5) - x^{19}(1 - x^5)(1 - x^6) - x^{24}(1 - x^5)(1 - x^6)(1 - x^7) - \dots$$

Quodsi porro ponatur

$$D = 1 - x^9 - Ex^{14},$$

reperietur simili modo

$$E = 1 - x^{11} - Fx^{17}$$

hincque ultra

$$\begin{aligned} F &= 1 - x^{13} - Gx^{20}, \\ G &= 1 - x^{15} - Hx^{23}, \\ H &= 1 - x^{17} - Ix^{26}, \\ &\vdots \end{aligned}$$

Restituamus iam successive hos valores eritque

$$\begin{aligned} s &= 1 - x - Ax^2, \\ Ax^2 &= x^2(1 - x^3) - Bx^7, \\ Bx^7 &= x^7(1 - x^5) - Cx^{15}, \\ Cx^{15} &= x^{15}(1 - x^7) - Dx^{26}, \\ Dx^{26} &= x^{26}(1 - x^9) - Ex^{40}, \\ &\vdots \end{aligned}$$

Quamobrem habebimus

$$s = 1 - x - x^2(1 - x^3) + x^7(1 - x^5) - x^{15}(1 - x^7) + x^{26}(1 - x^9) - x^{40}(1 - x^{11}) + \dots$$

sive id ipsum, quod demonstrari oportet,

$$s = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + x^{51} + \dots$$

unde simul lex exponentium supra indicata per differentias luculenter perspicitur.

XVA. The Pell equation $x^2 - Ny^2 = 1$ Andre Weil on the Pell equation ⁷⁰:

Fermat had stated that a triangular number $\frac{1}{2}n(n+1)$ cannot be a fourth power. Goldbach thought he had proved, in the *Acta Eruditorum* of 1724, that such a number cannot even be a square, and he had communicated his “proof” to Nicolas to Daniel Bernoulli in 1725 and to Euler in 1730. Euler pointed out the error at once; if one puts $x = 2n + 1$, the question amounts to $x^2 - 8y^2 = 1$ and is thus a special case of “Pell’s equation”. “Such problems,” he writes, “have been agitated between Wallis and Fermat . . . and the Englishman Pell devised for them a peculiar method described in Wallis’s works.” Pell’s name occurs frequently in Wallis’s *Algebra*, but never in connection with the equation $x^2 - Ny^2 = 1$ to which his name, because of Euler’s mistaken attribution, has remained attached.

Paper 29: De solutione problematum Diophanteorum per numeros integros (1732/33);

Paper 279: De resolutione formularum quadraticarum indeterminatarum per numeros integros (1762/63);

Paper 323 De usu novi algorithmi in problemate pelliano solvendo (1765)

Continued fraction expansion of square roots of integers

§9.

$$\sqrt{13} = 3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{6 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{6 + \dots}}}}}}}}$$

§10. Quo indoles harum operationum melius perspicatur, aliud exemplum prolixiorem calculum postulans adiungam. Proposita scilicet sit $\sqrt{61}$; cuius valor proxime minor cum sit 7, pono $\sqrt{61} = 7 + \frac{1}{a}$ et operationes sequenti modo erut instituendae:

I.	$a =$	$= \frac{1}{\sqrt{61}-7}$	$= \frac{\sqrt{61}+7}{12}$	$= 1 + \frac{1}{b}$	
	II.	$b =$	$= \frac{12}{\sqrt{61}-5}$	$= \frac{12(\sqrt{61}+5)}{36}$	$= \frac{\sqrt{61}+5}{3} = 4 + \frac{1}{c}$
	III.	$c =$	$= \frac{3}{\sqrt{61}-7}$	$= \frac{3(\sqrt{61}+7)}{12}$	$= \frac{\sqrt{61}+7}{4} = 3 + \frac{1}{d}$
	IV.	$d =$	$= \frac{4}{\sqrt{61}-5}$	$= \frac{4(\sqrt{61}+5)}{36}$	$= \frac{\sqrt{61}+5}{9} = 1 + \frac{1}{e}$
	V.	$e =$	$= \frac{9}{\sqrt{61}-4}$	$= \frac{9(\sqrt{61}+4)}{45}$	$= \frac{\sqrt{61}+4}{5} = 2 + \frac{1}{f}$
	VI.	$f =$	$= \frac{5}{\sqrt{61}-6}$	$= \frac{5(\sqrt{61}+6)}{25}$	$= \frac{\sqrt{61}+6}{5} = 2 + \frac{1}{g}$
	VII.	$g =$	$= \frac{5}{\sqrt{61}-4}$	$= \frac{5(\sqrt{61}+4)}{45}$	$= \frac{\sqrt{61}+4}{9} = 1 + \frac{1}{h}$
	VIII.	$h =$	$= \frac{9}{\sqrt{61}-5}$	$= \frac{9(\sqrt{61}+5)}{36}$	$= \frac{\sqrt{61}+5}{4} = 3 + \frac{1}{i}$
	IX.	$i =$	$= \frac{4}{\sqrt{61}-7}$	$= \frac{4(\sqrt{61}+7)}{12}$	$= \frac{\sqrt{61}+7}{3} = 4 + \frac{1}{k}$
	X.	$k =$	$= \frac{3}{\sqrt{61}-5}$	$= \frac{3(\sqrt{61}+5)}{36}$	$= \frac{\sqrt{61}+5}{12} = 1 + \frac{1}{l}$
	XI.	$l =$	$= \frac{12}{\sqrt{61}-7}$	$= \frac{12(\sqrt{61}+7)}{12}$	$= \sqrt{61} + 7 = 14 + \frac{1}{m}$
	XII.	$m =$	$= \frac{1}{\sqrt{61}-7}$		

⁷⁰Weil, *Number Theory*, pp.173–174.

ergo $m = a$ hincque porro $n = b$, $o = c$ etc. Ex quo indices pro fractione continua erunt

$$7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4, 3, 1, 2, \dots$$

neque opus est ipsam fractionem continuam hic exhibere.

§15 gives the continued fraction expansions of \sqrt{rn} for $n = 2, 3, \dots, 120$. For examples,

$$\begin{aligned}\sqrt{6} &= [2, 2, 4, 2, 4, 2, 4, \dots], \\ \sqrt{19} &= [4, 2, 1, 3, 1, 2, 8, 2, 1, 3, 1, 2, 8, \dots],\end{aligned}$$

§17. Euler gives the continued fraction expansions of several sequences of square roots:

$$\begin{aligned}\sqrt{n^2 + 1} &= [n, \overline{2n}], \\ \sqrt{n^2 + 2} &= [n, \overline{n, 2n}], \\ \sqrt{n^2 + n} &= [n, \overline{2, 2n}], \\ \sqrt{n^2 + 2n - 1} &= [n, \overline{1, n - 1, 1, 2n}], \\ \sqrt{4n^2 + 4} &= [2n, \overline{n, 4n}], \\ \sqrt{9n^2 + 3} &= [3n, \overline{2n, 6n}], \\ \sqrt{9n^2 + 6} &= [3n, \overline{n, 6n}].\end{aligned}$$

Euler's solution of Pell's equation Expositio calculi pro quolibet numero z ut fiat $p^2 = zq^2 + 1$

§38(Examples) I. Si $z = 6$, sunt indices 2, 2, 4; hinc operatio:

$$\begin{aligned}& \begin{array}{c} 2, \quad 2 \\ \frac{1}{0}, \quad \frac{2}{1}, \quad \frac{5}{2}, \end{array} \\ x &= 1 \cdot 5 + 0 \cdot 2, & p &= 5, \\ & & \text{ergo} & \\ y &= 1 \cdot 2 + 0 \cdot 1, & q &= 2.\end{aligned}$$

III. Si $z = 19$, sunt indices 4, 2, 1, 3, 1, 2, 8:

$$\begin{aligned}& \begin{array}{c} 4, \quad 2, \quad 1, \quad 3 \\ \frac{1}{0}, \quad \frac{4}{1}, \quad \frac{9}{2}, \quad \frac{13}{3}, \quad \frac{48}{11}, \end{array} \\ x &= 3 \cdot 48 + 2 \cdot 13, & p &= 170, \\ & & \text{ergo} & \\ y &= 3 \cdot 11 + 2 \cdot 3, & q &= 39.\end{aligned}$$

§39. Alterius vero generis, quo bini dantur indices medii in qualibet periodo, hace adiungo exempla. II. Si $z = 29$, sunt indices 5, 2, 1, 1, 2, 10:

$$5, 2, 1, \\ \frac{1}{0}, \frac{5}{1}, \frac{11}{2}, \frac{16}{3},$$

hinc

$$x = 3 \cdot 16 + 2 \cdot 11 = 70, \\ y = 3 \cdot 3 + 2 \cdot 2 = 13.$$

Ergo

$$p = 2x^2 + 1 = 9801, \\ q = 2xy = 1820.$$

IV. Si $z = 61$, indices sunt 7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14:

$$7, 1, 4, 3, 1, 2 \\ \frac{1}{0}, \frac{7}{1}, \frac{8}{1}, \frac{39}{5}, \frac{125}{16}, \frac{164}{21}, \frac{453}{58},$$

hinc fit

$$x = 58 \cdot 453 + 21 \cdot 164 = 29718, \\ y = 58 \cdot 58 + 21 \cdot 21 = 3805.$$

Ergo

$$p = 2x^2 + 1 = 1766319049, \\ q = 2xy = 226153980.$$

At the end of the paper, Euler lists the smallest positive solution of $p^2 = \ell q^2 + 1$ for ℓ between 1 and 100, and also those for $\ell = 103, 109, 113, 157$ and 367. These solutions are very large.

ℓ	p	q
103	227528	22419
109	158070671986249	15140424455100
113	1204353	113296
157	46698728731849	3726964292220
367	19019995568	992835687

XVB. Euler's proof of Fermat's Last Theorem for $n = 4$.⁷¹

Elements of Algebra, Part II, Chapter XIII: Of some expressions of the form $ax^4 + by^4$, which are not reducible to squares (§§202 – 205).

Paper 98: Theorematum quorundam arithmeticonum demonstrationes (1738)

Theorema 1. Summa duorum biquadratorum ut $a^4 + b^4$ non potest esse quadratum, nisi alterum biquadratum evanescat.

Demonstratio In theoremate hoc demonstrando ita versabor, ut ostendam, si uno casu fuerit $a^4 + b^4$ quadratum, quantumvis etiam magni fuerint numeri a et b , tum continuo minores numeros loca a et b assignari posse atque tandem ad minimos numeros integros perveniri oportere. Cum autem in minimis numeris tales non dentur, quorum biquadratorum summa quadratum constitueret, concludendum erit nec inter maximos numeros tales extare.

Ponamus ergo $a^4 + b^4$ esse quadratum atque a et b inter se esse numeros primos; nisi enim primi forent, per divisionem ad primos reduci possent. Sit a numerus impar, b vero par, quia necessario alter par, alter impar esse debet. Erit ergo

$$a^2 = p^2 - q^2 \quad \text{et} \quad b^2 = 2pq$$

numerique p et q inter se erunt primi eorumque alter par, alter impar. Cum autem sit $a^2 = p^2 - q^2$, necesse est, ut p sit numerus impar, quia alias $p^2 - q^2$ quadratum esse non posset. Erit ergo p numerus impar et q numerus par. Quia porro $2pq$ quadratum esse debet, necesse est, ut tam p quam $2q$ sit quadratum, quia p et $2q$ sunt numeri inter se primi. Ut vero $p^2 - q^2$ sit quadratum, necesse est, ut sit

$$p = m^2 + n^2 \quad \text{et} \quad q = 2mn$$

existentibus iterum m et n numeris inter se primis eorumque altero pari, altero impari. Sed quoniam $2q$ quadratum est, erit $4mn$ seu mn quadratum; unde tam m quam n quadrata erunt. Posito ergo

$$m = x^2 \quad \text{et} \quad n = y^2$$

erit

$$p = m^2 + n^2 = x^4 + y^4,$$

quod quadratum pariter esse deberet. Hinc ergo sequitur, si $a^4 + b^4$ foret quadratum, tum quoque $x^4 + y^4$ fore quadratum; manifestum autem est numeros x et y longe minores fore quam a et b . Pari igitur via ex biquadratis $x^4 + y^4$ denuo minora orientur, quorum summa esset quadratum, atque pergendo ad minima tandem biquadrata in integris pervenietur. Cum ergo non dentur minima biquadrata, quorum summa efficiet quadratum, palam est nec in maximis numeris talia dari. Si autem in uno biquadratorum pari alterum sit $= 0$, in omnibus reliquis paribus alterum evanescet, ita ut hinc nulli novi casus orientur. Q.E.D.

⁷¹See also Laubenbacher and Pengelley, *Mathematical Expeditions, Chronicles by the Explorers*, Springer, 1999, pp.179 – 185.

XVC. Sums of two squares

Paper 228: De numeris qui sunt aggregata duorum quadratorum (1752/3)

§5. Theorema Si p et q sint duo numeri, quorum uterque est summa duorum quadratorum, erit etiam eorum productum pq summa duorum quadratorum.

Proof. If $p = a^2 + b^2$ and $q = c^2 + d^2$, then $pq = (a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2$.

§8. Propositio 1. Si productum pq sit summa duorum quadratorum et alter factor p sit numerus primus pariterque duorum quadratorum summa, erit quoque alter factor q summa duorum quadratorum.

