A Tour of Triangle Geometry

Paul Yiu
Department of Mathematical Sciences
Florida Atlantic University

1. Introduction

Interests in triangle geometry have been rekindled in recent years with the availability of dynamic software such as the Geometer's Sketchpad and Cabri for geometric constructions. In this paper we outline some interesting results with illustrations made by such software. We shall center around the notions of reflection and isogonal conjugation, and introduce a number of interesting triangle centers, [1] lines, conics, and a few cubic curves. [2] Many results in triangle geometry can be discovered from such dynamic sketches, and proved either synthetically or by calculations. Although we do not present any proof, all results without references have been confirmed by calculations using barycentric coordinates. [3] The reader is invited to reproduce the figures in this paper as dynamic sketches using computer software, and to discover further results.

The conics in this paper are constructed by the five-point-conic command available in both Geometer's Sketchpad and Cabri. The location of the center of a rectangular hyperbola will be treated in detail in §**??** below. In

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[1]Most of the triangle centers in this paper appear in Kimberling's *Encyclopedia of Triangle Centers* [10], hereafter ETC. For example, the centroid, circumcenter, orthocenter, and incenter appear as X_2, X_3, X_4, and X_1 respectively. We shall reference occurrences in ETC of other triangle centers in footnotes.

[2]Most of the questions we consider are about the concurrency of three lines. We say that two triangles are perspective if the lines joining their corresponding vertices are concurrent. The point of concurrency is called the perspector of the triangles.

[3]For an introduction to the use of barycentric coordinates in triangle geometry, see [14, 15].

§10 we give a number of ruler-and-compass constructions for conics, which can be incorporated into efficient tools of the dynamic software. [4]

Part I: Some Interesting Triangle Centers

1.1. *The classical centers.* The most well known of triangle centers are certainly the centroid, the circumcenter, the orthocenter, and the incenter. The existence of each of these is due to the concurrency of three lines, respectively the medians, the perpendicular bisectors, the altitudes, and the internal angle bisectors.

Figure 1 shows the cirumcenter O and the orthocenter H. Note that the lines OA and HA are isogonal (or symmetric) with respect the sides AB and AC; similarly for OB, HB, and OC, HC.

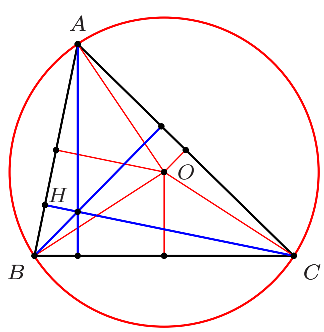

Figure 1

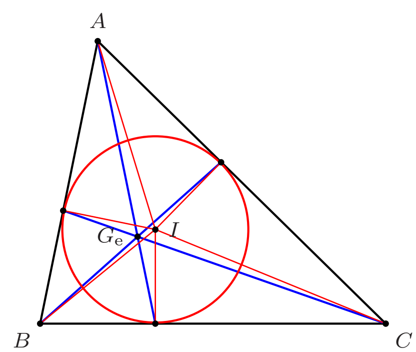

Figure 2

1.2. *The tritangent circles.* The tritangent circles are the one tangent to all three sidelines of triangle ABC. There are four of them, the incircle and the thee excircles.

The incircle touches each of the sides BC, CA, AB. Figure 2 shows the incenter I along with the Gergonne point G_e, the point of concurrency of the lines joining the point of tangency of the incircle with a side to the opposite vertex.

An excircle touches one side of the triangle and the extensions of the remaining two sides. The lines joining the points of tangency of the A-excircle with BC, the B-excircle with CA, and the C-excircle with AB, to the opposite vertices all pass through the Nagel point N_a.

1.3. *The symmedian point.*

[4]The intersections of conics and lines can be easily marked with Cabri, but not with the Geometer's Sketchpad. Figure 57, for example, is drawn with Cabri. Most of the sketches in this paper are drawn with the Geometer's Sketchpad.

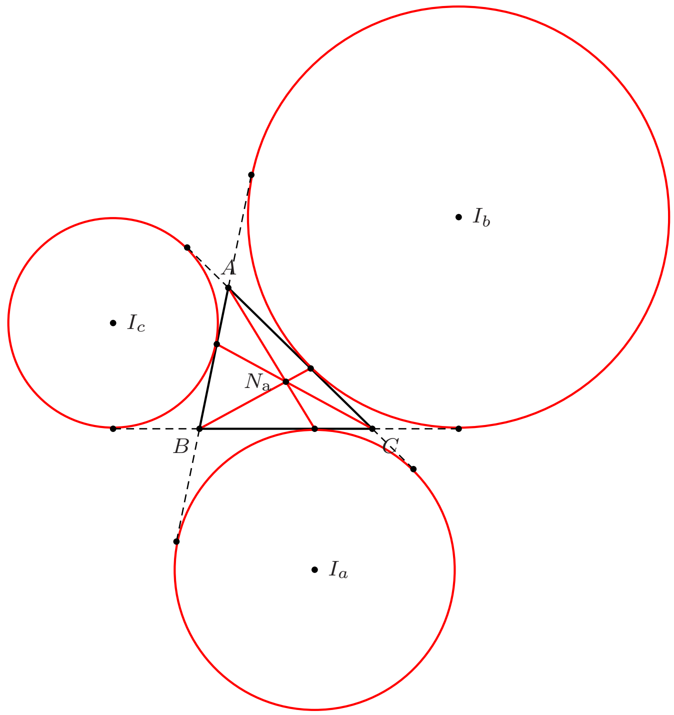

Figure 3. The excircles and the Nagel point

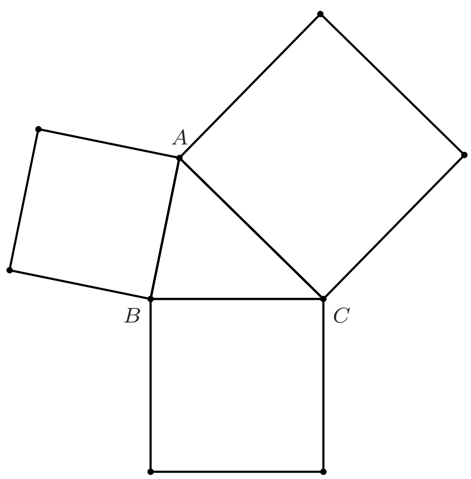

Figure 4.

2. Isogonal conjugates

Consider a point with reflections P_a, P_b, P_c in the sidelines BC, CA, AB. Let Q be a point on the line isogonal to AP with respect to angle

A, *i.e.*, the lines AQ and AP are symmetric with respect to the bisector of angle BAC. See Figure 5.

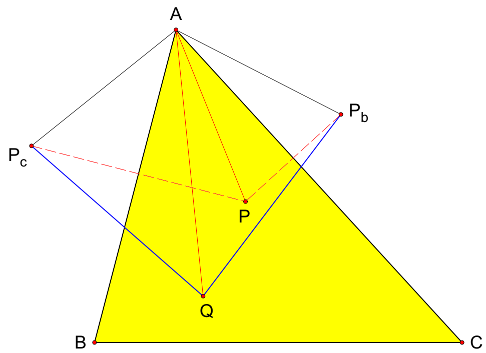

Figure 5.

It is easy to show that the triangles AQP_b and AQP_c are congruent, so that Q is equidistant from P_b and P_c. For the same reason, any point on a line isogonal to BP is equidistant from P_c and P_a. It follows that the intersection P^* of two lines isogonal to AP and BP is equidistant from the three reflections P_a, P_b, P_c. Furthermore, P^* is on a line isogonal to CP. For this reason, we call P^* the *isogonal conjugate* of P. It is the center of the circle of reflections of P. See Figure 6.

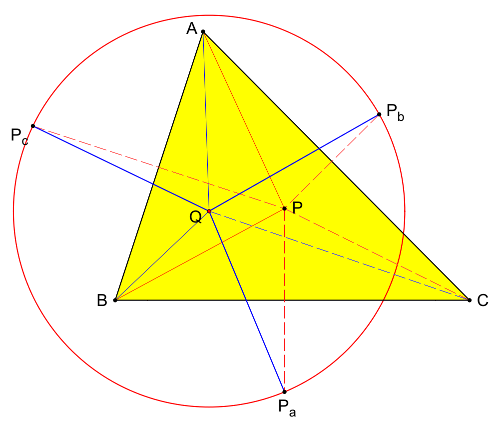

Figure 6.

Clearly, $(P^*)^* = P$. Moreover, the circles of reflections of P and P^* are congruent, since, in Figure 3, the trapezoid $PP^*P_a^*P_a$ being isosceles, $PP_a^* = P^*P_a$. It follows that the pedals of P and P^* on the sidelines all lie on the same circle with center the midpoint of PP^*. We call this the common **pedal circle** of P and P^*.

