

Conic Solution of Euler's Triangle Determination Problem

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Abstract. We study Euler's problem of determination of a triangle from its circumcenter, orthocenter, and incenter as a problem of geometric construction. While it cannot be solved using ruler and compass, we construct the vertices by intersecting a circle with a rectangular hyperbola, both easily constructed from the given triangle centers.

Keywords: circumcenter, orthocenter, incenter, Feuerbach point, rectangular hyperbola

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1. Euler's triangle determination problem

Leonhard EULER studied in his famous paper [2] the problem of determining a triangle ABC from its circumcenter O , orthocenter H , and incenter I . Euler constructed a cubic polynomial whose roots are the lengths of the sides, and whose coefficients are rational functions of the distances among the three given triangle centers. He showed that when $OI = IH$, the cubic polynomial factors nontrivially, and gave the roots explicitly. With a numerical example, Euler showed that in the general case, the solution reduces to the trisection of an angle. In this note, we address Euler's determination problem as a construction problem. While the problem cannot be solved with the traditional restriction to ruler and compass, we shall nevertheless give the vertices as the intersections of two conics, one a circle and the other a rectangular hyperbola easily constructed from O , H , and I .

Let G and N be the points which divide OH in the ratio

$$OG : GN : NH = 2 : 1 : 3.$$

These are the centroid and the nine-point center of the required triangle (if it exists). A necessary and sufficient condition for the existence of ABC is given by the following theorem. For details, see [4, 5, 6, 7].

Theorem 1 [A. GUINAND] *Let \mathcal{D} be the open circular disk with diameter HG . A triangle ABC exists with circumcenter O , orthocenter H , and incenter I if and only if $I \in \mathcal{D} \setminus \{N\}$.*

2. Ruler and compass construction of circumcircle and incircle

Let R and r denote respectively the circumradius and inradius of triangle ABC . In [2], EULER established, among other things, the famous relation

$$OI^2 = R(R - 2r). \quad (1)$$

With the help of the famous Feuerbach theorem [3], discovered half a century after Euler's paper [2], that the nine-point circle of a triangle is tangent internally to the incircle, we can easily construct the circumcircle, the incircle, the nine-point circle, and their point of tangency. According to the Feuerbach theorem,

$$NI = \frac{R}{2} - r. \quad (2)$$

Together with (1), this gives $2R \cdot NI = OI^2$, and suggests the following ruler-and-compass construction. See also [5], which studies the same problem from a paper-folding approach.

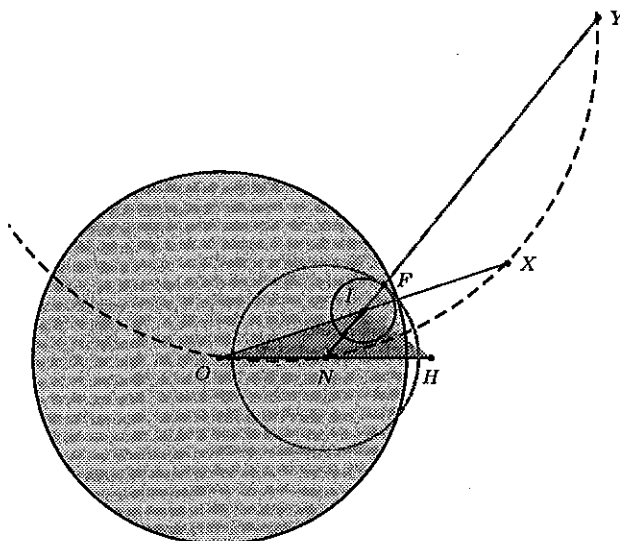


Figure 1: Incircle from O, H, I

Construction 2 Suppose O, H, I satisfy $I \in \mathcal{D} \setminus \{N\}$.