[If the product pq is a sum of two squares, and the factor p is a prime number which is a sum of two squares, then the quotient q is a sum of two squares.]

§14. Propositio 2. Si productum pq sit summa duorum quadratorum, eius factor autem q non sit summa duorum quadratorum, tum alter factor p , si sit numerus primus, non erit summa duorum quadratorum, sin autem non sit primus, saltem factorem certe habebit primum, qui non sit summa duorum quadratorum.

[If the product pq is a sum of two squares, and the factor q cannot be a sum of two squares, then the other factor p contains a prime factor which is not a sum of two squares.]

§19. Propositio 3. Si summa duorum quadratorum inter se primorum $a^2 + b^2$ divisibilis sit per numerum p , semper exhiberi poterit summa duorum aliorum quadratorum $c^2 + d^2$ divisibilis per eundem numerum p , ita ut ista summa $c^2 + d^2$ non sit maior quam $\frac{1}{2}p^2$.

[If a sum of two squares $a^2 + b^2$ (in which a and b are relatively prime) is divisible by a prime p , there are c and d such that $c^2 + d^2$ is divisible by p , and is not more than $\frac{1}{2}p^2$.]

§22. Propositio 4. Summa duorum quadratorum inter se primorum dividi nequit per ullum numerum, qui ipse non sit summa duorum quadratorum.

[A sum of two squares of relatively prime integers is not divisible by a number which is not a sum of two squares.]

§28[a] Propositio 5. Omnis numerus primus, qui unitate excedit multiplum quaternarii, est summa duorum quadratorum.

[Every prime number of the form $4n + 1$ is a sum of two squares.] The demonstration that Euler gives for this proposition is tentamen (= temptamen), “attempting”. An exact proof of this proposition is given in Paper 241, Demonstratio theorematis Fermatiani omnem numerum primum formae $4n + 1$ esse summam duorum quadratorum, (1754/55). See below.

§35. Propositio 6. Si numerus formae $4n + 1$ unico modo in duo quadrata inter se prima resolvi queat, tum certe est numerus primus.

[If a number $4n + 1$ is a sum of two relatively prime squares in a unique way, then it must be a prime number.]

§40. Propositio 7. Qui numerus duobus pluribusve diversis modis in duo quadrata resolvi potest, ille non est primus, sed ex duobus ad minimum factoribus compositus.

[If a number is a sum of two squares in more than one way, then it is not a prime, and is the product of at least two factors.]

Euler's proof that every prime $4n + 1$ is a sum of two squares

Paper 241: Demonstratio theorematis Fermatiani omnem numerum primum formae $4n + 1$ esse summam duorum quadratorum, (1754/55).

§3. Quodsi iam $4n + 1$ sit numerus primus, per eum omnes numeri in hac forma $a^{4n} - b^{4n}$ contenti erunt divisibiles, siquidem neuter numerorum a et b seorsim per $4n + 1$ fuerit divisibilis. Quare si a et b sint numeri minores quam $4n + 1$ (cyphra tamen excepta), numeris inde formatus $a^{4n} - b^{4n}$ sube ykka kunuatuibe oer byneryn orunyn orioisutyn $4n + 1$ erit divisibilis. Cum autem $a^{4n} - b^{4n}$ sit productum horum factorum $a^{2n} + b^{2n}$ et $a^{2n} - b^{2n}$, necesse est, ut alteruter horum factorum sit per $4n + 1$ divisibilis; fieri enim nequit, ut vel neuter vel uterque simul divisorem habeat $4n + 1$. Quodsi iam demonstrari posset dari casus, quibus forma $a^{2n} + b^{2n}$ sit divisibilis per $4n + 1$, quoniam $a^{2n} + b^{2n}$ ob exponentem $2n$ parem est summa duorum quadratorum, quorum neutrum seorsim per $4n + 1$ divisibile existit, inde sequeretur hunc numerum $4n + 1$ esse summam duorum quadratorum.

§4. Verum summa $a^{2n} + b^{2n}$ toties erit per $4n + 1$ divisibilis, quoties differentia $a^{2n} - b^{2n}$ per eundem numerum non est divisibilis. Quare qui negaverit numerum primum $4n + 1$ esse summam duorum quadratorum, is nagare cogitur ullum numerum huius formae $a^{2n} + b^{2n}$ per $4n + 1$ esse divisibilem; eundem propterea affirmare oportet omnes numeros in hac forma $a^{2n} - b^{2n}$ contento per $4n + 1$ esse divisibiles, siquidem neque a neque b per $4n + 1$ sit divisible. Quamobrem mihi hic demonstrandum est non omnes numeros in forma $a^{2n} - b^{2n}$ contento per $4n + 1$ esse divisibiles; hoc enim si praetitero, certum erit dari casus seu numeros pro a et b substituendos, quibus forma $a^{2n} - b^{2n}$ non sit per $4n + 1$ divisibilis; illis ergo casibus altera forma $a^{2n} + b^{2n}$ necessario per $4n + 1$ erit divisibilis. Unde, cum a^{2n} et b^{2n} sint numeri quadrati, conficietur id, quod proponitur, scilicet numerum $4n + 1$ esse summam duorum quadratorum.

§5. Ut igitur demonstrem non omnes numeros in hac forma $a^{2n} - b^{2n}$ contento seu non omnes differentias inter binas potestates dignitatis $2n$ esse per $4n + 1$ divisibiles, considerabo seriem harum potestatum ab unitate usque ad eam, quae a radice $4n$ formatur,

$$1, 2^{2n}, 3^{2n}, 4^{2n}, 5^{2n}, 6^{2n}, \dots, (4n)^{2n}$$

ac iam dico non omnes differentias inter binos terminos huius seriei esse per $4n + 1$ divisibiles. Si enim singulae differentiae primae

$$2^{2n} - 1, 3^{2n} - 2^{2n}, 4^{2n} - 3^{2n}, 5^{2n} - 4^{2n}, \dots, (4n)^{2n} - (4n - 1)^{2n}$$

per $4n + 1$ essent divisibiles, eetiam differentiae huius progressionis, quae sunt differentiae secundae illius seriei, per $4n + 1$ essent divisibiles; atque ob eandem rationem differentiae tertiae, quartae, quinae etc. omnes forent per $4n + 1$ divisibiles ac denique etiam differentiae ordinis $2n$, quae sunt, ut constat, omnes inter se aequales. Differentiae autem ordinis $2n$ sunt $= 1 \cdot 2 \cdot 3 \cdot 4 \cdots (2n)$, quae ergo pre numerum primum $4n + 1$ non sunt divisibiles, ex quo vicissim sequitur ne omnes quidem differentias primas per $4n + 1$ esse divisibiles.

XVIA. Euler's proof of Fermat's Last Theorem for $n = 3$ *Elements of Algebra*, Part II, Chapter XV, §243.

Theorem. It is impossible to find any two cubes, whose sum, or difference, is a cube.

[Introduction to Steps 3.] Euler begins with the equation $x^3 + y^3 = z^3$, observes that x and y can be assumed to be *odd* numbers without common divisors, so that if $p = \frac{x+y}{2}$ and $q = \frac{x-y}{2}$, then

$$x^3 + y^3 = 2p(p^2 + 3q^2)$$

. It is sufficient to demonstrate the impossibility of $2p(p^2 + 3q^2)$ being a cube.

[Step 4.] If therefore $2p(p^2 + 3q^2)$ were a cube, that cube would be even, and therefore divisible by 8. ... $p^2 + 3q^2$ must be an odd number, ..., $\frac{p}{4}$ must be a whole number.

[Step 5.] But in order that the product $\frac{1}{4}(p^2 + 3q^2)$ may be a cube, each of these factors, unless they have a common divisor, must separately be a cube. ... [T]he question here is, to know if the factors p and $p^2 + 3q^2$, might not have a common divisor. ... [s]ince p and q are prime to each other, these numbers ... can have no other common divisor than 3, which is the case when p is divisible by 3.

Case 1. p not divisible by 3.

[Step 7.] [The] two factors $\frac{p}{4}$ and $p^2 + 3q^2$ are prime to each other; so that each must separately be a cube. Now, in order that $p^2 + 3q^2$ may become a cube, we have only, as we have seen before⁷², to suppose

$$p + q\sqrt{-3} = (t + u\sqrt{-3})^3 \quad \text{and} \quad p - q\sqrt{-3} = (t - u\sqrt{-3})^3,$$

which gives $p^2 + 3q^2 = (t^2 + 3u^2)^3$, which is a cube, and gives us

$$p = t^3 - 9tu^2 = t(t^2 - 9u^2),$$

also,

$$q = 3t^2u - 3u^3 = 3u(t^2 - u^2).$$

Since therefore q is an odd number, u must be odd, and ..., t must be even.

[Step 8.] Having transformed $p^2 + 3q^2$ into a cube, and having found $p = t(t^2 - 9u^2) = t(t + 3u)(t - 3u)$, it is also required that $\frac{p}{4}$, and consequently, $2p$ be a cube; or, which comes to the same, that ... $2t(t + 3u)(t - 3u)$ be a cube.

But here it must be observed that ... the factors $2t$, $t + 3u$, and $t - 3u$, are prime to each other, and each of them must separately be a cube. If ..., $t + 3u = f^3$, and $t - 3u = g^3$, we shall have $2t = f^3 + g^3$.

⁷²The principle here that Euler used is stated in Part II, Chapter XII, §182. §182: Let the formula $x^2 + cy^2$ be proposed, and let it be required to make it a square. As it is composed of the factors $(x + y\sqrt{-c}) \times (x - y\sqrt{-c})$, these factors must either be squares, or squares multiplied by the same number. The explicit formulas for cubes are given in §189. Euler here assumes the *unique* prime factorization property of numbers of the form $x + y\sqrt{-c}$, which mathematicians in the 19th century realized do not always hold. However, for the proof of Fermat's Last Theorem, numbers of the type $x + y\sqrt{-3}$ do have the unique factorization property, though it needs a justification.

So that, if $2t$ is a cube, we shall have two cubes, f^3 and g^3 , whose sum would be a cube, and which would evidently be much less than the cube x^3 and y^3 assumed at first.

[Step 9.] If, therefore, there could be found in great numbers two such cubes as we require, we should also be able to assign in less numbers two cubes, whose sum would make a cube, and in the same manner we should be led to cube always less. Now, as it is very certain that there are no such cubes among small numbers, it follows, that there are not any among greater numbers. This conclusion is confirmed by that which the second case furnishes, and which will be seen to be the same.

Case 2. p divisible by 3.

[Step 10.] Suppose that p is divisible by 3, and that q is not so, and $\dots p = 3r$; our formula will then become

$$\frac{3r}{4}(9r^2 + 3q^2) = \frac{9r}{4}(3r^2 + q^2),$$

and the two factors are prime to each other, since $3r^2 + q^2$ is neither divisible by 2 nor 3, and r must be even as well as p ; therefore each of these two factors must separately be a cube.

[Step 11.] Now by transforming the second factor $3r^2 + q^2$, we find, in the same manner as before,

$$q = t(t^2 - 9u^2), \quad \text{and} \quad r = 3u(t^2 - u^2);$$

and \dots since q was odd, t must be \dots odd \dots and u must be even.

[Step 12.] But $\frac{9r}{4}$ must also be a cube, or $2u(t+u)(t-u)$ a cube; and as these three factors are prime to each other, each must of itself be a cube. Suppose therefore $t+u = f^3$ and $t-u = g^3$, we shall have $2u = f^3 - g^3$; that is to say, if $2u$ were a cube, $f^3 - g^3$ would be a cube. We should consequently have two cube, f^4 and g^3 , much smaller than the first, whose difference would be a cube, and that would enable us also to find two cubes whose sum would be a cube. \dots Thus, the foregoing conclusion is fully confirmed.

XVIB. Sums of fourth powers

Euler opens Paper 428⁷³ with the statement

Inter theoremata, quae circa proprietates numerorum versantur, id quidem demonstrari solet trium biquadratorum summam nullo modo quoque esse biquadratum sive

$$a^4 + b^4 + c^4 = d^4$$

aequationem esse impossibilem.

⁷³Observationes circa bina biquadrata quorum summam in duo alia biquadrata resolvere liceat, Nova commentarii academiae scientiarum Petropolitanae **17** (1772), 1773, pp.64–69; I3, 211–217.