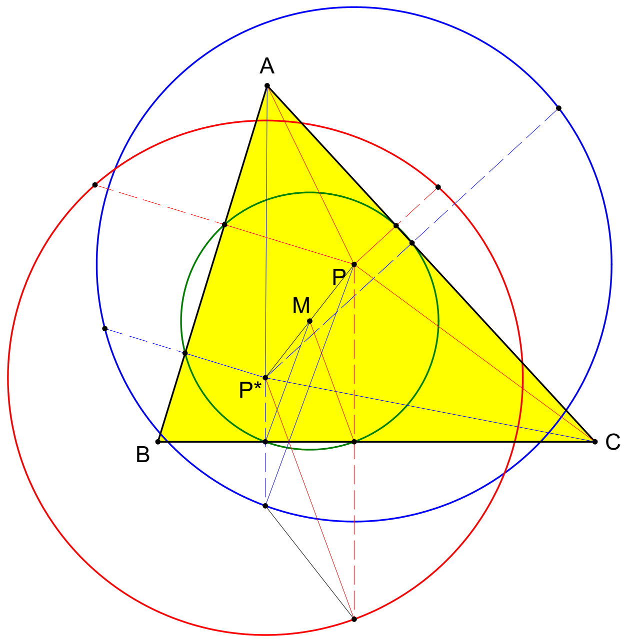

Figure 7.

2.1. *The circumcenter O and the orthocenter H.* The circumcenter O and the orthocenter H are isogonal conjugates. Since $OO_a = AH$, AOO_aH is a parallelogram, and $HO_a = AO$. See Figure 4. This means that the circle of reflections of O is congruent to the circumcircle. Therefore, the circle of reflections of H is the circumcircle, and the reflections of H lie on the circumcircle.

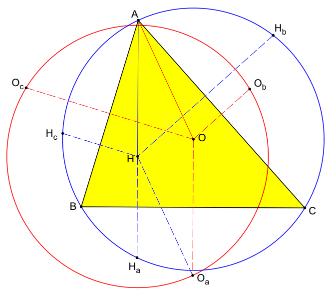

Figure 8.

2.1.1. *The nine-point circle and the Euler line.* The common pedal circle of O and H has center N at the midpoint of OH. [5] See Figure 5. This circle is usually called the *nine-point circle* since it also passes through the midpoints of the three segments AH, BH, and CH. The line containing O and H is called the *Euler line* of triangle ABC. It contains, apart from the nine-point center N, also the centroid G and the deLongchamps point L, [6] which is the reflection of H in O, and

$$HN : NG : GO : OL = 3 : 1 : 2 : 6.$$

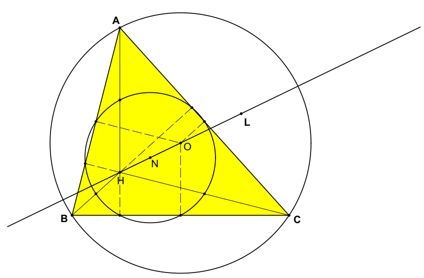

Figure 9.

[5]The nine-point center N is X_5 in ETC.
[6]The deLongchamps point is X_{20} in ETC.

2.2. *The Feuerbach point* F_e . The remarkable Feuerbach theorem asserts that the nine-point circle of a triangle is tangent internally to the incircle and externally to each of the excircles. The Feuerbach point F_e is the point of tangency with the incircle. [7]

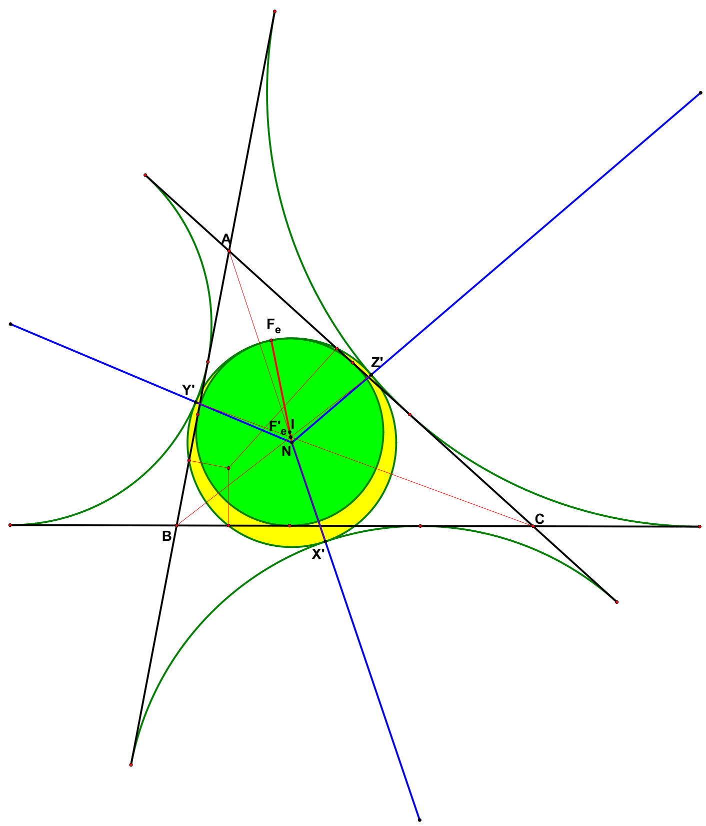

Figure 10.

If X', Y', Z' are the point of tangency of the nine-point circle with the excircles, then AX', BY', CZ' are concurrent at a point F_e' on the line joining I to N. [8]

[7]The Feuerbach point is X_{11} in ETC.
[8]This is the point X_{12} in ETC.

2.3. *The symmedian point K.* The isogonal conjugate of the centroid G is called the symmedian point K. [9] Honsberger [6, p.53] calls it "a crown jewel of modern geometry". It has many interesting properties.

2.3.1. *The symmedian point K and the tangential triangle.* The symmedian point K is the perspector of the tangential triangle: if $A'B'C'$ is the triangle bounded by the tangents to the circumcircle at the vertices, the lines AA', BB', CC' intersect at K. See Figure 11.

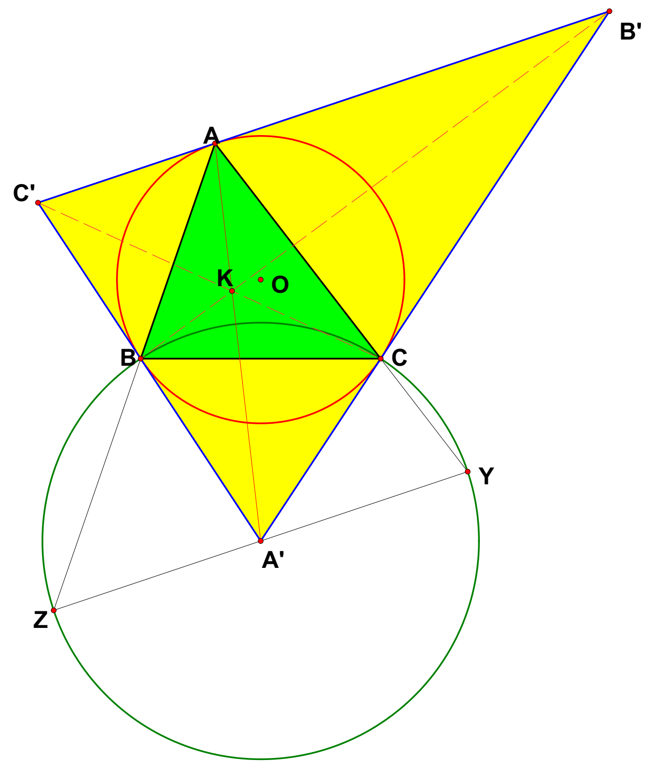

Figure 11.

The circumcenter of the tangential triangle is a point on the Euler line. [10] For further properties of K, see §9.1.

[9] The symmedian point is X_6 in ETC.
[10] This is the point X_{26} in ETC.

2.4. *The Gergonne point and the internal center of similitude of the circumcircle and incircle* . It is clear that the incenter is the isogonal conjugate of itself. The Gergonne point G_e is the perspector of the intouch triangle: if the incircle of triangle ABC is tangent to the sides BC, CA, AB at X, Y, Z respectively, then AX, BY, CZ are concurrent at G_e. [11] The isogonal conjugate of G_e is the internal center of similitude of the circumcircle and the incircle. [12] See Figure 12. The reflections $\begin{matrix} X' \\ Y' \\ Z' \end{matrix}$ of $\begin{matrix} X \\ Y \\ Z \end{matrix}$ in the bisectors of angles $\begin{matrix} A \\ B \\ C \end{matrix}$ all lie on the incircle. A simple calculation shows that the oriented angle between IY' and BC is $\frac{\pi}{2} - A$. Likewise, the oriented angle between IZ' and CB is also $\frac{\pi}{2} - A$. This means that IY' and IZ' are isogonal with respect to the line BC. Let R and r be respectively the circumradius and the inradius of triangle ABC. Since $IY' = IZ'$, $Y'Z'$ is parallel to BC and has length $2r \sin A$. Since $a = 2R \sin A$, the ratio of homothety is $r : R$. It follows that the isogonal conjugate of G_e is the internal center of similitude of the circumcircle and the circle.