- (1) Extend OI to X such that $OX = 2OI$; construct the circle ONX , and extend NI to intersect the circle again at Y . The length of IY is twice the circumradius, and four times the radius of the nine-point circle.
- (2) Construct the circumcircle (O) and the nine-point circle (N).
- (3) Construct the intersection F of the circle (N) with the half line NI , and the circle, center I , passing through F . This is the incircle and F is the (Feuerbach) point of tangency with the nine-point circle (see Fig. 1).

Starting with an arbitrary point A on (O), by drawing tangents, we can complete a triangle ABC with incircle (I) and circumcircle (O). The locus of the orthocenter of the variable triangle ABC is the circle, center P , passing through H . We determine the specific triangle with H as orthocenter. First we consider a special case where the constructibility with ruler and compass is evident from EULER's calculations.

3. The case $OI = IH$

If $OI = IH$, consider the intersection A of the half line NI with the circumcircle (see Fig. 2). We complete a triangle ABC with (O) as circumcircle and (I) as incircle. The orthocenter of triangle ABC lies on the reflection of AO in the line AI . This is the line AH . Since angle AHN is a right angle, for the midpoint M of AH , we have $NM = \frac{1}{2} AH = \frac{1}{2} AO = \frac{R}{2}$. This means that M is a point on the nine-point circle. It is the midpoint of the segment joining A to the orthocenter. It follows that the orthocenter must be H .

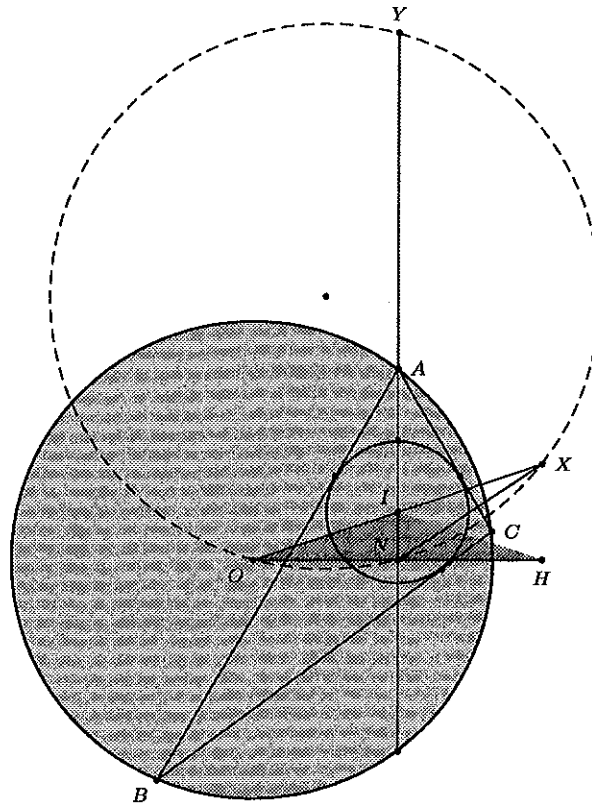


Figure 2: Triangle from O, H, I with $OI = IH$

Remark: One referee has kindly pointed us to [1], which shows that in this case the angle BAC must be $\frac{\pi}{3}$, and the Euler line cuts off an equilateral triangle with the sides AB and AC .

4. The general case: construction with the aid of a conic

Set up a cartesian coordinate system with origin at O . Assume $H = (k, 0)$ and $I = (p, q)$. Since $\angle HAI = \angle OAI$, and likewise for B and C , the vertices A, B, C are on the locus of the point P for which $\angle HPI = \angle OPI$. If $P = (x, y)$, a routine calculation shows that the locus of P is a curve K : $K(x, y) = 0$, where

$$\begin{aligned}
 K(x, y) := & 2qx^3 - (2p - k)x^2y + 2qxy^2 - (2p - k)y^3 \\
 & - 2(p + k)qx^2 + 2(p^2 - q^2)xy + 2(p - k)qy^2 \\
 & + 2kpqx - k(p^2 - q^2)y.
 \end{aligned}$$

Note that $K(k, 0) = 0$, i.e., \mathcal{K} contains the point H .