He immediately proceeded to consider the equation

$$A^4 + B^4 = C^4 + D^4.$$

§3. Rewriting the equation as $A^4 - D^4 = C^4 - B^4$, and writing

$$A = p + q, \quad D = p - q, \quad C = r + s, \quad B = r - s,$$

he transforms the equation into

$$pq(p^2 + q^2) = rs(r^2 + s^2).$$

§4. He writes

$$p = ax, \quad q = by, \quad r = kx, \quad s = y,$$

to bring the equation into the form

$$ab(a^2x^2 + b^2y^2) = k(k^2x^2 + y^2)$$

so that

$$\frac{y^2}{x^2} = \frac{k^3 - a^3b}{ab^3 - k}.$$

§5. In this expression, set $k = ab(1 + z)^{74}$ to transform it into

$$\frac{y^2}{x^2} = a^2 \cdot \frac{b^2 - 1 - 3b^2z + 3b^2z^2 + b^2z^3}{b^2 - 1 - z},$$

and

$$\frac{y}{x} = a \cdot \frac{\sqrt{(b^2 - 1)^2 + (b^2 - 1)(3b^2 - 1)z + 3b^2(b^2 - 2)z^2 + b^2(b^2 - 4)z^3 - b^2z^4}}{b^2 - 1 - z}.$$

Now, we try to make the expression under the $\sqrt{\quad}$ sign a square, say, the square of $(b^2 - 1) + fz + gz^2$ for some appropriate choice of f and g . The square of such a quadratic is

$$(b^2 - 1)^2 + 2(b^2 - 1)fz + 2(b^2 - 1)gz^2 + 2fgz^3 + g^2z^4 + f^2z^2$$

Comparison of the second terms suggests putting

$$f = \frac{3b^2 - 1}{2},$$

and the third terms determines g :

$$3b^2(b^2 - 2) = 2(b^2 - 1)g + \frac{9b^4 - 6b^2 + 1}{4},$$

⁷⁴In the preceding section, Euler had observed that $k = ab$ would lead to a trivial solution.

from which

$$g = \frac{3b^4 - 18b^2 - 1}{8(b^2 - 1)}.$$

Now determine the value of z for which the sum of the last two terms in the two expressions are equal:

$$(g^2 + b^2)z = b^2(b^2 - 4) - 2fg,$$

from which

$$z = \frac{b^2(b^2 - 4) - 2fg}{b^2 + g^2}.$$

§6. Euler now makes a summary: from a value of b we determine f, g, z as above, and choose

$$x = b^2 - 1 - z, \quad y = a(b^2 - 1 + fz + gz^2),$$

leading to

$$p = a(b^2 - 1 - z), \quad q = ab(b^2 - 1 + fz + gz^2); \quad r = ab(1 + z)(b^2 - 1 - z), \quad s = a(b^2 - 1 + fz + gz^2).$$

Since these are all divisible by a , we simply take

$$p = b^2 - 1 - z, \quad q = b(b^2 - 1 + fz + gz^2); \quad r = b(1 + z)(b^2 - 1 - z), \quad s = b^2 - 1 + fz + gz^2.$$

These leads to a set of rational numbers $p + q, p - q, r + s, r - s$. Dividing by the gcd, one obtains a set of integers satisfying $A^4 + B^4 = C^4 + D^4$.

Then Euler gave some numerical examples. In §7. With $b = 2$,

$$f = \frac{11}{2}, \quad g = \frac{-25}{24}, \quad z = \frac{6600}{2929},$$

from which

$$p = \frac{2187}{2929}, \quad q = \frac{173369646}{8579041}, \quad r = \frac{41679846}{8579041}, \quad s = \frac{86684823}{8579041}.$$

These give

$$(A, B, C, D) = (2219449, -555617, 1584749, -2061283).$$

The signs certainly can be dropped.

§8. With $b = 3$, we have $f = 13, g = \frac{5}{4}, h = \frac{200}{169}$, and

$$p = \frac{1152}{169}, \quad q = \frac{2153664}{28561}, \quad r = \frac{1275264}{28561}, \quad s = \frac{717888}{28561};$$

From these,⁷⁵

$$(A, B, C, D) = (12231, 2903, 10381, -10203).$$

⁷⁵Euler actually made some mistakes in his computation, and obtained $A = 477069, B = 8497, C = 310319, D = 428397$. These were corrected by the editor Ferdinand Rudio.

In Paper 776⁷⁶, Euler found a smaller set $A = 542, B = 359, C = 514, D = 103$.

Exercise Find a set of integers A, B, C, D satisfying $A^4 + B^4 = C^4 + D^4$ by using $b = 5$.⁷⁷

Euler begins his paper 716⁷⁸ with

Pluribus autem insignibus Geometris visum est haec theoremata latius extendi posse. Quemadmodum enim duo cubi exhiberi nequeunt, quorum summa vel differentia sit cubus, ita etiam certum est nequidem exhiberi posse tria biquadrata, quorum summa sit pariter biquadratum, sed ad minimum quatuor biquadrata requiri, ut eorum summa prodire queat biquadratum, quamquam nemo adhuc talia quatuor biquadrata assignare potuerit. Eodem modo etiam affirmari posse videtur non exhiberi posse quatuor potestates quintas, quarum summa etiam esset potestas quinta; similique modo res se habebit in altioribus potestatibus; unde sequentes quoque postiones omnes pro impossibilibus erunt habendae:

$$\begin{aligned} a^3 + b^3 &= c^3, \\ a^4 + b^4 + c^4 &= d^4, \\ a^5 + b^5 + c^5 + d^5 &= e^5, \\ a^6 + b^6 + c^6 + d^6 + e^6 &= f^6, \\ &\vdots \end{aligned}$$

Plurimum igitur scientia numerorum promoveri esset censenda, si demonstrationem desideratam etiam ad has formulas extendere liceret.⁷⁹

In 1911, R.Norrie found a sum of four biquadrates equal to a biquadrate:

$$30^4 + 120^4 + 272^4 + 315^4 = 353^4.$$

In 1966, L.J. Lander and T.R. Parkin found a counterexample of Euler's conjecture for fifth powers:

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5.$$

⁷⁶Dilucidationes circa binas summas duorum biquadratorum inter se aequales, Mémoires de l'académie des sciences de St. Pétersbourg (11), 1830, pp.49–57; I5, 135–145.

⁷⁷ $(A, B, C, D) = (2367869, 1834883, 2533177, -1123601)$.

⁷⁸Resolutio formulae diophanteae $ab(maa + nbb) = cd(mcc + ndd)$ per numeros racionales, Nova acta academiae scientiarum Petropolitanae **13** (1795/6), 1802, pp.45–63; I4, 329–351.

⁷⁹Translation in Dickson, *History*, vol. II, P.648: "It has seemed to many geometers that this (Fermat's last) theorem may be generalized. Just as there do not exist two cubes whose sum or difference is a cube, it is certain that it is impossible to exhibit three biquadrates whose sum is a biquadrate, but that at least four biquadrates are needed if their sum is to be a biquadrate, although no one has been able up to the present to assign four such biquadrates. In the same manner it would seem to be impossible to exhibit four fifth powers whose sum is a fifth power, and similarly for higher powers".

Much more sensational was the discovery in 1988 of Noam Elkies⁸⁰ of a counterexample for fourth powers:

$$2682440^4 + 15365639^4 + 18796760^4 = 20615673^4,$$

by studying elliptic curves.

XVIC. On Euler's fearsome foursome

Over the course of his career, Euler addressed number theoretic matters of profound importance as well as those of considerably less significance. Among the latter was a challenge to find four different whole numbers, the sum of any two of which is perfect square. With his fearsome foursome of 18530, 38114, 45986 and 65570, Euler supplied a correct, if utterly non - intuitive answer.⁸¹

This fearsome foursome appeared in *Opera Omnia*, vol. 5, pp. 337 – 339, as a supplement to Paper 796,⁸² To find four numbers A, B, C, D the sum of any two of which is a perfect square, Euler wrote⁸³

$$A + B = r^2, \quad A + C = q^2, \quad B + C = p^2.$$

Then,

$$A = \frac{-p^2 + q^2 + r^2}{2}, \quad B = \frac{p^2 - q^2 + r^2}{2}, \quad C = \frac{p^2 + q^2 - r^2}{2}.$$

Now, if $B + D = v^2$, then

$$D = \frac{-p^2 + q^2 - r^2 + 2v^2}{2}.$$

We require

$$\begin{aligned} A + D &= -p^2 + q^2 + v^2 = u^2, \\ C + D &= q^2 - r^2 + v^2 = w^2, \end{aligned}$$

for integers u and w . From this,

$$p^2 + u^2 = q^2 + v^2 = r^2 + w^2.$$

It is therefore enough to find a number which can be decomposed into a sum of two squares in three different ways.

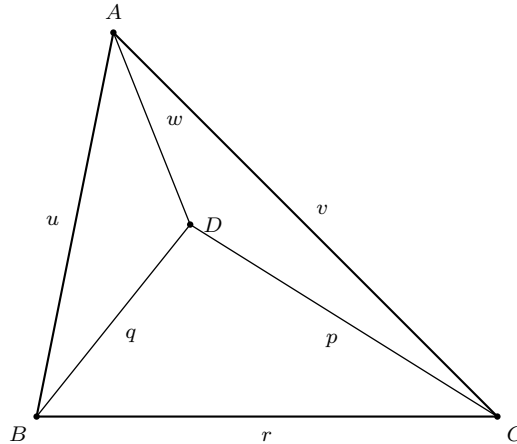
⁸⁰Noam Elkies, On $A^4 + B^4 + C^4 = D^4$, *Math. Comput.*, 51 (1988) pp. 825–835.

⁸¹Dunham, p.7.

⁸²*Recherches sur le problème de trois nombres carrés tels que la somme de deux quelconques moins le troisième fasse un nombre carré*, published in 1781.

⁸³I am here making a slight change of the notation so that the final result may appear more symmetric.

A convenient way to picture the situation is to think of a tetrahedron $ABCD$ with a base triangle of sides p, q, r , and opposite edges u, v, w (satisfying the condition above). Now, we require p^2, q^2, r^2 to satisfy the triangle inequality. This means that the base is an acute - angled triangle. The other faces of the tetrahedron have sides (p, v, w) , (q, w, u) , and (r, u, v) respectively. Each of these should be an acute - angled triangle.



This is a familiar theme since ancient times. Diophantus had addressed the problem of writing a number as a sum of two squares in two different ways, and made use of the composition formula

$$(a^2 + b^2)(x^2 + y^2) = (ax - by)^2 + (bx + ay)^2.$$

Now, Euler knew that every prime of the form $4k + 1$ is a sum of two squares in a unique way. The product of two primes of this form would be a sum of two squares in two different ways. To achieve an expression in three different ways, it is enough to take three such numbers. He made use of the numbers

$$5, 13, 17, 29, 37, 41, 53, 61, 73, 89, 97 \dots$$

and produced numerous examples.

Euler began with

$$5 = 1^2 + 2^2, \quad 13 = 2^2 + 3^2, \quad 17 = 1^2 + 4^2.$$

By repeatedly using the composition formula, we find the different ways of writing $5 \cdot 13 \cdot 17 = 1005$ as a sum of two squares:

$$4^2 + 33^2 = 9^2 + 32^2 = 12^2 + 31^2 = 23^2 + 24^2.$$

More than enough! There are even four pairs. However, it is easy to see that there is no way to form an acute - angled triangle with these edges, one with length 4.

We are therefore limited to

$$9^2 + 32^2 = 12^2 + 31^2 = 23^2 + 24^2.$$

Again, it is not possible to form from these two acute - angled triangles with one edge of length 9. Here, Euler doubled each of these sums, rewriting them using

$$2(a^2 + b^2) = (a + b)^2 + (a - b)^2.$$

In other words,

$$29^2 + 37^2 = 23^2 + 41^2 = 19^2 + 43^2 = 1^2 + 47^2.$$

Obviously, he discarded the last pair and considered

$$29^2 + 37^2 = 23^2 + 41^2 = 19^2 + 43^2.$$

Now, this time, there is an acute tetrahedron:

$$(19, 23, 29; 43, 41, 37).$$

From these, Euler obtained

$$A = \frac{-p^2 + q^2 + r^2}{2}, \quad B = \frac{p^2 - q^2 + r^2}{2}, \quad C = \frac{p^2 + q^2 - r^2}{2}, \quad D = \frac{u^2 + v^2 - r^2}{2}.$$

These are

$$(A, B, C, D) = \left(\frac{1009}{2}, \frac{673}{2}, \frac{49}{2}, \frac{2689}{2}\right).$$

Magnifying by a factor 4, he finally obtained

$$(A, B, C, D) = (2018, 1346, 98, 5378),$$

with

$$\begin{aligned} B + C &= 38^2, & C + A &= 46^2, & A + B &= 58^2, \\ A + D &= 86^2, & B + D &= 82^2, & C + D &= 74^2. \end{aligned}$$

Using the same method, Euler then obtained a few more examples.

<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>
18	3231	7378	30258
10018	19566	32418	101538
4482	6217	8514	21762
18	882	2482	4743

The “fearsome foursome” mentioned by Dunham Euler considered in a subsequent note,⁸⁴ in which he dealt with the same problem with one extra requirement, namely, $A + B = C + D$.⁸⁵ This means $w = r$, and

$$p^2 + u^2 = q^2 + v^2 = 2r^2. \quad (1)$$

Here, it is enough to find a number expressible in two ways as a sum of two squares. Euler made use of

$$65 = 5 \cdot 13 = 1^2 + 8^2 = 4^2 + 7^2.$$

From this,

$$65^2 = (1^2 + 8^2)(4^2 + 7^2) = 39^2 + 52^2 = 25^2 + 60^2.$$

Then,

$$2 \cdot 65^2 = 13^2 + 91^2 = 35^2 + 85^2.$$

Euler actually used

$$\begin{aligned} 65^2 &= (1^2 + 8^2)^2 = 16^2 + 63^2, \\ &= (4^2 + 7^2)^2 = 33^2 + 56^2, \end{aligned}$$

and subsequently

$$2 \cdot 65^2 = 47^2 + 79^2 = 23^2 + 89^2.$$

Here, we would require all four triangles

$$(65, 47, 23), (65, 47, 89), (65, 79, 23), (65, 79, 89)$$

to be acute - angled. But only the last one is!