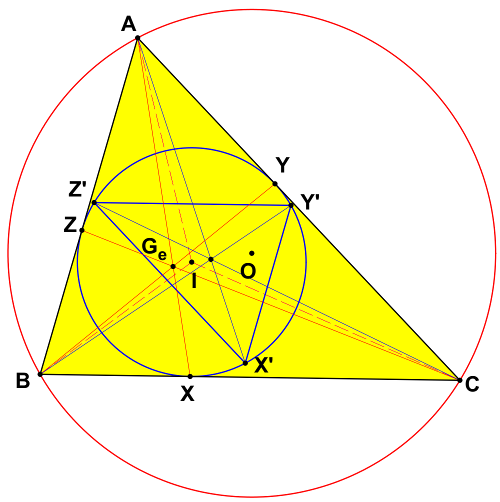

Figure 12.

If A', B', C' are the midpoints of the sides BC, CA, AB respectively, then the lines $A'X'$, $B'Y'$, $C'Z'$ intersect at the Feuerbach point F_e. [13]

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[11] The Gergonne point is X_7 in ETC.

[12] The internal center of similitude of the circumcircle and incircle is X_{55} in ETC.

[13] International Mathematical Olympiad, 1982, Problem 2.

2.5. *The Nagel point and the external center of similitude of the circumcircle and incircle.* The Nagel point $N_{\rm a}$ is the perspector of the extouch triangle: if the excircle on the side $\begin{matrix}BC\\CA\\AB\end{matrix}$ of triangle ABC is tangent to this side at $\begin{matrix}X\\Y\\Z\end{matrix}$, then AX, BY, CZ are concurrent at $N_{\rm a}$.[14] If $\begin{matrix}X'\\Y'\\Z'\end{matrix}$ is the reflection of $\begin{matrix}X\\Y\\Z\end{matrix}$ in the bisector of angle $\begin{matrix}A\\B\\C\end{matrix}$, the lines AX', BY', CZ' all intersect OI at the point which divides it in the ratio $R : -r$. This is the external center of similitude of the circumcircle and the circle, and is the isogonal conjugate of $N_{\rm a}$.[15]

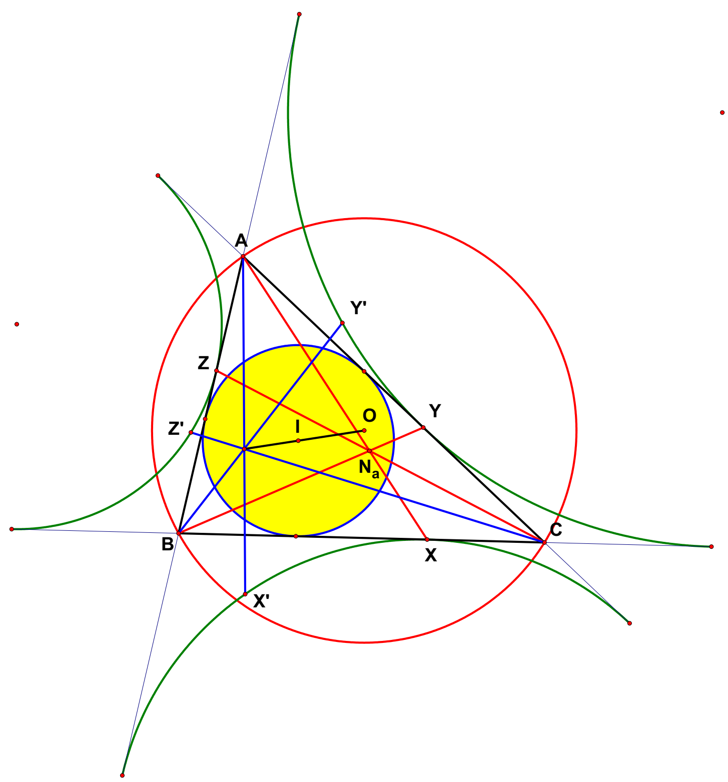

Figure 13.

[14]The Nagel point is X_8 in ETC.

[15]The external center of similitude of the circumcircle and incircle is X_{56} in ETC.

2.6. *Gergonne and Nagel points as isotomic conjugates* . The Gergonne point G_e and the Nagel point N_a form a pair of isotomic conjugates. See Figure 14 below. In general, let P be a point with traces X, Y, Z on the sidelines of ABC. If X', Y', Z' are the reflections of X, Y, Z in the midpoints of the respective sides, then the lines AX', BY', CZ' are concurrent at a point P' which we call the *isotomic conjugate* of P. Like isogonal conjugation, we have $(P')' = P$ for every point P. The centroid is clearly the isotomic conjugate of itself.

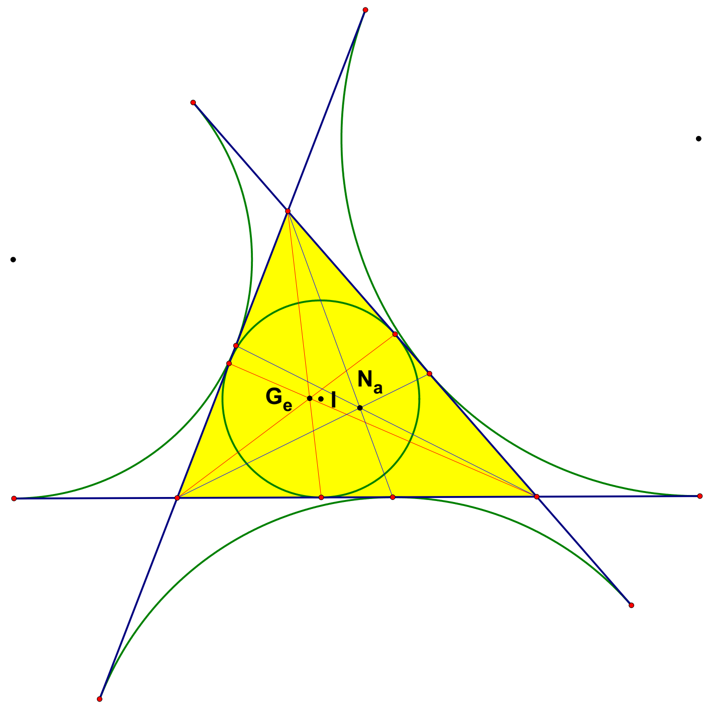

Figure 14.

2.7. *The isogonal conjugate of a point on the circumcircle.* To find the isogonal conjugate of a point P on the circumcircle, we reflect the lines AP, BP, CP in the respective angle bisectors. These reflected lines do not intersect at a finite point. They are parallel. We say that the isogonal conjugate of P is an infinite point (which is the common point of a family of parallel lines). See Figure 15.

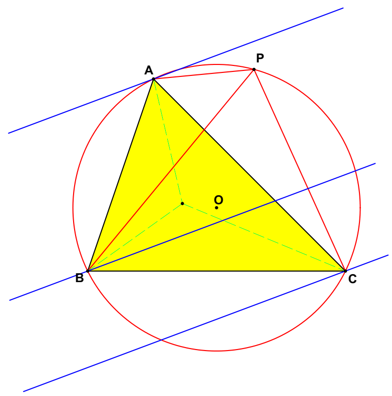

Figure 15. Isogonal conjugates of points on the circumcircle

2.8. *Isotomic conjugates of infinite points* . The isotomic conjugate of an infinite point lies on the circum-ellipse with center G. See Figure 16. This is called the Steiner circum-ellipse.

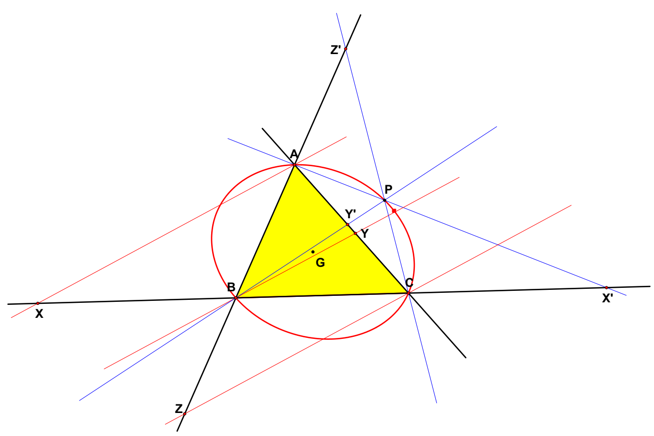

Figure 16.