By computing the circumradius R , we easily obtain the equation of the circumcircle $(O) : G(x, y) = 0$, where

$$G(x, y) := x^2 + y^2 - \frac{(p^2 + q^2)^2}{(2p - k)^2 + 4q^2}.$$

It is possible to find a linear function $L(x, y)$ such that

$$Q(x, y) := K(x, y) - L(x, y)G(x, y)$$

does not contain third degree terms. For example, by choosing

$$L(x, y) := 2qx - (2p - k)y - 2qk,$$

we have

$$\begin{aligned} Q(x, y) &= -2pqx^2 + 2(p^2 - q^2)xy + 2pqy^2 \\ &\quad - \frac{k^2(k - 4p)(p^2 - q^2) + k(3p^2 - 5q^2)(p^2 + q^2) + 2p(p^2 + q^2)^2}{(2p - k)^2 + 4q^2} \cdot x \\ &\quad + \frac{2q(kp((2p - k)^2 + 4q^2) + (p^2 + q^2)^2)}{(2p - k)^2 + 4q^2} \cdot y - \frac{2k(p^2 + q^2)^2q}{(2p - k)^2 + 4q^2}. \end{aligned}$$

The finite intersections of (O) with \mathcal{K} are precisely the same with the conic \mathcal{C} defined by $Q(x, y) = 0$. Note that the coefficients of x and y in L are dictated by the elimination of the third degree terms in $K - L \cdot G$. We have chosen the constant term such that $L(k, 0) = 0$, so that $L(x, y) = 0$ represents the line HP parallel to NI . It follows that $Q(k, 0) = 0$, and the conic \mathcal{C} contains the vertices and the orthocenter of the required triangle ABC . It is necessarily a rectangular hyperbola. This fact also follows from the factorization of the quadratic part of Q , namely, $-2(px + qy)(qx - py)$. This means that \mathcal{C} is a rectangular hyperbola whose asymptotes have slopes q/p and $-p/q$. These are parallel and perpendicular to the segment OI .

To construct the rectangular hyperbola \mathcal{C} , we identify its center O' . This is the point with coordinates (u, v) for which the quadratic polynomial $Q(x - u, y - v)$ has no first degree terms in x and y . A routine calculation gives

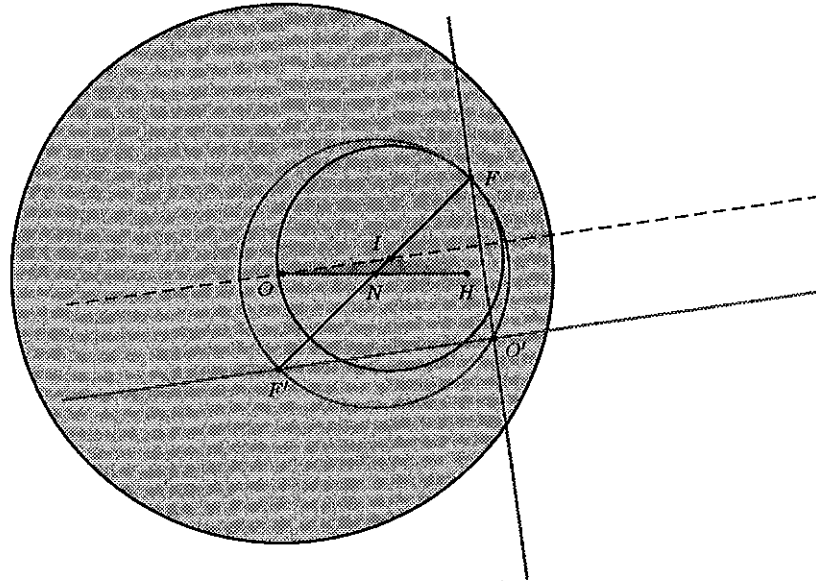
$$\begin{aligned} O' &= \left(\frac{k((2p - k)^2 - p^2 + 5q^2) + 2p(p^2 + q^2)}{2((2p - k)^2 + 4q^2)}, \frac{(p^2 + q^2 - kp)q}{(2p - k)^2 + 4q^2} \right) \\ &= \left(\frac{k}{2} + \frac{(2p - k)(p^2 + q^2) + 2kq^2}{2((2p - k)^2 + 4q^2)}, \frac{(p^2 + q^2)q - kpq}{(2p - k)^2 + 4q^2} \right). \end{aligned} \quad (3)$$