Therefore, Euler tried another pair. This time, $r = 5 \cdot 17 = 85$, but to no avail. Then, he tried the next pair: $r = 5 \cdot 29 = 1^2 + 12^2 = 8^2 + 9^2$.

$$\begin{aligned} 145^2 &= (1^2 + 12^2)^2 = 24^2 + 143^2, \\ &= (8^2 + 9^2)^2 = 17^2 + 144^2. \end{aligned}$$

Thus,

$$2 \cdot 145^2 = 119^2 + 167^2 = 127^2 + 161^2.$$

This time, the four triangular faces of the tetrahedron

$$(119, 127, 145; 167, 161, 145)$$

⁸⁴*Opera Omnia*, vol. 5, pp. 337 – 339.

⁸⁵Again, I am changing notation.

are all acute - angled. These lead to

$$(A, B, C, D) = \left(\frac{22993}{2}, \frac{19057}{2}, \frac{9265}{2}, \frac{32785}{2} \right).$$

Magnifying by a factor 4, Euler obtained the fearsome foursome

$$(A, B, C, D) = (45986, 38114, 18530, 65570),$$

with

$$A + B = C + D = 290^2, \quad A + C = 254^2, \quad A + D = 334^2, \quad B + C = 238^2, \quad B + D = 322^2.$$

Further Reading

Now that the technical mathematics in Euler's works being over, you should continue keeping the wonderful book

- W. Dunham, *Euler, the Master of Us All*, Math. Assoc. Amer., 1999
under your pillow, and from time to time enjoy the stories and mathematics there more leisurely. After all, learning from a master is a life long endeavour. The modern English edition of Euler's
- *Elements of Algebra*, translated by John Hewlett, (1840), Springer, 1984
begins with an essay by C. Trusdell,
- *Leonard Euler, Supreme Geometer*, reprinted from C. Trusdell, *An Idiot's Fugitive Essays on Science*, Springer, 1984.

In this article, Trusdell points out that

It was Euler who first in the western world wrote mathematics openly, so as to make it easy to read. He taught his era that the infinitesimal calculus was something any intelligent person could learn, with application, and use. He was justly famous for his clear style and for his honesty to the reader about such difficulties as there were.

Practically, every history of mathematics book contains a chapter or at least long sections on Euler. More noteworthy are

- C.B.Boyer (2nd ed, revised by U.Merzbach) *A History of Mathematics*, Wiley, 1991.
- M.Kline, *Mathematics from Ancient to Modern Times*, Oxford University Press, 1972. Now available in 3 paperback volumes.
- E.T.Bell, *Men of Mathematics*, Simon Schuster, 1937.

While my own copy of Euler's biography

- E.A. Fellman, *Leonhard Euler*, Taschenbuch, 1995
is still on the way, there is a short introduction to Euler's work in
- R. Fueter, *Leonhard Euler*, Beihefte Nr. 3, *Elemente der Mathematik*, Birkhäuser, 1948.

The entire November issue of *Mathematics Magazine*, Volume 56, 1983, was devoted to commemorate of bicentenary of Euler's death. There, you find articles like

- J.J. Burckhardt, *Leonard Euler, 1707 – 1783*, pp. 262 – 273.
- G.L. Alexanderson, *Euler as Writer and Teacher*, pp. 274 – 278.
- P. Erdős and U. Dudley, *Some remarks and problems in number theory related to the work of Euler*, pp. 292 – 298.
- M. Kline, *Euler and infinite series*, pp.307–315.

It even contains a 10-page glossary of mathematical terms, equations, formulas, techniques, and theorems attributed to Euler.

On number theory, one can gain a sense of Euler's vast contribution by looking up the indices of the 3 monumental volumes of

- L.E. Dickson, *History of the Theory of Numbers*, 3 volumes, Chelsea, 1919, 1920, 1923.

A major portion of the classic work

- A. Weil, *Number Theory, an Approach through history, from Hammurapi to Legendre*, Birkhäuser, 1984

is devoted to Euler. Here, one finds a comprehensive survey of Euler's work in number theory by a 20th century master. Weil's

- *Two Lectures in Number Theory, past and present*, L'Enseignement Mathématique, 20 (1974) 87 – 110; reprinted in *Collected Works*, volume 2, pp. 279 – 302

begins with interesting accounts of Euler's work.

The introductions to the volumes in Euler's *Opera Omnia* by the various editors put Euler's works in excellent historical perspective. The most interesting of all is certainly the *Opera Omnia* itself. Here is how Trusdell ends his article on Euler:

Only recently have we been able, by study of the manuscripts he left behind, to determine the course of Euler's thought. We now know, for example, that many of the manuscript memoirs published in the two volumes of posthumous works in 1862 he wrote while still a student in Basel and himself withheld from publication for a reason – which usually was some hidden error or an unacceptable or unconvincing result. . . . The most interesting of all Euler's remains is his first notebook, written when he was eighteen or nineteen and still a student of John Bernoulli. It could nearly be described as being all his 800 books and papers in little. Much of what he did in his long life is an outgrowth of the projects he outlined in these years of adolescence. Later, he customarily worked in some four domains of mathematics and physics at once, but he kept changing these from year to year. Typically he would develop something as far as he could, write eight or ten memoirs on various aspects of it, publish most of them, and drop the subject. Coming back to it ten or fifteen years later, he would repeat the pattern but from a deeper point of view, incorporating everything he had done before but presenting it more simply and in a broader conceptual framework. Another ten or fifteen years would see the pattern repeated again. To learn the subject, we need consult only his last works upon it, but to learn his course of thought, we must study the earliest ones, especially those he did not himself publish.

Topics in History of Mathematics (Yiu) Summer B 1999*The Elementary Mathematical Works of Leonhard Euler***Exercises**

I.1. Solve the cubic equation $x^3 + 12x + 12 = 0$ completely.

I.2. If a cubic equation $ax^3 + bx^2 + cx + d = 0$ has a rational root, this must be of the form $\frac{p}{q}$, where p divides d and q divides a . Each of the following cubic equations Euler gave has a rational root. Make use of this fact to solve the equations completely.⁸⁶

(A) $x^3 - 6x^2 + 13x - 12 = 0$.

(B) $y^3 + 30y = 117$.

(C) $y^3 - 36y = 91$.

I.3. Solve the system of equations for x , y , and z :⁸⁷

$$x(y + z) = a, \quad y(z + x) = b, \quad z(x + y) = c.$$

I.4. The following are worked examples on cubic equations in Cardano's *Ars Magna*. The area of a triangle, in terms of its sides a , b , c , is given by the Heron formula:

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = \frac{1}{2}(a + b + c)$.

(A) There is a triangle the difference between the first and second sides of which is 1 and between the second the third side of which is also 1, and the area of which is 3. Find the sides of the triangle.

(B) ABC is a right triangle and AD is perpendicular to its base. Its side, AB plus BD is 36, and AC plus CD is 24. Find its area.

(C) Let, again, ABC be a right triangle and let AD be its perpendicular, and let $AB + CD = 29$, and $AC + BD = 31$. The area is to be found.

II.1. Decompose each of the following quartic polynomials into the product of two quadratic polynomials, each with real coefficients.

(A) $x^4 + a^4$.⁸⁸

(B) $z^4 + 4z^3 + 8z^2 + 4z + 1$.⁸⁹

(C) $x^4 - 4x^3 + 2x^2 + 4x + 4$.

⁸⁶*Elements of Algebra*, Sect. IV, Chap. XII., §747 and Questions for Practice.

⁸⁷In a posthumous paper (808), Euler solved the system of equations

$$v(x + y + z) = a, \quad x(v + y + z) = b, \quad y(v + x + z) = c, \quad z(v + y + z) = d.$$

⁸⁸J. Bernoulli, letter to Euler, 1742.

⁸⁹A. de Moivre, 1730.

II.2. In §§765 – 770 of the *Elements of Algebra*, Euler taught the solution of fourth degree equations by “the rule of Bombelli”,⁹⁰ and with an illustrative example. The basic idea is to rewrite a given fourth degree equation

$$x^4 + ax^3 + bx^2 + cx + d = 0$$

in the form of

$$\left(x^2 + \frac{1}{2}ax + p\right)^2 - (qx + r)^2 = 0$$

for appropriate choice of p, q, r .

(a) For $f(x) = x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$, write the difference $(x^2 - 5x + p)^2 - f(x)$ as a quadratic function, in *descending* order of x .

(b) Show that the quadratic in (a) is the square of a linear function if and only if

$$2p^3 - 35p^2 + 202p - 385 = 0.$$

(c) The cubic equation in (b) has three *positive* rational roots. What are these?

(d) Make use of any one of the rational roots you found in (c) to solve the fourth degree equation in (a) completely.

II.3(a). Construct a cubic equation whose roots α, β and γ satisfy

$$\alpha + \beta + \gamma = 3, \quad \alpha^2 + \beta^2 + \gamma^2 = 5, \quad \alpha^3 + \beta^3 + \gamma^3 = 7.$$

Find the value of $\alpha^4 + \beta^4 + \gamma^4$.

II.3(b)*. More generally, given the sum S_1 of n numbers, the sum S_2 of their squares, the sum S_3 of their cubes, and so on, up to S_n the sum of their n -th powers, find the sum S_{n+1} of their $(n+1)$ -st powers (in terms of S_1, S_2, \dots, S_n).

III.1. Follow Euler’s method to resolve $\frac{1}{z^3(1-z)^2(1+z)}$ completely into partial fractions.

III.2(a) Follow Euler’s method of undetermined coefficients to express

$$\frac{1 + 2z}{1 - z - z^2}$$

as an infinite series.

(b) Find the recurrent relation for the coefficients of this series.

III.3. Find the series expansion of $\frac{1-z}{1-z-2z^2}$ by resolving it into partial fractions. What is the coefficient of the general term?

III.4 If the human population increases annually by $\frac{1}{100}$, we would like to know how long it will take for the population to become ten times as large.

⁹⁰The editor remarked that “[t]his rule rather belongs to Louis Ferrari. It is improperly called the Rule, in the same manner as the rule discovered by Scipio Ferreo has been ascribed to Cardan.

IV.1(=III.5) (a) In §123 of *Introductio*, I, Euler used the series for

$$a = \log \frac{3}{2}, \quad b = \log \frac{4}{3}, \quad \text{and} \quad c = \log \frac{5}{4}$$

to compute the logarithms of the integers from 2 to 10, except $\log 7$. Explain how these can be done.

(b) In the same section, Euler gave the value of $\log 7$ as

$$1.94591\ 01490\ 55313\ 30510\ \mathbf{54639}$$

Here, the last 4 digits are not correct. What should these be?

IV.2. Use division to find the series of $\tan x = \frac{\sin x}{\cos x}$ up to the term of x^5 .

IV.3(a) Express $\frac{1}{2}(2^i + 2^{-i})$ and $\frac{1}{2i}(2^i - 2^{-i})$ as real numbers.

(b) Find the values of i^i .

(c) Show that if $\log(x + yi) = a + bi$, then $x = e^a \cos b$ and $y = e^a \sin b$.⁹¹

IV.4. Show that

$$\tan(a + bi) = \frac{2e^{2b} \sin 2a + (e^{4b} - 1)i}{e^{4b} + 2e^{2b} \cos 2a + 1}.$$

V.1. Justify Euler's claim that

$$\int \frac{1}{1 + \sqrt{x}} dx = 2\sqrt{x} - 2 \log(1 + \sqrt{x})$$

by using the substitution $u = 1 + \sqrt{x}$.

V.2. Show that

$$\int z^n \log z dz = C - \frac{z^{n+1}}{(n+1)^2} + \frac{z^{n+1}}{n+1} \log z.$$

V.3. In his paper 20, Euler made use of

$$\log 2 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 8} + \frac{1}{4 \cdot 16} + \cdots.$$

What is the general term of this series? How did this series arise?

VI.1. Show that⁹²

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - + \cdots = \frac{\pi^3}{32}.$$

⁹¹Paper 170, §103.

⁹²§11 of Paper 41.

VI.2. Use the general binomial theorem to show that⁹³

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{1 \cdot 3}{2 \cdot 4}x^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^6 + \dots$$

Find the general term of the series.

VI.3*. Follow Euler's method in his Paper 73 to show that

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \dots = \frac{\pi^4}{90}.$$

Hint: Find the series expansion for $\frac{1}{6}(\arcsin x)^3$.

VII.1. Bernoulli noted that with his table (Note, p.55), it took him less than half of a quarter of an hour to find the sum the tenth powers of the first 1000 numbers. What is this sum precisely?

VII.2. Prove that

$$\csc x = \cot \frac{1}{2}x - \cot x,$$

and make use of it to find the series expansion of $\csc x$.⁹⁴

VIII.1. Use the recurrence for the Bernoulli numbers to determine the value of B_4 .

VIII.2. Use the recurrence for the Euler numbers to determine the values of E_{10} , E_{12} and E_{14} .

VIII.3. Find the sum

$$1 + \frac{1}{3^{2k}} + \frac{1}{5^{2k}} + \frac{1}{7^{2k}} + \dots$$

in terms of Bernoulli numbers.

VIII.4. Make use of the relation

$$\cot x + \tan \frac{x}{2} = \frac{1}{\sin x}$$

to show that

$$\frac{x}{\sin x} = 1 + \sum_{k=0}^{\infty} \frac{(2^{2k} - 2)B_{2k}}{(2k)!} x^{2k}.$$

IX.1. Write the partial quotients of the continued fraction

$$a_0 + \frac{\alpha_1}{a_1 + \frac{\alpha_2}{a_2 + \frac{\alpha_3}{a_3 + \frac{\alpha_4}{a_4 + \dots + \frac{\alpha_n}{a_n + \frac{\alpha_{n+1}}{a_{n+1} + \dots}}}}}$$

⁹³Euler integrated this to obtain $s = \arcsin x$ in his "forgotten" paper 73.