3. Simson line and line of reflections

3.1. *Simson line.* Let P be a point on the circumcircle of triangle ABC, with pedals X, Y, Z on the sidelines. It is well known that X, Y, Z are collinear. The line containing them is the *Simson line* of P. The converse is also true: if the pedals of a point on the sidelines are collinear, then the point lies on the circumcircle. Now, the reflections of P in the sidelines are the images of the pedals under the homothety $\mathsf{h}(P, 2)$. It follows that the reflections of P are collinear if and only if P lies on the circumcircle. It is remarkable that the line of reflections (of a point on the circumcircle) always passes through the orthocenter H. (See [6, pp.43–46]).

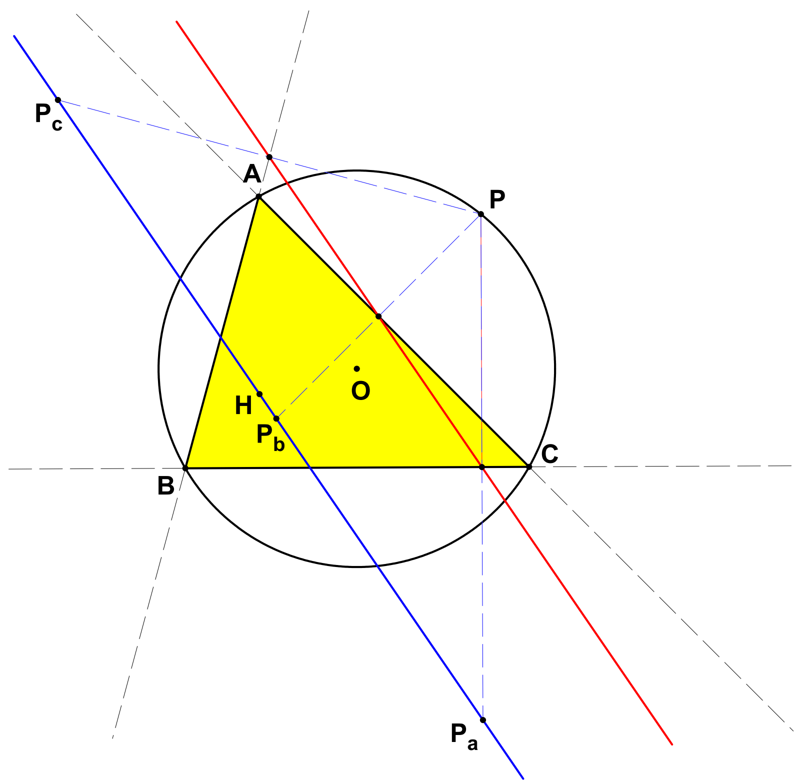

Figure 17. Simson and reflection lines

The Simson lines of antipodal points intersect orthogonally on the nine-point circle.

3.2. *Line of reflections and reflections of line.* Given a line ℓ through H, what is the point on the circumcircle whose line of reflections is ℓ ? This question is most elegantly answered by the following theorem of Collings and Longuet-Higgins [2, 11]: *The reflections of a line ℓ through H in the sidelines of triangle ABC intersect at a point F on the circumcircle whose line of reflections is ℓ.* See Figure 18.

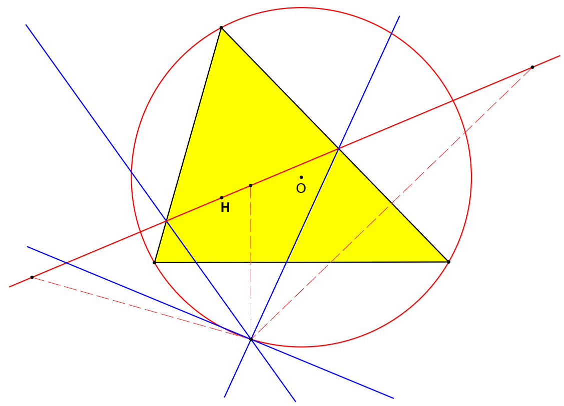

Figure 18.

In particular, the reflections of the Euler line OH in the sidelines intersect at the *Euler reflection point E* on the circumcircle. [16]

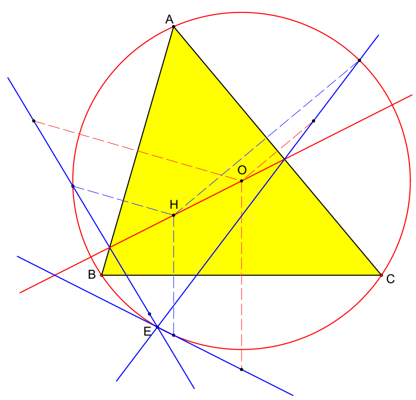

Figure 19.

[16]The Euler reflection point is X_{110} in ETC. It is also called the Kiepert focus. For more about this point, see [12].

3.2.1. Consider a line through the circumcenter O, intersecting the circumcircle at antipodal points Q and Q', and the sidelines BC, CA, AB at X, Y, Z respectively. The three circles with diameters AX, BY, CZ have two common points, T on the circumcircle and W on the nine-point circle. The point W is the intersection of the orthogonal Simson lines of Q and Q'. The line TW passes through the orthocenter H. See Figure 20.

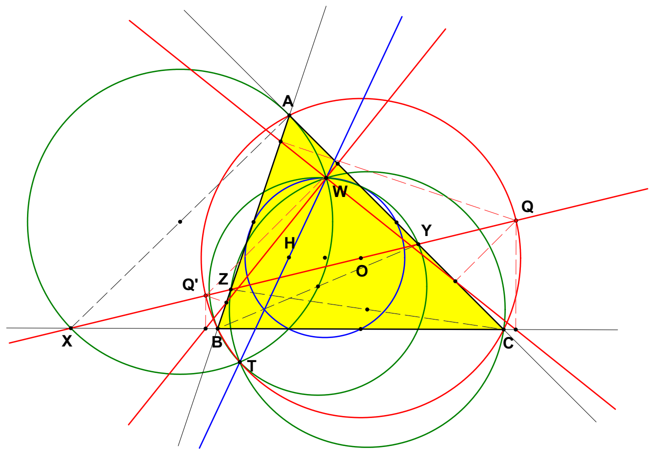

Figure 20.

The line TW is the line of reflections of the reflection of T in QQ'.

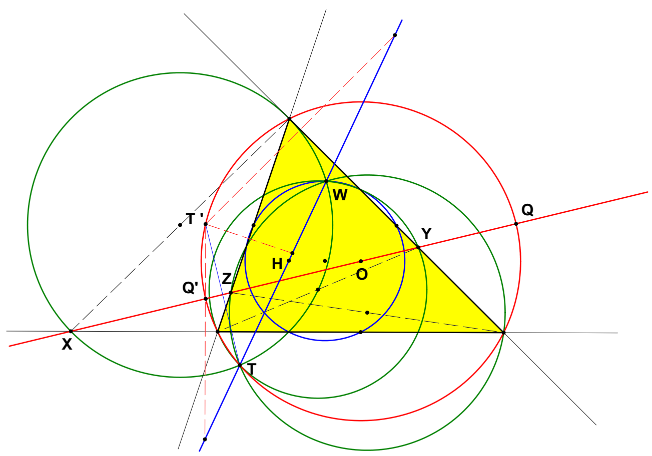

Figure 21.

4. Rectangular circum-hyperbolas

4.1. *Brianchon-Poncelet theorem.* A conic through the vertices of a triangle is a rectangular hyperbola if and only if it passes through the orthocenter of the triangle. In this case, the center of the conic lies on the nine-point circle. The perpendicular asymptotes of the rectangular hyperbola are the Simson lines of two antipodal points on the circumcircle. The conjugates of these points are infinite points, which are the infinite points of the hyperbola. The rectangular hyperbola can be considered as the locus of the isogonal conjugates of points on a line through the circumcenter. [17]

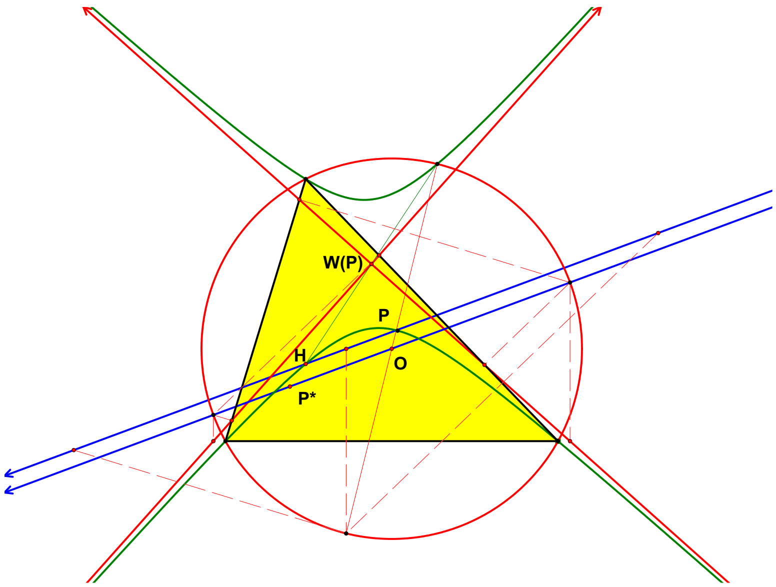

Figure 22.