Since \mathcal{C} is a rectangular hyperbola, its center O' lies on the nine-point circle (N) . The following observation leads to a very simple construction of the center.

Proposition 3 *The Feuerbach point F lies on the asymptote perpendicular to OI .*

Proof: The nine-point circle has the equation

$$\left(x - \frac{k}{2}\right)^2 + y^2 - \frac{(p^2 + q^2)^2}{4((2p - k)^2 + 4q^2)} = 0. \quad (4)$$

Figure 3: Center O' of rectangular hyperbola \mathcal{C}

This intersects the line NI at two points, the Feuerbach point and its antipode (on the nine-point circle). The Feuerbach point is the point

$$F = \left(\frac{k}{2} + \frac{(2p-k)(p^2+q^2)}{2((2p-k)^2+4q^2)}, \frac{(p^2+q^2)q}{(2p-k)^2+4q^2} \right).$$

It is easy to see that the line $O'F$ has slope $-\frac{p}{q}$, and is perpendicular to the line OI . \square

Corollary 4 *The center O' of the rectangular hyperbola \mathcal{C} is the second intersection of the nine-point circle (N) with the perpendicular from F to OI (Fig. 3).*

It is well known that if \mathcal{C} is a rectangular hyperbola passes through the vertices of a triangle ABC , its fourth intersection with the circumcircle at the reflection of the orthocenter of the triangle in the center of the hyperbola. Therefore, one of the intersections of \mathcal{C} with the circle (O) is the reflection of H in O' . The other three are the vertices of the required triangle (see Fig. 4).

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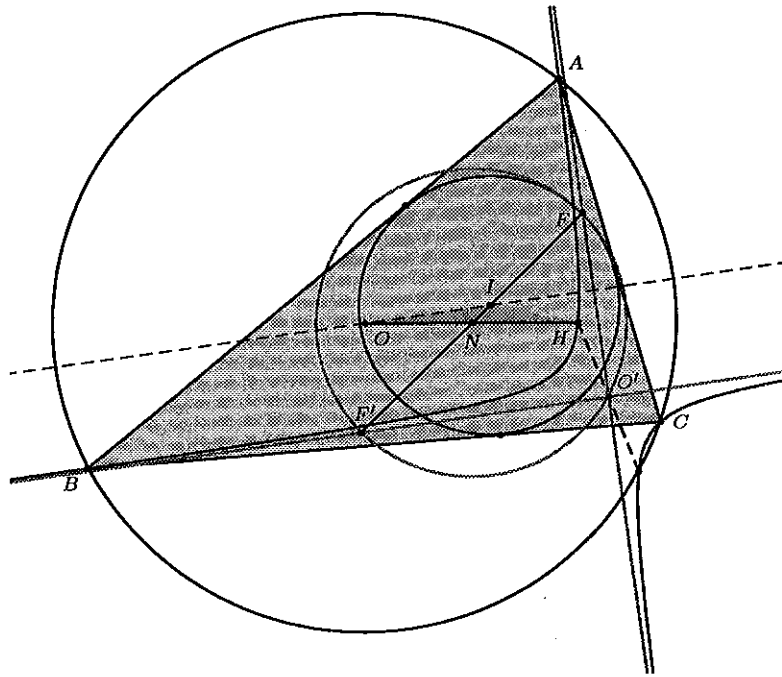


Figure 4: Triangle ABC with given O, H, I

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