⁹⁴Euler's *Institute de Calculi Differentialis*, §223. Euler wrote cosec x for $\csc x$.

in the form

$$\frac{P_n}{Q_n} = a_0 + \frac{\alpha_1}{a_1 + \frac{\alpha_2}{a_2 + \frac{\alpha_3}{a_3 + \frac{\alpha_4}{\dots + \frac{\alpha_n}{a_n}}}}.$$

It is clear that $P_0 = a_0$ and $Q_0 = 1$. Set $P_{-1} = 1$ and $Q_{-1} = 0$. Write down the recurrence relations for P_{n+2} and Q_{n+2} .

IX.2. What is the number represented by the periodic continued fraction⁹⁵

$$a + \frac{1}{b + \frac{1}{2a + \frac{1}{b + \frac{1}{2a + \frac{1}{b + \frac{1}{2a + \dots}}}}}}?$$

IX.3. Show that the continued fraction

$$a + \frac{1}{b + \frac{1}{c + \frac{1}{d + \frac{1}{b + \frac{1}{c + \frac{1}{d + \dots}}}}}}$$

is a root of the quadratic equation⁹⁶

$$(1 + cd)x^2 - (bcd + 2acd + 2a + b - c + d)x + (abcd + a^2cd + a^2 + ab - ac + ad - bc - 1) = 0.$$

IX.4. Convert a series for $\frac{\pi}{4}$ to show that

$$\frac{\pi}{4} = \frac{1}{1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}}}}}.$$

X.1. A rectangle $ADEB$ is constructed externally on the hypotenuse AB of a right triangle ABC . The line CD and CE intersect the line AB in the points F and G respectively.⁹⁷

- (a) If $DE = AD\sqrt{2}$, show that $AG^2 + FB^2 = AB^2$.
 (b) If $AD = DE$, show that $FG^2 = AF \cdot GB$.

X.2(a) Calculate the area of the triangle with sides 13, 14, 15, and determine the altitudes.

(b) Calculate the area of the triangle with sides 25, 34, 39, and determine the altitudes.

(c) Calculate the area of the triangle with sides 9, 10, and 17.

⁹⁵Paper 71, §20.

⁹⁶This has been corrected.

⁹⁷*Pi Mu Epsilon Journal*, Problem 317 (1974 Spring), proposed by Leon Bankoff. Part (a) is Fermat's theorem. Euler solved (a) in the beginning of his paper 135, *Variae demonstrationes geometriae*.

X.3*. Heronian triangles with 3 consecutive integers as sides. Examples:

$$(3, 4, 5; 6), (13, 14, 15; 84), (51, 52, 53; 1170), \dots$$

More generally, these triangles form a recurrent sequence: if $(b - 1, b, b + 1; \Delta)$ is a Heronian triangle, then b is a term of the sequence

$$b_{n+2} = 4b_{n+1} - b_n, \quad b_1 = 4, \quad b_2 = 14.$$

What is the recurrent relation for the area of such triangles?

XI.1(a) Show that a triangle is equilateral if and only if two of its orthocenter, centroid, and circumcenter coincide.

(b) What can you say about a triangle whose incenter also lies on the Euler line?

XI.2. Let O and H be respectively the circumcenter and orthocenter of triangle ABC . Show that the bisector of angle BAC also bisects angle OAH .

XI.3. Let R , ρ and s denote the circumradius, inradius, and semiperimeter of a triangle with orthocenter H , centroid G , circumcenter O , and incenter I . Show that Euler's distance formulas can be rewritten as

$$\begin{aligned} HG^2 &= \frac{4}{9}(9R^2 + 8R\rho + 2\rho^2 - 2s^2), \\ HI^2 &= 4R^2 + 4R\rho + 3\rho^2 - s^2, \\ HO^2 &= 9R^2 + 8R\rho + 2\rho^2 - 2s^2, \\ GI^2 &= \frac{1}{9}(5\rho^2 - 16R\rho + s^2), \\ GO^2 &= \frac{1}{9}(9R^2 + 8R\rho + 2\rho^2 - 2s^2), \\ IO^2 &= R^2 - 2R\rho. \end{aligned}$$

XII.1(a) Describe precisely the steps for the ruler - and - compass construction of a triangle ABC given its centroid G , circumcenter O , and vertex A .

(b) Same as (a), but with given orthocenter H , circumcenter O , and vertex A .

(c) Same as (a), but with given incenter I , circumcenter O , and vertex A .

XII.2. Let ABC be an isosceles triangle. Show that the incenter lies between the orthocenter and the centroid.⁹⁸

XII.3 Make use of Euler's solution to show that if triangle OIH is isosceles, triangle ABC contains a 60 degree angle.

⁹⁸Make use of the proportions of the side lengths in §25 of Euler's paper 135.

XIII.1(a) Euler had found that $2^{29} - 1 = 536870911$ is divisible by 1103. Find a smaller prime divisor.

(b) Find a prime divisor of $2^{83} - 1$.

XIII.2. In the beginning of his paper 26, Euler observed that if $d^n + 1$ is prime, then n must be a power of 2. Justify this.

XIII.3. Complete the following steps show that if N is an *odd* perfect number, it cannot be a square.

(i) The sum of all divisors of N , including 1 and N itself, is even.

(ii) If N is a square, the number of divisors, including 1 and N itself, is odd.

XIV.1(a) Make use of the multiplicative property of the sum of divisor function to find the sum of all divisors of 100.

(b) Make use of Euler's "extraordinary relation" to compute the sum of divisors of 100.

XV.1. Follow Euler's presentation to find the continued fraction expansion of $\sqrt{67}$.

XV.2. Show that the continued fraction expansion of $\sqrt{n^2 + 2n - 1}$ is

$$n + \frac{1}{1 + \frac{1}{n - 1 + \frac{1}{1 + \frac{1}{2n + \frac{1}{1 + \frac{1}{n - 1 + \frac{1}{1 + \frac{1}{2n + \dots}}}}}}}}$$

XV.3. Make use of the continued fraction expansion

$$\sqrt{31} = [5, \overline{1, 1, 3, 5, 3, 1, 1, 10}]$$

to find the smallest positive solution of $p^2 - 31q^2 = 1$.

Solution to Exercises

I.1. Write $x = \sqrt[3]{A} + \sqrt[3]{B}$. Then $AB = (-\frac{12}{3})^3 = -64$ and $A + B = -12$. The numbers A and B are the roots of $x^2 + 12x - 64 = 0$; $(x + 16)(x - 4) = 0$. We choose $A = -16$ and $B = 4$, and obtain the real root $-2\sqrt[3]{2} + \sqrt[3]{4}$. The other two roots are the imaginary numbers $-2\sqrt[3]{2}\omega + \sqrt[3]{4}\omega^2$ and $-2\sqrt[3]{2}\omega^2 + \sqrt[3]{4}\omega$, where $\omega = \frac{1}{2}(-1 + \sqrt{3}i)$ and $\omega^2 = \frac{1}{2}(-1 - \sqrt{3}i)$.

I.2.

	rational root	Factorization	Remaining roots
(a)	3	$(x - 3)(x^2 - 3x + 4)$	$\frac{1}{2}(3 \pm \sqrt{7}i)$
(b)	3	$(y - 3)(y^2 + 3y + 39)$	$\frac{1}{2}(3 \pm 7\sqrt{3}i)$
(c)	7	$(y - 7)(y^2 + 7y + 13)$	$\frac{1}{2}(7 \pm \sqrt{3}i)$

I.3. Adding the three equations together, we have $2(xy + yz + zx) = a + b + c$. This means $xy + yz + zx = \frac{1}{2}(a + b + c)$. By writing $s = \frac{1}{2}(a + b + c)$, we obtain

$$yz = s - a, \quad zx = s - b, \quad xy = s - c.$$

Multiplying these together, $x^2y^2z^2 = (s - a)(s - b)(s - c)$, and $xyz = \pm\sqrt{(s - a)(s - b)(s - c)}$. It follows that

$$x = \pm\sqrt{\frac{(s - b)(s - c)}{s - a}}, \quad y = \pm\sqrt{\frac{(s - c)(s - a)}{s - b}}, \quad z = \pm\sqrt{\frac{(s - a)(s - b)}{s - c}}.$$

I.4(A) Cardano, *Ars Magna*, Chapter XXXII, Problem I. Now assume x to be the second side. The first side will be $x - 1$ and the third $x + 1$. Follow the rule for the triangles given in the following book⁹⁹, and this makes $\sqrt{\frac{3}{16}x^4 - \frac{3}{4}x^2}$ equal to 3. Therefore

$$\frac{3}{16}x^4 = \frac{3}{4}x^2 + 9$$

and, therefore,

$$x^4 = 4x^2 + 48,$$

and x will be $\dots\sqrt{\sqrt{52} + 2}$, and this is the second side. Add and subtract 1, therefore, and you will have the remaining sides.

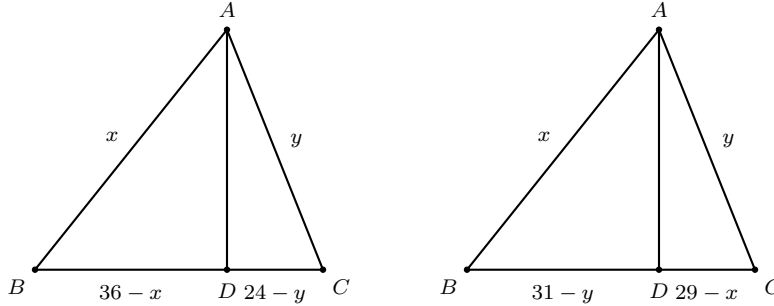
I.4(B) Cardano, *Ars Magna*, Chapter XXXVIII, Problem XIII. The hypotenuse BC is 25; the sides of the triangles are 15 and 20, and the area is 150.

Suppose $AB = x$, $AC = y$. Then $BD = 36 - x$ and $CD = 24 - y$. By computing the length of AD in two ways, we have

$$x^2 - (24 - x)^2 = y^2 - (36 - y)^2,$$

⁹⁹Heron's formula.

and $576 - 48x = 1296 - 72y$, or $3y = 2x + 30$. Now, the length of $BC = (24 - x) + (34 - y) = (60 - x - y)$. We have $(60 - x - y)^2 = x^2 + y^2$, from which $3600 - 120(x + y) + 2xy = 0$. Substituting $x = 15 - \frac{3y}{2}$, we have $(y - 20)(y - 90) = 0$. Clearly, y must be 20. From this, $x = 15$, and $BC = 60 - 15 - 20 = 25$. The triangle being right angled, its area is $\frac{1}{2} \cdot 15 \cdot 20 = 150$.



I.4(C). Cardano, *Ars Magna*, Chapter XXXVIII, Problem XIII. The same triangle as in the preceding problem.

Again, let $AB = x$ and $AC = y$ so that $BD = 31 - y$ and $CD = 29 - x$. From the length of AD , we have $x^2 - (31 - y)^2 = y^2 - (29 - x)^2$, and $2x^2 - 2y^2 + 58x - 62y = 31^2 - 29^2$,

$$x^2 - y^2 + 29x - 31y = 60. \tag{2}$$

Also, since the length of BC is $60 - x - y$, we again have $3600 - 120(x + y) + 2xy = 0$, $xy - 60(x + y) + 1800 = 0$, $y = \frac{60(30-x)}{60-x}$. Substituting into equation (1), we have

$$\frac{2(x - 20)(x^3 - 129x^2 + 2700x + 54000)}{(60 - x)^2} = 0,$$

from which $x = 20$, and we obtain the same triangle as in (B).

Remark. If we put $x = y + 43$ into the cubic factor, this becomes $y^3 - 2847y + 11086$. With $a = 2847$ and $b = -11086$, we have $\frac{a^3}{27} + \frac{b^2}{4} > 0$. This means that the cubic polynomial has one real and two imaginary roots. Their product being -11086 , the real root cannot be positive. Thus, there is only one admissible solution to the problem, namely, $x = 20$.

II.1. (A) $x^4 + a^4 = (x^2 + a^2)^2 - 2a^2x^2 = (x^2 - \sqrt{2}ax + a^2)(x^2 + \sqrt{2}ax + a^2)$.

(B) The roots of the quartic equation $z^4 + 4z^3 + 8z^2 + 4z + 1 = 0$ are

$$\begin{aligned} z_1 &= -1 + \frac{1}{\sqrt{2}} - i\sqrt{\frac{3-2\sqrt{2}}{2}}, & z_2 &= -1 + \frac{1}{\sqrt{2}} + i\sqrt{\frac{3-2\sqrt{2}}{2}}, \\ z_3 &= -1 - \frac{1}{\sqrt{2}} - i\sqrt{\frac{3+2\sqrt{2}}{2}}, & z_4 &= -1 - \frac{1}{\sqrt{2}} + i\sqrt{\frac{3+2\sqrt{2}}{2}}. \end{aligned}$$

The real quadratic factors are

$$(z - z_1)(z - z_2) = \left(z + 1 - \frac{1}{\sqrt{2}}\right)^2 + \frac{3 - 2\sqrt{2}}{2} = z^2 + (2 - \sqrt{2})z + (3 - 2\sqrt{2}),$$

and

$$(z - z_3)(z - z_4) = \left(z + 1 + \frac{1}{\sqrt{2}}\right)^2 + \frac{3 + 2\sqrt{2}}{2} = z^2 + (2 + \sqrt{2})z + (3 + 2\sqrt{2}).$$

(C) Proposed by Nicholas Bernoulli (1687 – 1759) to Euler, in letters dated October 24, 1742 and April 6, 1743. Euler communicated his results to Christian Goldbach on December 15, 1742 and February 26, 1743. Answer: $x^2 - (2 + \sqrt{4 + 2\sqrt{7}})x + 1 + \sqrt{7} + \sqrt{4 + 2\sqrt{7}}$ and $x^2 - (2 - \sqrt{4 + 2\sqrt{7}})x + 1 + \sqrt{7} - \sqrt{4 + 2\sqrt{7}}$.