4.1.1. Given a point P not on the sidelines and other than the orthocenter, we denote by $\mathcal{H}(P)$ the rectangular circum-hyperbola through P. It is the isogonal conjugate of the line OP^*. The fourth intersection of $\mathcal{H}(P)$ with the circumcircle is the isogonal conjugate of the infinite point of OP^*. It is the antipode of the orthocenter H on the hyperbola $\mathcal{H}(P)$. The antipode of this fourth intersection on the circumcircle is the intersection of the reflections of the line through H parallel to OP^* in the sidelines of triangle ABC. See Figure 22.

[17] More generally, the isogonal conjugate of a line is a circumconic (through the vertices) of triangle. It is an ellipse, a parabola, or a hyperbola according as the line intersects the circumcircle at 0, 1, 2 real points.

4.1.2. In §3.2.1, the line TW passes through orthocenter H. It intersects the circumcircle again at the fourth intersection of the rectangular circum-hyperbola which is the isogonal conjugate of QQ'. The point T is the intersection of the reflections of the tangent of the hyperbola at H in the sidelines of triangle ABC. [18]

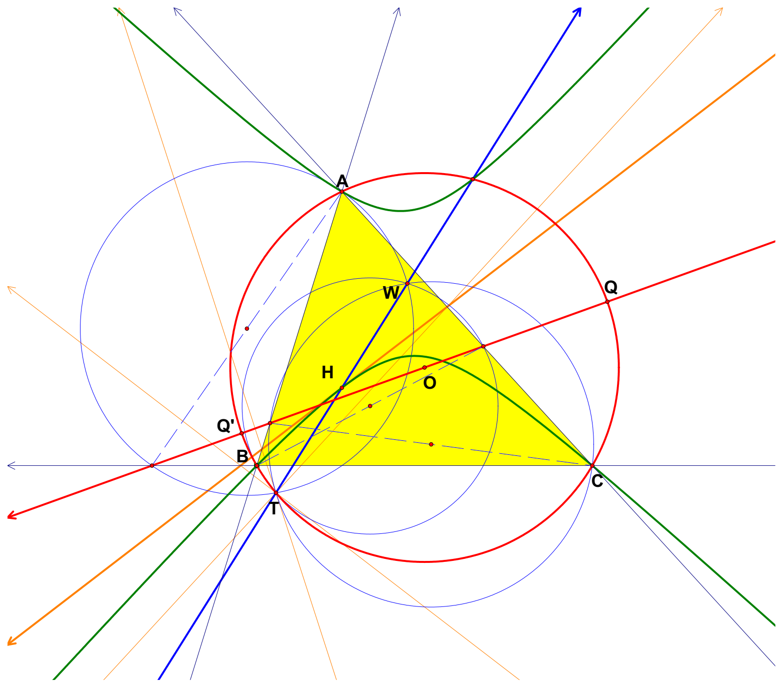

Figure 23.

[18]For a construction of the tangent to a conic, see §10.

4.2. *Some examples of rectangular circum-hyperbolas.*

4.2.1. *The Jerabek hyperbola* . The Jerabek hyperbola is the isogonal conjugate of the Euler line. It intersects the circumcircle at the antipode of the Euler reflection point E. [19] Its center J_e [20] is also the intersection of the Euler lines of triangles AYZ, BZX, and CXY, where XYZ is the orthic triangle. [21]

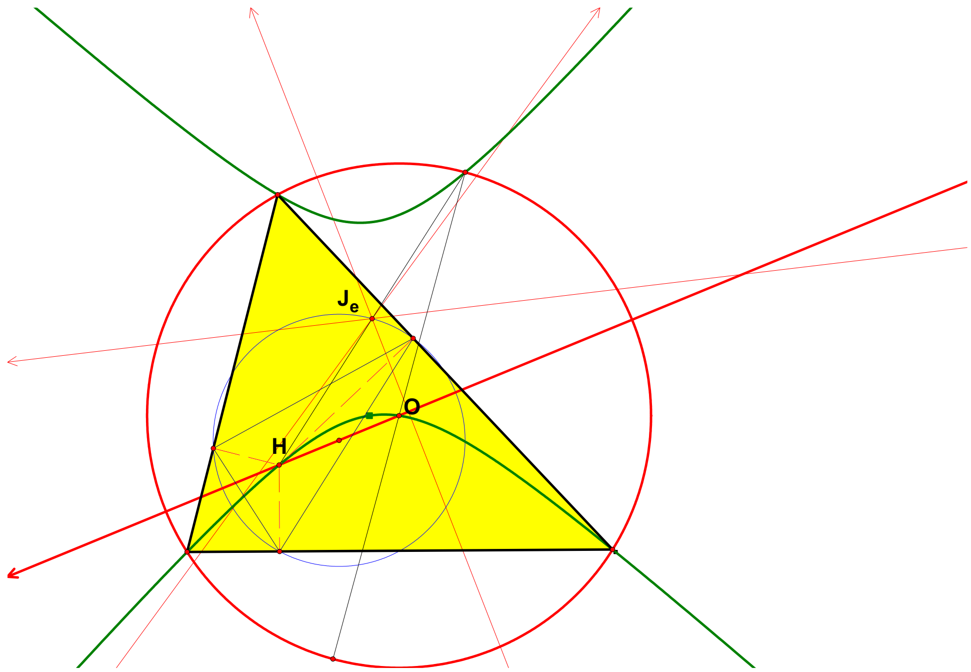

Figure 24.

Given a point P, let X, Y, Z be the intersections of the circumcircle with the lines AP, BP, CP respectively, [22] and X', Y', Z' their reflections in the respective sidelines. The lines AX', BY', CZ' are concurrent if and only if P lies on the Jerabek hyperbola. [23]

[19]The antipode of the Euler reflection point is X_{74} in ETC.
[20]The center of the Jerabek hyperbola is X_{125} in ETC.
[21]Thébault's theorem; see also [3].
[22]XYZ is called the circumcevian triangle of P.
[23]For the locus of the point of concurrency, see §8.3.

4.2.2. *The Kiepert hyperbola.* The Kiepert hyperbola is the isogonal conjugate of the Brocard axis OK. Its center $K_{\rm i}$ [24] is also the intersection of the Brocard axes of triangles AYZ, BZX, and CXY, where XYZ is the orthic triangle. [25] It is also the midpoint of the Fermat points.

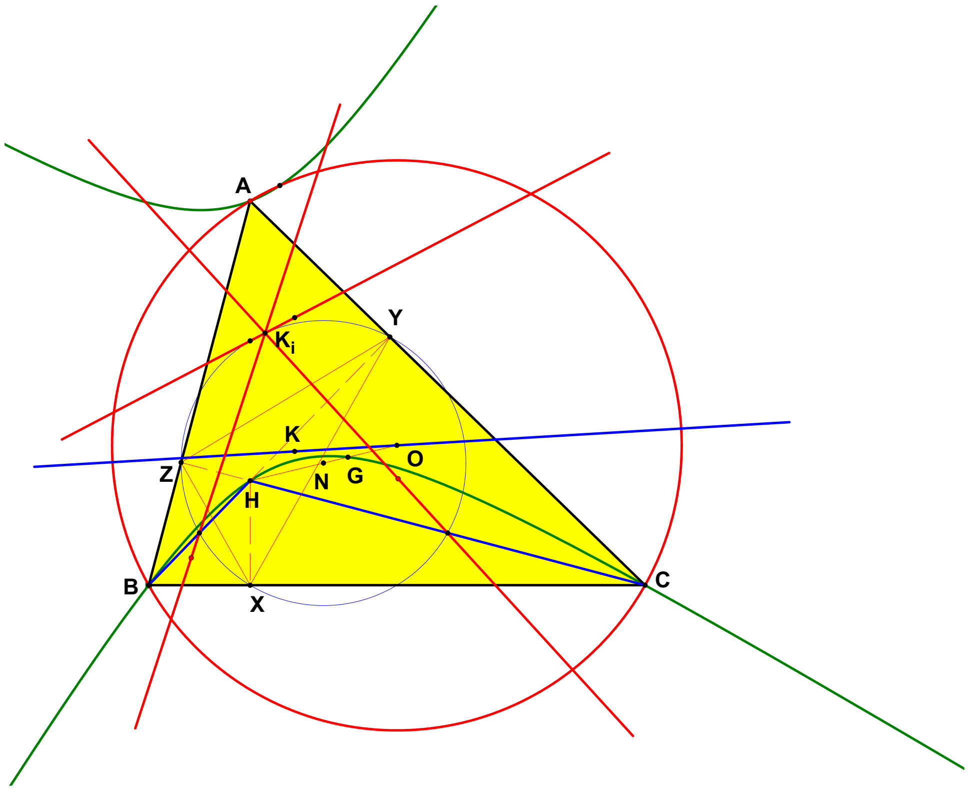

Figure 25. The Kiepert hyperbola

The centers of the Jerabek and Kiepert hyperbolas can be constructed as the intersections of the nine-point circle and the common pedal circles of G and K. See §6.1 and Figure **??** below.