II.2. (a)

$$\begin{aligned} & (x^2 - 5x + p)^2 - f(x) \\ = & (x^2 - 5x + p)^2 - (x^4 - 10x^3 + 35x^2 - 50x + 24) \\ = & (x^4 - 10x^3 + (2p + 25)x^2 - 10px + p^2) - (x^4 - 10x^3 + 35x^2 - 50x + 24) \\ = & 2(p - 5)x^2 - 10(p - 5)x + (p^2 - 24) \end{aligned}$$

(b) The quadratic in (a) is a perfect square if and only if $0 = 100(p - 5)^2 - 8(p - 5)(p^2 - 24) = -4(p - 5)[2(p^2 - 24) - 25(p - 5)] = -4(p - 5)(2p^2 - 25p + 77) = 0$.

(c) This further factors as $-4(p - 5)(p - 7)(2p - 11)$. The quadratic has three rational roots 5, 7, and $\frac{11}{2}$.

(d) By taking $p = 5$, we have

$$\begin{aligned} f(x) &= (x^2 - 5x + 5)^2 - 1 \\ &= (x^2 - 5x + 4)(x^2 - 5x + 6) \\ &= (x - 1)(x - 4)(x - 2)(x - 3). \end{aligned}$$

It follows that the roots are $x = 1, 2, 3, 4$.

Remark: The rational roots $p = 7$ and $\frac{11}{2}$ would certainly lead to the same rational roots.

II.3(a)

$$\begin{aligned} \sigma_1 &= \alpha + \beta + \gamma = 3, \\ \sigma_2 &= \frac{1}{2}[(\alpha + \beta + \gamma)^2 - (\alpha^2 + \beta^2 + \gamma^2)] = 2. \end{aligned}$$

Since $S_3 - \sigma_1 S_2 + \sigma_2 S_1 - 3\sigma_3 = 0$,

$$\sigma_3 = \frac{1}{3}[S_3 - \sigma_1 S_2 + \sigma_2 S_1] = \frac{1}{6}(2s_3 - 3S_1 S - 2 + S_1^3) = \frac{1}{3}[7 - 3 \cdot 5 + 2 \cdot 3] = -\frac{2}{3}.$$

Also,

$$S_4 = \sigma_1 S_3 - \sigma_2 S_2 + \sigma_3 S_1 = 3 \cdot 7 - 2 \cdot 5 + \left(-\frac{2}{3}\right) \cdot 3 = 9.$$

(b) More generally, given S_1, S_2, \dots, S_n of n quantities $\alpha_1, \alpha_2, \dots, \alpha_n$, we use Newton's relations to find the

elementary symmetric functions:

$$\begin{aligned}
 \sigma_1 &= S_1, \\
 S_1\sigma_1 - 2\sigma_2 &= S_2, \\
 S_2\sigma_1 - S_1\sigma_2 + 3\sigma_3 &= S_3, \\
 S_3\sigma_1 - S_2\sigma_2 + S_1\sigma_3 - 4\sigma_4 &= S_3, \\
 &\vdots \\
 S_{n-1}\sigma_1 - S_{n-2}\sigma_2 + S_{n-3}\sigma_3 - S_{n-4}\sigma_4 + \cdots + (-1)^{n-1}n\sigma_n &= S_n.
 \end{aligned}$$

From these, $\sigma_1, \sigma_2, \dots, \sigma_n$ can be successively determined, and

$$S_{n+1} = S_n\sigma_1 - S_{n-1}\sigma_2 + S_{n-2}\sigma_3 - S_{n-3}\sigma_4 + \cdots + (-1)^{n-1}S_1\sigma_n.$$

Here are S_{n+1} in terms of S_1, \dots, S_n , for $n \leq 5$:

n	S_{n+1}
1	S_1^2
2	$-\frac{1}{2}S_1(S_1^2 - 3S_2)$
3	$\frac{1}{6}(S_1^4 - 6S_1^2S_2 + 3S_2^2 + 8S_1S_3)$
4	$\frac{1}{24}9(-S_1^5 + 10S_1^3S_2 - 15S_1S_2^2 - 20S_1^2S_3 + 20S_2S_3 + 30S_1S_4)$
5	$\frac{1}{120}(S_1^6 - 15S_1^4S_2 + 45S_1^2S_2^2 - 15S_2^3 + 40S_1^3S_3 - 120S_1S_2S_3 + 40S_3^2 - 90S_1^2S_4 + 90S_2S_4 + 144S_1S_5)$

III.1. Introductio, I, §46, Example.

First we take the singular factor of the denominator $1 + z$, which gives $\frac{p}{q} = -1$, while $M = 1$ and $Z = z^3 - 2z^4 + z^5$. In order to find the fraction $\frac{A}{1+z}$, we let

$$A = \frac{1}{z^3 - 2z^4 + z^5}$$

when $z = -1$. Hence $A = -\frac{1}{4}$ and from the factor $1 + z$ there arises the partial fraction $\frac{-1}{4(1+z)}$. Now, take the quadratic factor $(1 - z)^2$, which gives $\frac{p}{q} = 1$, $M = 1$, and $Z = z^3 + z^4$. We let the partial fractions arising from this be $\frac{A}{(1-z)^2} + \frac{B}{1-z}$, then $A = \frac{1}{z^3+z^4}$ when $z = 1$, so that $A = \frac{1}{2}$. We let

$$P = \frac{M - \frac{1}{2}Z}{1 - z} = \frac{1 - \frac{1}{2}z^3 - \frac{1}{2}z^4}{1 - z} = 1 + z + z^2 + \frac{1}{2}z^3$$

so that

$$B = \frac{P}{Z} = \frac{1 + z + z^2 + \frac{1}{2}z^3}{z^3 + z^4}$$

when $z = 1$, hence $B = \frac{7}{4}$ and the desired partial fractions are

$$\frac{1}{2(1-z)^2} + \frac{7}{4(1-z)}.$$

Finally, the cubic factor z^3 gives $\frac{P}{Q} = 0$, $M = 1$, and $Z = 1 - z - z^2 + z^3$. We let the corresponding partial fractions be $\frac{A}{z^3} + \frac{B}{z^2} + \frac{C}{z}$. Then $A = \frac{M}{Z} = \frac{1}{1-z-z^2+z^3}$ so that $B = \frac{P}{Z}$ when $z = 0$, hence $B = 1$. We let

$$Q = \frac{P - Z}{z} = 2 - z^2$$

so that $C = \frac{Q}{Z}$ when $z = 0$, hence $C = 2$. Thus, the given function $\frac{1}{z^3(1-z)^2(1+z)}$ is expressed in the form

$$\frac{1}{z^3} + \frac{1}{z^2} + \frac{2}{z} + \frac{1}{2(1-z)^2} + \frac{7}{4(1-z)} - \frac{1}{4(1+z)}.$$

There is no polynomial part since the given function is not improper.

III.2. Introductio, I, §61, Example.

If the given function is $\frac{1+2z}{1-z-z^2}$, and this is set equal to the series

$$A + Bz + Cz^2 + Dz^3 + \dots,$$

since $a = 1$, $b = 2$, $\alpha = 1$, $\beta = -1$, and $\gamma = -1$, we have $A = 1$, $B = 3$. Then

$$C = B + A, \quad D = C + B, \quad E = D + C, \quad F = E + D, \dots,$$

so that any coefficient is the sum of the two immediately preceding it. If P and Q are known successive coefficients and R is the next coefficient, then $R = P + Q$. Since the first two coefficients A and B are known, the given rational function $\frac{1+2z}{1-z-z^2}$ is transformed into the infinite series

$$1 + 3z + 4z^2 + 7z^3 + 11z^4 + 18z^5 + \dots,$$

which can be continued as long as desired with no trouble.

III.3. Introductio, I, §216, Example 1.

$$\frac{1-z}{1-z-2z^2} = \frac{1}{3} \left(\frac{2}{1+z} + \frac{1}{1-2z} \right) = \frac{1}{3} \left(2 \sum_{n=0}^{\infty} (-1)^n z^n + \sum_{n=0}^{\infty} 2^n z^n \right) = \frac{1}{3} \sum_{n=0}^{\infty} (2^n + 2(-1)^n) z^n.$$

Remark. Euler expresses the coefficient of z^n as $\frac{1}{3}(2^n \pm 2)$, where the positive sign is used when n is even and the negative sign when n is odd.

III.4. Introductio, I, §111, Example I: We suppose that this will occur after x years and that the initial population is n . Hence after x years the population will be $(\frac{101}{100})^x n = 10n$, so that $(\frac{101}{100})^x = 10$, and $x \log \frac{101}{100} = \log 10$.¹⁰⁰ From this we have

$$x = \frac{\log 10}{\log 101 - \log 100} = \frac{10000000}{43214} = 231.$$

After 231 years the human population will be ten times as large with an annual increase of only $\frac{1}{100}$.

¹⁰⁰Logarithms of any base would work. Here, Euler uses the common logarithms, base 10.

Euler further remarks: "It also follows that after 462 years the population will be one hundred times as large, and after 693 years the population will be one thousand time as large".

IV.1. (a) Let $a = \log \frac{3}{2}$, $b = \log \frac{4}{3}$, and $c = \log \frac{5}{4}$.

$$\begin{aligned} \log 2 &= \log \frac{3}{2} \cdot \frac{4}{3} = a + b, \\ \log 3 &= \log 2 \cdot \frac{3}{2} = (a + b) + a = 2a + b, \\ \log 4 &= 2 \log 2 = 2a + 2b, \\ \log 5 &= \log 4 \cdot \frac{5}{4} = 2a + 2b + c, \\ \log 6 &= \log 2 + \log 3 = 3a + 2b, \\ \log 8 &= \log 2^3 = 3 \log 2 = 3a + 3b, \\ \log 9 &= \log 3^2 = 2 \log 3 = 4a + 2b, \\ \log 10 &= \log 2 + \log 5 = 3a + 3b + c. \end{aligned}$$

(b) The value of $\log 7$, to 22 places of decimal, is

$$1.94591\ 01490\ 55313\ 30510\ 53527\ 43443 \dots$$

Euler's **4639** should be 3527.

IV.2

$$1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} \left| \begin{array}{r} x + \frac{x^3}{3} + \frac{2}{15}x^5 \\ x - \frac{x^3}{6} + \frac{x^5}{120} \\ x - \frac{x^3}{2} + \frac{x^5}{24} \\ \hline \frac{x^3}{3} - \frac{x^5}{30} \\ \frac{x^3}{3} - \frac{x^5}{6} \\ \hline \frac{2x^5}{15} \\ \frac{2x^5}{15} \end{array} \right.$$

Remark. The next term is $\frac{17}{315}x^7$.

IV.3. (a)

$$\frac{1}{2}(2^i + 2^{-i}) = \frac{1}{2}(e^{i \log 2} + e^{-i \log 2}) = \cos \log 2.$$

Similarly,

$$\frac{1}{2i}(2^i - 2^{-i}) = \frac{1}{2i}(e^{i \log 2} - e^{-i \log 2}) = \sin \log 2.$$

(b) Begin with the value of $\log i$. Since $i = \cos(\frac{\pi}{2} + 2k\pi) + i \sin(\frac{\pi}{2} + 2k\pi) = e^{(\frac{\pi}{2} + 2k\pi)i}$ for integers k , we have $\log i = (\frac{\pi}{2} + 2k\pi)i$. It follows that $i \log i = -(\frac{\pi}{2} + 2k\pi)$. This gives $i^i = e^{-(\frac{\pi}{2} + 2k\pi)}$ for integer values of k .

Remark. In particular, with $k = 0$, Euler gave the real value $i^i = e^{-\frac{\pi}{2}} = 0.2078795763507$ in *Introductio*, I, §97.

(c) $x + yi = e^{a+bi} = e^a \cdot e^{bi} = e^a(\cos b + i \sin b)$. From this the result is clear.