[24]The center of the Kiepert hyperbola is X_{115} in ETC.
[25]Floor van Lamoen, Hyacinthos, message 1251, 8/19/00.

4.2.3. *The Feuerbach hyperbola.* The Feuerbach hyperbola is the isogonal conjugate of the line OI. Its center is the Feuerbach point F_{e}. Its tangent at I passes through O.

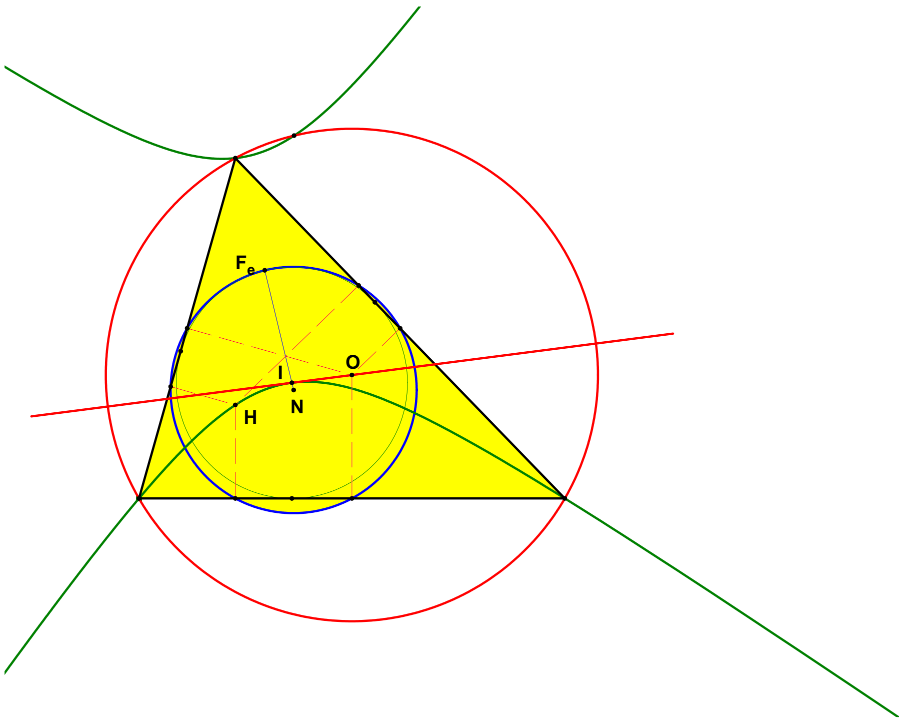

Figure 26. The Feuerbach hyperbola

The pedal circles of points on the Feuerbach hyperbola pass through the Feuerbach point.

4.2.4. *The Stammler hyperbola* . The Stammler hyperbola is the one that passes through the circumcenter, the incenter, and the excenters. This is a rectangular hyperbola since the incenter is the orthocenter of the triangle formed by the excenters. Its center is the Euler reflection point E. The asymptotes are the lines joining E to the intersections of the Euler line with the circumcircle. [26] The Stammler hyperbola is tangent to the Euler line. It also passes through the symmedian point K.

The Stammler hyperbola also contains the vertices of the tangential triangle. It is the Feuerbach hyperbola of the tangential triangle.

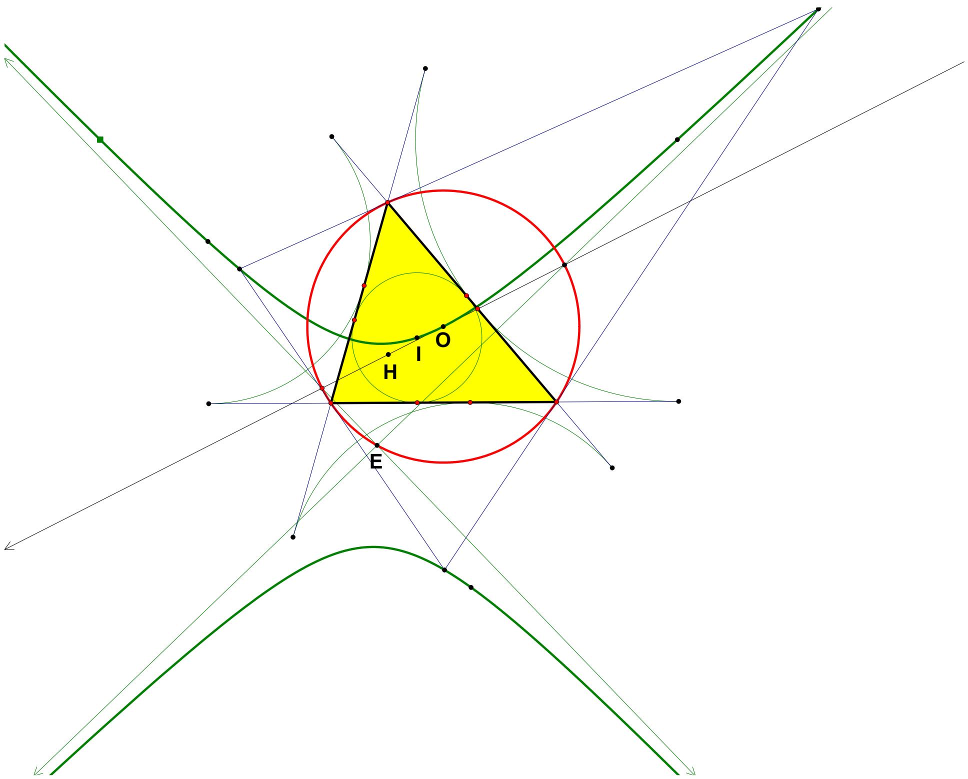

Figure 27. The Stammler hyperbola

The Stammler hyperbola is the locus of P whose pedals on the perpendicular bisectors are perspective. [27]

[26] The intersections of the Euler line with the circumcircle are X_{1113} and X_{1114} in ETC.

[27] See §7.3 for the locus of the perspector.

5. Conics

5.1. *Circumconics with given center.* Given a point P, it is easy to construct the conic through the vertices of triangle ABC and with center P. This conic, which we denote by $\mathcal{C}_c(P)$, also contains the reflections of A, B, C in P. The circumconic $\mathcal{C}_c(G)$ is called the Steiner circum-ellipse. [28] It intersects the circumcircle at the Steiner point S_t, [29] which can be constructed by extending the segment K_iG such that $K_iS_t = 3K_iG$.

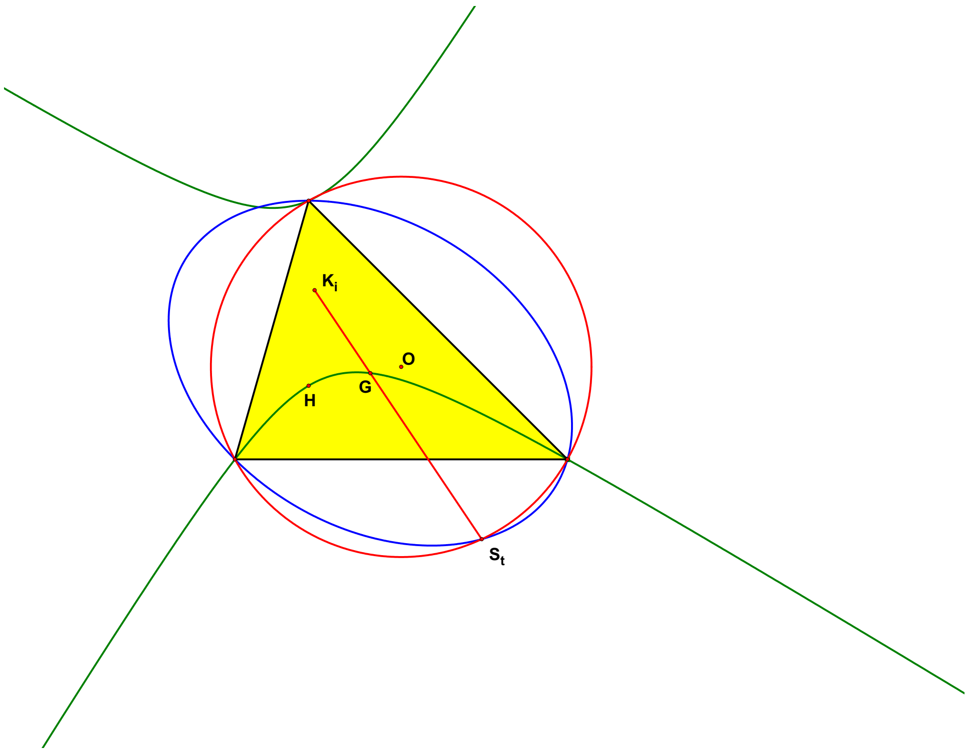

Figure 28.

[28]See §2.8.
[29]The Steiner point X is X_{99} in ETC.

5.1.1. *Fourth intersection of* $\mathcal{C}_{\rm c}(P)$ *and circumcircle* . More generally, the fourth interesection of a circumconic and the circumcircle can be constructed as follows. Let A', B', C' be the antipodes of A, B, C in the circumconic $\mathcal{C}_{\rm c}(P)$. The circles $AB'C'$, $BC'A'$, $CA'B'$ intersect at the fourth intersection T with the circumcircle.

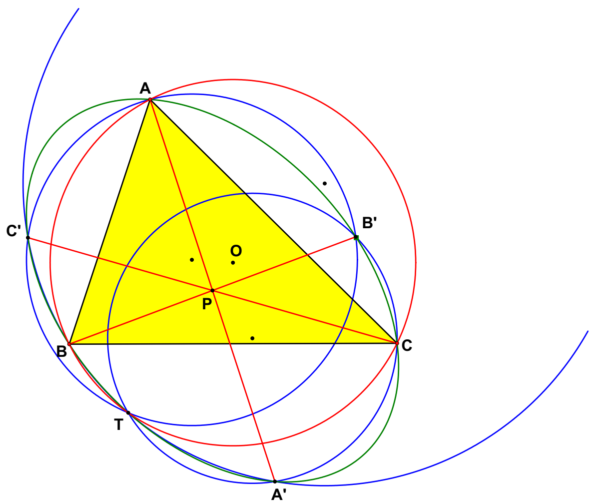

Figure 29.

The circles ABC', BCA', CAB', on the other hand, intersect at the antipode of T' in $\mathcal{C}_{\rm c}(P)$.

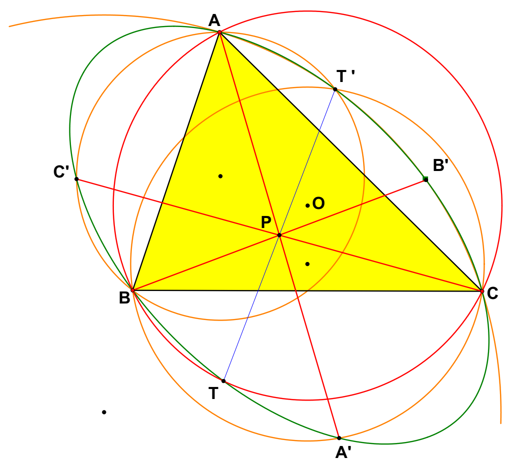

Figure 30.

5.2. *Conics through the traces of two points.* Given two points P and Q, there is a unique conic through the traces of these two points on the sidelines of the reference triangle ABC. We denote this conic by $\mathcal{C}(P,Q)$. The simplest example is the nine-point circle; it is $\mathcal{C}(G,H)$.

5.2.1. Let P be a point on the circumcircle, the conic $\mathcal{C}(G,P)$ is a rectangular hyperbola since it passes through O, the orthocenter of the triangle formed by the midpoints of the sides.

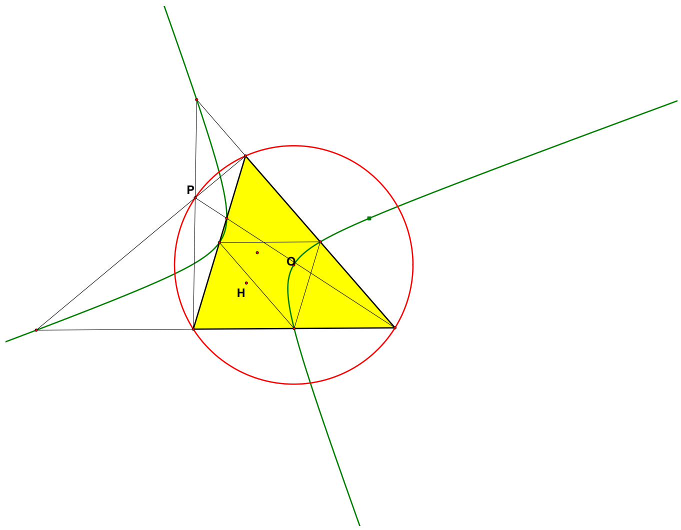

Figure 31.

5.2.2. *Central conic cevian complement.* It is easy to construct a conic through the traces of a point P which is also the center of the conic. This is $\mathcal{C}(P,Q)$ for some point Q. We call Q the central conic cevian complement of P.

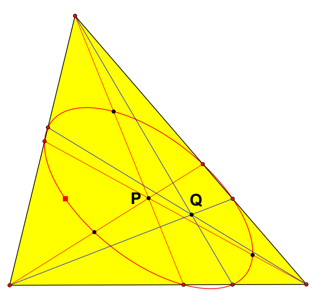

Figure 32.

Here are two interesting examples.

(i) If P is the incenter, Q is the homothetic center of the excentral and intouch triangles. [30] It is a point on the line joining the circumcenter to the incenter.

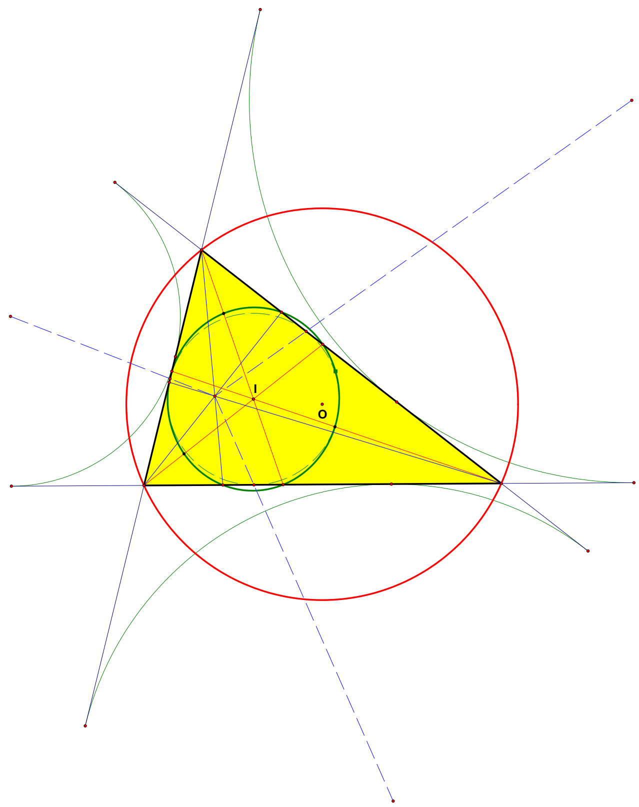

Figure 33.

(ii) If P is the symmedian point, Q is the homothetic center of the tangential and orthic triangles. [31] It is a point on the Euler line. See §9.1.1 and Figure 52.

[30]The homothetic center of the excentral and intouch triangle is X_{57} in ETC.

[31]The homothetic center of X_{25} in ETC.

5.3. *Inscribed conics.* Given a pair of isogonal conjugates P and P^*, there is a conic with these as foci and tangent to the sidelines of triangle ABC. The center of the conic is the midpoint M of PP^*, the center of the common pedal circle of P and P^*. To construct the conic (using the five-point conic command), we find the points of tangency with the sidelines. For this, extend MG to Q' such that $MQ' = 3MG$. Let X', Y', Z' be the traces of Q'. The points of tangency with the sidelines are the reflections of triad X'Y'Z' in the midpoints of $\begin{matrix}BC\\CA\\AB\end{matrix}$. Their reflections in M are also on the conic.