IV.4. Beginning with

$$\begin{aligned}\cos(a + bi) + i \sin(a + bi) &= e^{(a+bi)i} = e^{-b+ai} = e^{-b}(\cos a + i \sin a), \\ \cos(a + bi) - i \sin(a + bi) &= e^{-(a+bi)i} = e^{b-ai} = e^b(\cos a - i \sin a),\end{aligned}$$

we have

$$\begin{aligned}\cos(a + bi) &= \frac{1}{2}((e^{-b} + e^b) \cos a + i(e^{-b} - e^b) \sin a) = \frac{1}{2e^b}[(e^{2b} + 1) \cos a - i(e^{2b} - 1) \sin a], \\ \sin(a + bi) &= \frac{1}{2}(-(e^{-b} + e^b) \sin a - i(e^{-b} - e^b) \cos a) = \frac{1}{2e^b}[(e^{2b} + 1) \sin a + i(e^{2b} - 1) \cos a].\end{aligned}$$

From these, we have

$$\begin{aligned}\tan(a + bi) &= \frac{(e^{2b} + 1) \sin a + i(e^{2b} - 1) \cos a}{(e^{2b} + 1) \cos a - i(e^{2b} - 1) \sin a} \\ &= \frac{[(e^{2b} + 1) \sin a + i(e^{2b} - 1) \cos a][e^{2b} + 1] \cos a + i(e^{2b} - 1) \sin a}{[e^{2b} + 1] \cos a - i(e^{2b} - 1) \sin a}[e^{2b} + 1] \cos a + i(e^{2b} - 1) \sin a} \\ &= \frac{[(e^{2b} + 1) \sin a + i(e^{2b} - 1) \cos a][e^{2b} + 1] \cos a + i(e^{2b} - 1) \sin a}{(e^{2b} + 1)^2 \cos^2 a + (e^{2b} - 1)^2 \sin^2 a} \\ &= \frac{[(e^{2b} + 1)^2 - (e^{2b} - 1)^2] \sin a \cos a + i(e^{2b} + 1)(e^{2b} - 1)(\cos^2 a + \sin^2 a)}{(e^{4b} + 1)(\cos^2 a + \sin^2 a) + 2e^{2b}(\cos^2 a - \sin^2 a)} \\ &= \frac{2e^{2b} \sin 2a + (e^{4b} - 1)i}{e^{4b} + 2e^{2b} \cos 2a + 1}.\end{aligned}$$

V.1. With this substitution, $du = \frac{1}{2\sqrt{x}}dx$, and $dx = 2(u - 1)du$. It follows that the integral

$$= \int \frac{2(u - 1)}{u} du = \int \left(2 - \frac{2}{u}\right) du = 2u - 2 \log u = 2(1 + \sqrt{x}) - 2 \log(1 + \sqrt{x}) + C$$

for some constant C .

V.2. In the method of integration by parts, set $u = \log z$ and $dv = z^n dz$. We have $du = \frac{1}{z} dz$ and $v = \frac{1}{n+1} z^{n+1}$. Thus,

$$\int z^n \log z dz = \frac{1}{n+1} z^{n+1} \log z - \int \frac{1}{n+1} z^n dz = \frac{1}{n+1} z^{n+1} \log z - \frac{1}{(n+1)^2} z^{n+1} + C.$$

V.3. The n -th term of the series is $\frac{1}{n \cdot 2^n}$. This series results from putting $x = \frac{1}{2}$ in the expansion of $\log(1 - x)$.

VI.1. §11 of Paper 41.

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - + \cdots = \frac{\pi^3}{32}.$$

Solution. The roots of $\sin x = 1$ are, according to Note, p.45,

$$\frac{\pi}{2}, \frac{\pi}{2}, \frac{-3\pi}{2}, \frac{-3\pi}{2}, \frac{5\pi}{2}, \frac{5\pi}{2}, \frac{-7\pi}{2}, \frac{-7\pi}{2}, \frac{9\pi}{2}, \frac{9\pi}{2}, \dots$$

If P = sum of the reciprocals of the roots, Q = sum of the reciprocals of the squares of the roots, and R = sum of the reciprocals of the cubes of the roots, with $y = 1$ from the relations on p.45,

$$P = 1, \quad Q = 1, \quad R = Q - \frac{1}{2} = \frac{1}{2},$$

It follows that

$$2\left(\frac{8}{\pi^3} - \frac{8}{3^3\pi^3} + \frac{8}{5^3\pi^3} - \frac{8}{7^3\pi^3} + \dots\right) = \frac{1}{2},$$

and the result follows.

VI.2. By the binomial theorem,

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} = 1 + \sum_{n=1}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} x^{2n}.$$

Here,

$$\binom{-\frac{1}{2}}{n} = \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\dots(-\frac{2n-1}{2})}{n!} = (-1)^n \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2^n \cdot n!} = (-1)^n \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n)}.$$

From this the result follows.

VII.1. 91 409 924 241 424 243 424 241 924 242 500.

VII.2. We begin with an expression for $\cot \frac{1}{2}x$:

$$\cot \frac{x}{2} = \frac{\cos \frac{1}{2}x}{\sin \frac{1}{2}x} = \frac{2 \cos^2 \frac{1}{2}x}{2 \sin \frac{1}{2}x \cos \frac{1}{2}x} = \frac{1 + \cos x}{\sin x} = \csc x + \cot x.$$

This establishes the required relation. From the formulas in Note, p.63,

$$\cot x = \frac{1}{x} - \frac{2^2 \mathcal{A}x}{2!} - \frac{2^4 \mathcal{B}x^3}{4!} - \frac{2^6 \mathcal{C}x^5}{6!} - \frac{2^8 \mathcal{D}x^7}{8!} - \dots,$$

$$\cot \frac{1}{2}x = \frac{2}{x} - \frac{2\mathcal{A}x}{2!} - \frac{2\mathcal{B}x^3}{4!} - \frac{2\mathcal{C}x^5}{6!} - \frac{2\mathcal{D}x^7}{8!} - \dots,$$

we obtain

$$\csc x = \frac{1}{x} + \frac{2(2-1)\mathcal{A}x}{2!} + \frac{2(2^3-1)\mathcal{B}x^3}{4!} + \frac{2(2^5-1)\mathcal{C}x^5}{6!} + \dots.$$

VIII.1. The recurrence relation to obtain B_{14} is $(1+B)^{15} - B^{15} = 0$. Since $B_1 = -\frac{1}{2}$, $B_3 = B_5 = \dots = B_{13} = 0$, this gives

$$1 + 15B_1 + 105B_2 + 1365B_4 + 5005B_6 + 6435B_8 + 3003B_{10} + 455B_{12} + 15B_{14} = 0.$$

Using the values

$$B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730}, \dots$$

from p.68, we obtain $B_{14} = \frac{7}{6}$.

VIII.2. The beginning Euler numbers are, from Notes, p.68,

$$E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad E_6 = -61, \quad E_8 = 1385.$$

The subsequent Euler numbers are generated from the relations

$$\begin{aligned} 0 &= E_{10} + 45E_8 + 210E_6 + 210E_4 + 45E_2 + E_0, \\ 0 &= E_{12} + 66E_{10} + 495E_8 + 924E_6 + 495E_4 + 66E_2 + E_0, \\ 0 &= E_{14} + 91E_{12} + 1001E_{10} + 3003E_8 + 3003E_6 + 1001E_4 + 91E_2 + E_0. \end{aligned}$$

From these, we obtain, in succession,

$$E_{10} = -50521, \quad E_{12} = 2702765, \quad E_{14} = -199360981.$$

VIII.3. According to Note, p.62,

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{2^{2k-1}}{(2k)!} \cdot |B_{2k}| \cdot \pi^{2k}.$$

We have

$$\begin{aligned} 1 + \frac{1}{3^{2k}} + \frac{1}{5^{2k}} + \frac{1}{7^{2k}} + \dots &= \sum_{n=1}^{\infty} \frac{1}{n^{2k}} - \sum_{n=1}^{\infty} \frac{1}{(2n)^{2k}} \\ &= \left(1 - \frac{1}{2^{2k}}\right) \sum_{n=1}^{\infty} \frac{1}{n^{2k}} \\ &= \frac{2^{2k} - 1}{2^{2k}} \cdot \frac{2^{2k-1}}{(2k)!} \cdot |B_{2k}| \cdot \pi^{2k} \\ &= \frac{2^{2k} - 1}{2(2k)!} \cdot |B_{2k}| \cdot \pi^{2k}. \end{aligned}$$

VIII.4. Making use of Notes, pp. 63–64,

$$\begin{aligned} \cot x &= \frac{1}{x} - \frac{2^2 \mathcal{A}x}{2!} - \frac{2^4 \mathcal{B}x^3}{4!} - \frac{2^6 \mathcal{C}x^5}{6!} - \frac{2^8 \mathcal{D}x^7}{8!} - \dots, \\ \tan x &= \frac{2^2(2^2 - 1)\mathcal{A}x}{2!} + \frac{2^4(2^4 - 1)\mathcal{B}x^3}{4!} + \frac{2^6(2^6 - 1)\mathcal{C}x^5}{6!} + \frac{2^8(2^8 - 1)\mathcal{D}x^7}{8!} + \dots, \end{aligned}$$

we have

$$x \cot x = 1 - \frac{2^2 \mathcal{A}x^2}{2!} - \frac{2^4 \mathcal{B}x^4}{4!} - \frac{2^6 \mathcal{C}x^6}{6!} - \frac{2^8 \mathcal{D}x^8}{8!} - \dots,$$

¹⁰¹ and follow Theorem 1 of Notes, p.74 to obtain

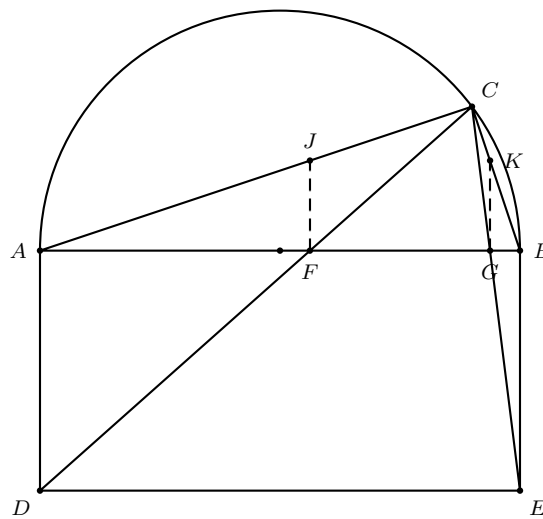
$$\frac{4}{\pi} = 1 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \frac{7^2}{2 + \dots}}}}$$

From this,

$$\frac{\pi}{4} = \frac{1}{1 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \dots}}}}$$

X.1.

Solution. Let the perpendiculars to AB at F and G intersect AC and BC respectively at J and K respectively.



(i) By the similarity of triangles CJF and CAD , and of CFG and CDE , we have

$$AD : JF = CD : CF = DE : FG.$$

Similarly,

$$BE : KG = DE : FG.$$

¹⁰¹In Paper 71, Euler pointed out that this expression for $\frac{\pi}{4}$ was given by Brouncker.

It follows that $JF = GK$. Also, $JF : FG = AD : DE$.

Now, the right triangles AJF and KBG are similar. $JF : AF = GB : KG = GB : JF$. Therefore,

$$\begin{aligned} AG^2 + BF^2 &= (AF + FG)^2 + (GB + FG)^2 \\ &= AF^2 + GB^2 + 2(AF + FG + GB)FG \\ &= (AF + GB)^2 - 2AF \cdot GB + 2AB \cdot FG \\ &= (AB - FG)^2 - 2FG^2 + 2AB \cdot FG \\ &= AB^2 + FG^2 - 2JF^2. \end{aligned}$$

(a) is clear now. If $DE = \sqrt{2}AD$, then $FG = \sqrt{2}JF$, and $AG^2 + BF^2 = AB^2$.

(b) If $AD = DE$, then $JF = FG$, and $AF \cdot GB = JF^2 = FG^2$.

X.2. (a) 84. The altitudes are $\frac{2 \cdot 84}{13} = \frac{168}{13}$, $\frac{2 \cdot 84}{13} = \frac{168}{13}$, and $\frac{2 \cdot 84}{15} = \frac{56}{5}$.

(b) 420. The altitudes are $\frac{840}{25} = \frac{168}{5}$, $\frac{840}{34} = \frac{420}{17}$, and $\frac{840}{39} = \frac{280}{13}$.

Remark. Since none of the altitudes is an integer, this Heronian triangle cannot be constructed by glueing two Pythagorean triangles along a common side.

(c) The area is 36.

Remark. Fermat, however, has shown that there is no Pythagorean triangle whose area is a square.

X.3. The next triangle has $b_4 = 4 \cdot 52 - 14 = 194$. It is the Heron triangle (193, 194, 195 with area 16296.

Assuming the areas also satisfy a second order linear recurrence relation of the form

$$\Delta_{n+2} = p\Delta_{n+1} + q\Delta_n.$$

Then from $\Delta_1 = 6$, $\Delta_2 = 84$, $\Delta_3 = 1170$, and $\Delta_4 = 16296$, we have

$$\begin{aligned} 84p + 6q &= 1170, \\ 1170 + 84q &= 16296. \end{aligned}$$

Solving these, $p = 14$, $q = -1$.

XI.1 (a) The necessity part (\implies) is trivial.

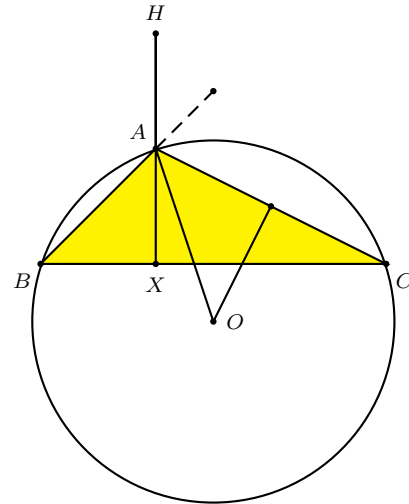
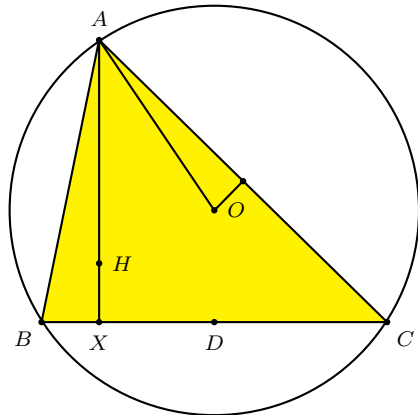
Suppose two of the points H , G , and O coincide. Since $HG : GO : HO = 2 : 1 : 3$, the three points all coincide. This means each altitude is also a median, and is therefore the perpendicular bisector of a side. From this, any two sides are equal in length, and the triangle is equilateral.

(b) A triangle with incenter on the Euler line is necessarily isosceles. To justify this, we make use of the result of XI.2. Suppose A and B are acute angles. Let H be the orthocenter. Then $AH = 2R \cos A$. Since AH and AO are symmetric about the bisector of angle A , $HI : OI = AH : AO = 2R \cos A : R$. Likewise, for angle B , we have $HI : OI = 2R \cos B : R$. From this it follows that $\cos A = \cos B$, and the triangle is isosceles.

XI.2. Let O and H be respectively the circumcenter and orthocenter of triangle ABC . Show that the bisector of angle BAC also bisects angle OAH .

In both cases, $\angle AOE = \angle ABC$. It follows that if $\angle A$ is acute, $\angle BAH = \angle EAO$, and the lines AH and AO are symmetric about the bisector of angle A .

If $\angle A$ is obtuse, then $\angle HAZ = \angle BAX = 90^\circ - \angle B = \angle OAE$. It follows that AH and AO are symmetric about the *external* bisector of angle A .

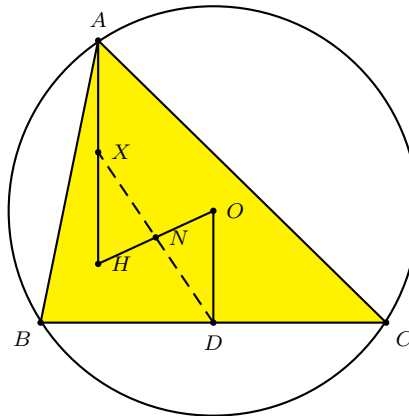
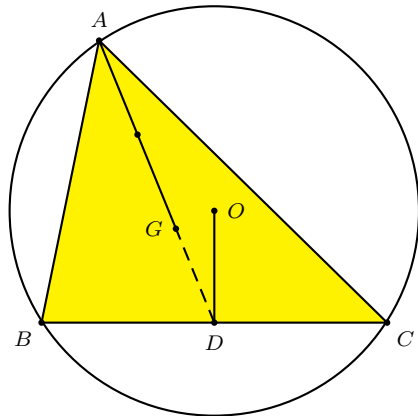


XI.3. Use the relations given in Notes, p.90.

XII.1. (a) (i) Construct the circle, center O , passing through A . This is the circumcircle.

(ii) Extend AG to D such that $AG : GD = 2 : 1$.

(iii) Construct the perpendicular to OD at D , to intersect the circumcircle at B and C , the remaining two vertices of the required triangle.



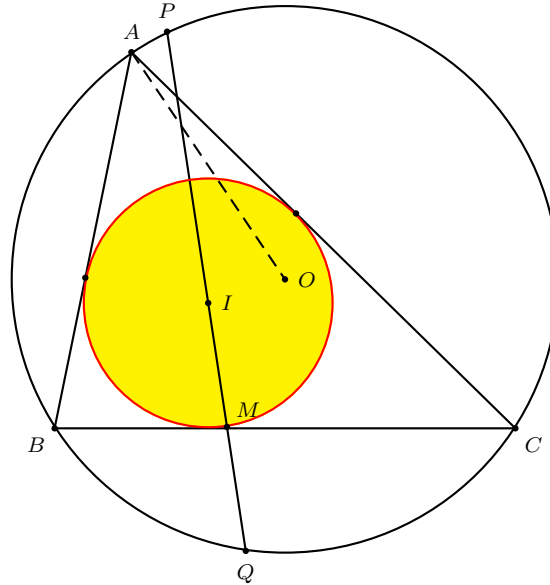
(b) (i) Construct the circle, center O , passing through A . This is the circumcircle.

(ii) Mark the midpoints X and N of the segments AH and OH respectively.

(iii) Extend XN to D so that $XN = ND$.

(iv) Construct the perpendicular to OD at D , to intersect the circumcircle at B and C , the remaining two vertices of the required triangle.

- (c) (i) Construct the circle, center O , passing through A . This is the circumcircle.
 (ii) Mark a point P on the circumcircle such that IP has the same length as the radius of the circumcircle.
 (iii) Extend PI to intersect the circumcircle again at Q , and mark the midpoint of the segment OQ .
 (iv) Construct the circle, center I , passing through M . This is the incircle, since the radius ρ of this circle satisfies $OI^2 = R(R - 2\rho)$.
 (v) Construct the tangents from A to the incircle, and let these tangents intersect the circumcircle again at B and C , the remaining two vertices of the required triangle.



XII.2. If triangle ABC is isosceles, then $a : b : c = f : f : 2f - 3h$, where $f = IO$ and $h = IG$. (See Notes, p.96). By the triangle inequality, h must be positive. This means that I is between G and H .

XII.3. If triangle OIH is isosceles, the lengths of the sides of triangle ABC are

$$a = \frac{\sqrt{3}(k^2 - 2f^2)}{2\sqrt{4f^2 - k^2}} + \frac{k}{2}, \quad b = \frac{\sqrt{3}(k^2 - 2f^2)}{2\sqrt{4f^2 - k^2}} - \frac{k}{2}, \quad c = \frac{\sqrt{3}f^2}{\sqrt{4f^2 - k^2}}$$

as given in Notes, p.98. Here,

$$\begin{aligned} a^2 + b^2 - ab &= 2\left(\frac{\sqrt{3}(k^2 - 2f^2)}{2\sqrt{4f^2 - k^2}}\right)^2 + 2\left(\frac{k}{2}\right)^2 - \left(\frac{\sqrt{3}(k^2 - 2f^2)}{2\sqrt{4f^2 - k^2}}\right)^2 + \left(\frac{k}{2}\right)^2 \\ &= \left(\frac{\sqrt{3}(k^2 - 2f^2)}{2\sqrt{4f^2 - k^2}}\right)^2 + 3\left(\frac{k}{2}\right)^2 \\ &= \frac{3(k^2 - 2f^2)^2}{4(4f^2 - k^2)} + \frac{3k^2}{4} = \frac{3[(k^2 - 2f^2)^2 + k^2(4f^2 - k^2)]}{4(4f^2 - k^2)} = \frac{3f^4}{4f^2 - k^2} = c^2. \end{aligned}$$

From the cosine formula, the angle between a and b is 60° .

XIII.1. (a) A prime divisor of $2^{29} - 1$ is of the form $58k + 1$. For $k = 1, 4$, these are 59 and 233 respectively. Now, 233 divides 536870911. This gives a prime smaller than the one Euler found, namely, 1103.

Remark: Since $536870911 = 233 \times 1103 \times 2089$, and it is easy to check that 2089 is a prime, this gives a complete factorization of M_{29} .

(b) A prime divisor of $M_{83} = 2^{83} - 1$ is of the form $166k + 1$. Now,

$$2^{83} - 1 = 9671406556917033397649407 = 167 \times 57912614113275649087721.$$

Remark. (1) To check that M_{83} is indeed divisible by 167, we make use of the following

k	1	2	4	8	16	32	64
$2^k \pmod{167}$	2	4	16	89	72	7	49

Since $83 = 64 + 16 + 2 + 1$,

$$2^{83} \equiv 49 \times 72 \times 4 \times 2 \equiv 1 \pmod{167}.$$

(2) It is much harder to confirm that the second factor is a prime.

XIII.2. Suppose n is not a power of 2, it must have an odd divisor. Write $n = mk$ for an odd number $k > 1$. Then $a^n + 1 = (a^m)^k + 1$ is divisible by the *smaller* number $a^m + 1 > 1$. This shows that $a^n + 1$ cannot be prime.

XIII.3. If N is an odd square, it has exactly an odd number of divisors, each of which is odd. The sum of divisors is therefore an odd number, and cannot be $2N$. This means that N cannot be a perfect number.

XIV.1. (a) Since $100 = 2^2 \cdot 5^2$ and $\int 2^2 = 1 + 2 + 4 = 7$, $\int 5^2 = 1 + 5 + 25 = 31$, by the multiplicative property of the sum of divisor function,

$$\int 100 = \int 2^2 \cdot \int 5^2 = 7 \cdot 31 = 217.$$

(b) Making use of Euler's extraordinary relation, and the data on Note, p.113, we have

$$\begin{aligned} \int 100 &= \int 99 + \int 98 - \int 95 - \int 93 + \int 88 + \int 85 - \int 78 - \int 74 + \int 65 + \int 60 - \int 49 - \int 43 \\ &\quad + \int 30 + \int 23 - \int 8 - \int 0 \\ &= 156 + 171 - 120 - 128 + 180 + 108 - 168 - 114 + 84 + 168 - 57 - 44 + 72 + 24 - 15 - 100 \\ &= 963 - 746 = 217. \end{aligned}$$

XV.1. First of all, $\sqrt{67} = 8 + \frac{1}{a}$. Following Euler's calculation, we have

$$\begin{aligned}
 \text{I. } a &= \frac{1}{\sqrt{67}-8} = \frac{\sqrt{67}+8}{3} = 5 + \frac{1}{b} \\
 \text{II. } b &= \frac{3}{\sqrt{67}-7} = \frac{3(\sqrt{67}+7)}{18} = \frac{\sqrt{67}+7}{6} = 2 + \frac{1}{c} \\
 \text{III. } c &= \frac{6}{\sqrt{67}-5} = \frac{6(\sqrt{67}+5)}{42} = \frac{\sqrt{67}+5}{7} = 1 + \frac{1}{d} \\
 \text{IV. } d &= \frac{7}{\sqrt{67}-2} = \frac{7(\sqrt{67}+2)}{63} = \frac{\sqrt{67}+2}{9} = 1 + \frac{1}{e} \\
 \text{V. } e &= \frac{9}{\sqrt{67}-7} = \frac{9(\sqrt{67}+7)}{18} = \frac{\sqrt{67}+7}{2} = 7 + \frac{1}{f} \\
 \text{VI. } f &= \frac{2}{\sqrt{67}-7} = \frac{2(\sqrt{67}+7)}{18} = \frac{\sqrt{67}+7}{9} = 1 + \frac{1}{g} \\
 \text{VII. } g &= \frac{9}{\sqrt{67}-2} = \frac{9(\sqrt{67}+2)}{63} = \frac{\sqrt{67}+2}{7} = 1 + \frac{1}{h} \\
 \text{VIII. } h &= \frac{7}{\sqrt{67}-5} = \frac{7(\sqrt{67}+5)}{42} = \frac{\sqrt{67}+5}{6} = 2 + \frac{1}{i} \\
 \text{IX. } i &= \frac{6}{\sqrt{67}-7} = \frac{6(\sqrt{67}+7)}{18} = \frac{\sqrt{67}+7}{3} = 5 + \frac{1}{k} \\
 \text{X. } k &= \frac{3}{\sqrt{67}-8} = \frac{3(\sqrt{67}+8)}{3} = \sqrt{67} + 8 = 16 + \frac{1}{\ell} \\
 \text{XII. } \ell &= \frac{1}{\sqrt{67}-8}.
 \end{aligned}$$

From these,

$$\sqrt{67} = [8, \overline{5, 2, 1, 1, 7, 1, 1, 2, 5, 16}].$$

XV.2. $\sqrt{n^2 + 2n - 1} = n + \frac{1}{a}$ where

$$\begin{aligned}
 a &= \frac{1}{\sqrt{n^2+2n-1}-n} = \frac{\sqrt{n^2+2n-1}+n}{2n-1} = 1 + \frac{1}{b} \\
 b &= \frac{2n-1}{\sqrt{n^2+2n-1}-n+1} = \frac{\sqrt{n^2+2n-1}+n-1}{2} = (n-1) + \frac{1}{c} \\
 c &= \frac{2}{\sqrt{n^2+2n-1}-n+1} = \frac{\sqrt{n^2+2n-1}+n-1}{2n-1} = 1 + \frac{1}{d} \\
 d &= \frac{2n-1}{\sqrt{n^2+2n-1}-n} = \sqrt{n^2 + 2n - 1} + n = 2n + \frac{1}{e} \\
 e &= \frac{1}{\sqrt{n^2+2n-1}-n}
 \end{aligned}$$

At this point, the pattern recurs, and we obtain the continued fraction expansion of

$$\sqrt{n^2 + 2n - 1} = [n, \overline{1, n-1, 1, 2n}].$$

XV.3. We compute the convergents of the continued fraction of $\sqrt{31}$.

$$\begin{array}{cccccccccccc}
 & & 5 & 1 & 1 & 3 & 5 & 3 & 1 & 1 & 10 & \dots \\
 0 & 1 & 5 & 6 & 11 & 39 & 206 & 657 & 863 & 1520 & & \\
 1 & 0 & 1 & 1 & 2 & 7 & 37 & 118 & 155 & 273 & &
 \end{array}$$

Here, $p = 1522$ and $q = 273$ satisfy

$$p^2 - 31q^2 = 2310400 - 31 \cdot 74529 = 2310400 - 2310399 = 1.$$

$(p, q) = (1520, 273)$ is the smallest positive solution of the given Pell equation.