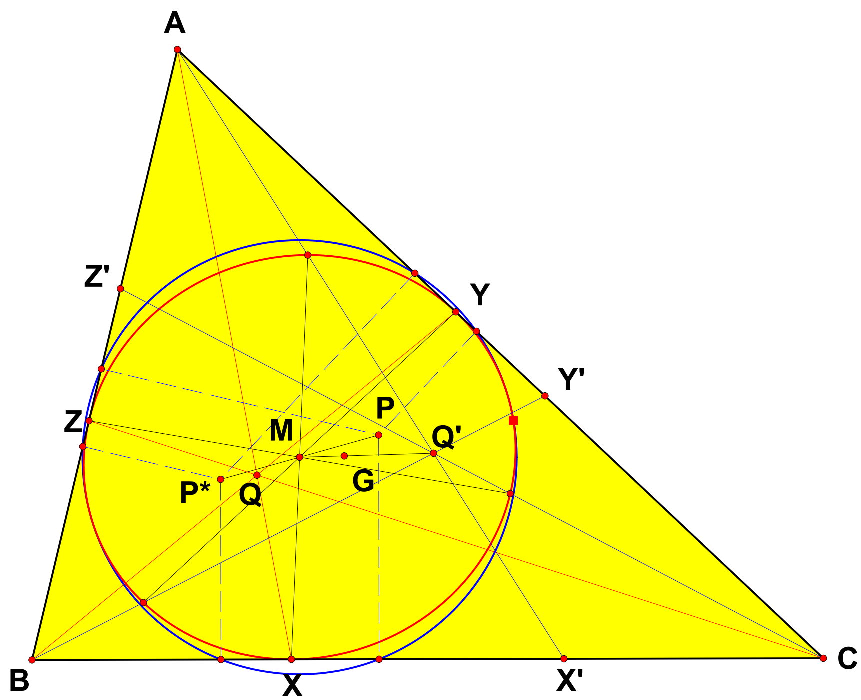

Figure 34.

The lines AX, BY, CZ are concurrent at a point Q, called the perspector (or the Brianchon point) of the inscribed conic. This is the isotomic conjugate of the point Q' constructed above. The common pedal circle of P and P^* is the auxiliary circle of the inscribed conic.

5.3.1. The inscribed conic with foci O and H has center N. It is the envelope of the perpendicular bisector of segment HP, for P on the circumcircle.

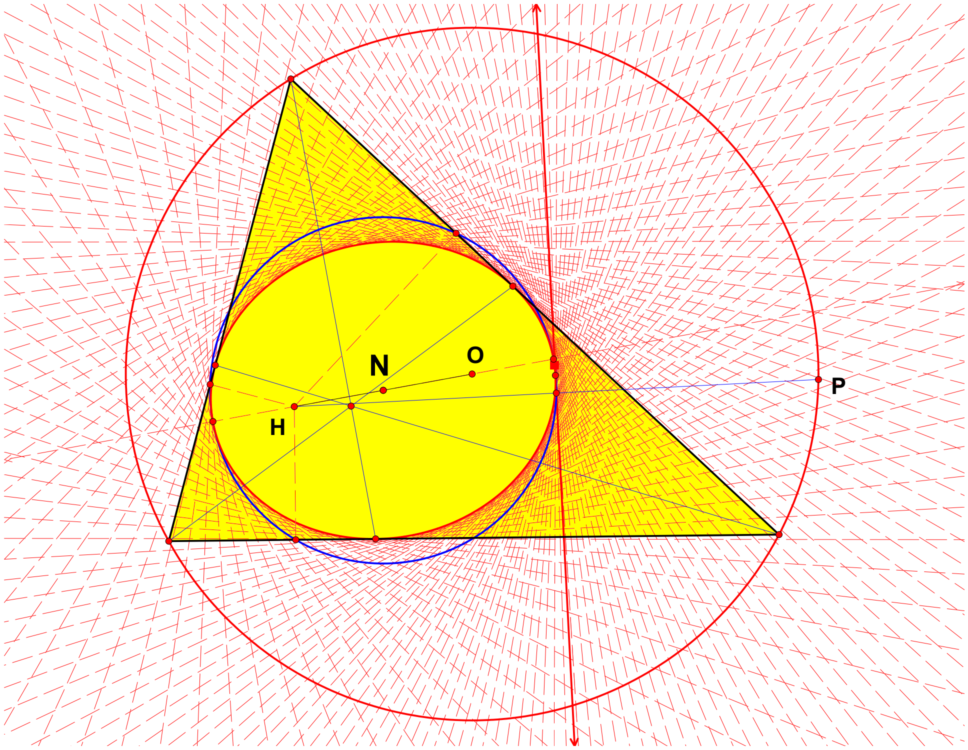

Figure 35.

5.3.2. The inscribed conic with center K touches the sideslines at the vertices of the orthic triangle.

5.4. *Inscribed parabolas.* If F is a point on the circumcircle, its isogonal conjugate is an infinite point. The inscribed conic with F as a focus is therefore a parabola. The directrix $\mathcal{L}$ is the line of reflections of F. The points of tangency X, Y, Z with the sidelines are such that XF_a, YF_b, ZF_c are perpendicular to $\mathcal{L}$. The axis is the perpendicular from F to $\mathcal{L}$. This is enough to construct the parabola (by the 5-point conic command).

5.4.1. *The perspector of an inscribed parabola.* The perspector of the inscribed parabola, *i.e.*, the intersection of the lines AX, BY, CZ, is a point on the Steiner circum-ellipse, the circumconic with centroid G. It is indeed the second intersection with the line joining F to the Steiner point.

If the focus is E, then the directrix is the Euler line. The perspector is the Steiner point $S_{\rm t}$.

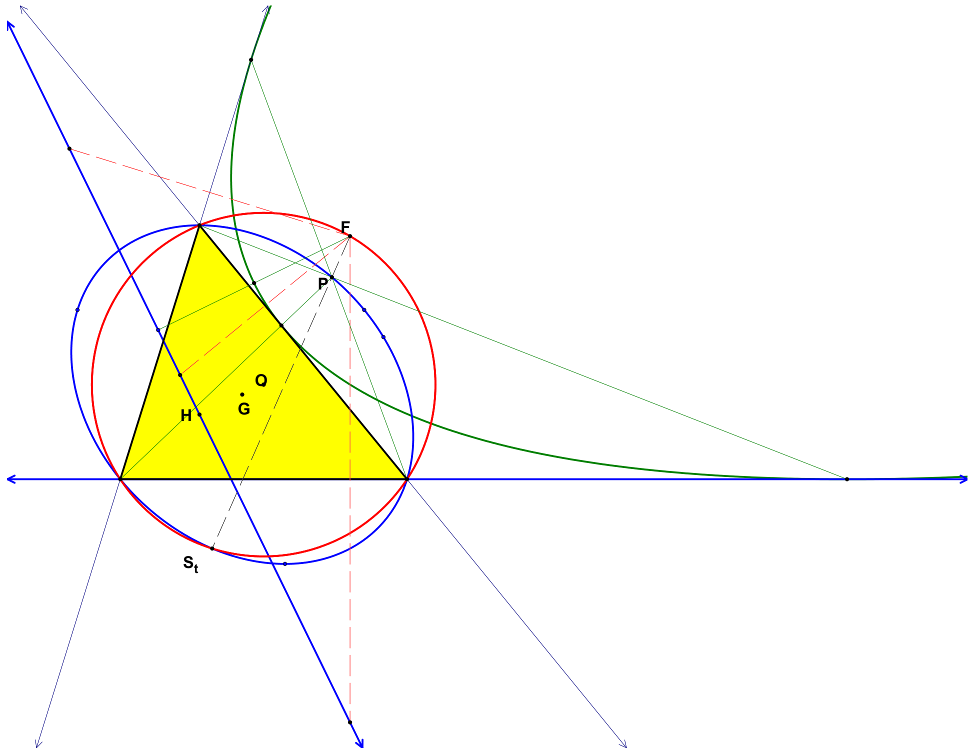

Figure 36.

5.4.2. *Inscribed parabola tangent to a given line.* Given a line ℓ, there is a unique inscribed parabola tangent to ℓ and to the sidelines of the reference triangle.

Suppose the line ℓ intersects the sidelines BC, CA, AB at X, Y, Z respectively. Let X', Y', Z' be the reflections of these points in the midpoint of the respective sides. Then X', Y', Z' are also collinear. The focus F of the parabola is the isogonal conjugate of the infinite point of the line containing them. This can be easily constructed.

To find the point of tangency with ℓ, reflect F in ℓ. This reflection lies on the line of reflections of F, which is the directrix of the parabola. The perpendicular to the line of reflection at this point intersects ℓ at the point of tangency.

6. Further examples of reflections

6.1. *The reflection conjugate* . Let $P \neq H$ be a point not on the circumcircle, nor any of the circles through H with centers O_a, O_b, O_c. The circles P_aBC, P_bCA, and P_cAB intersect at a point $r(P)$ on the circle of reflections of P. We call $r(P)$ the *reflection conjugate* of P, since $r(r(P)) = P$.

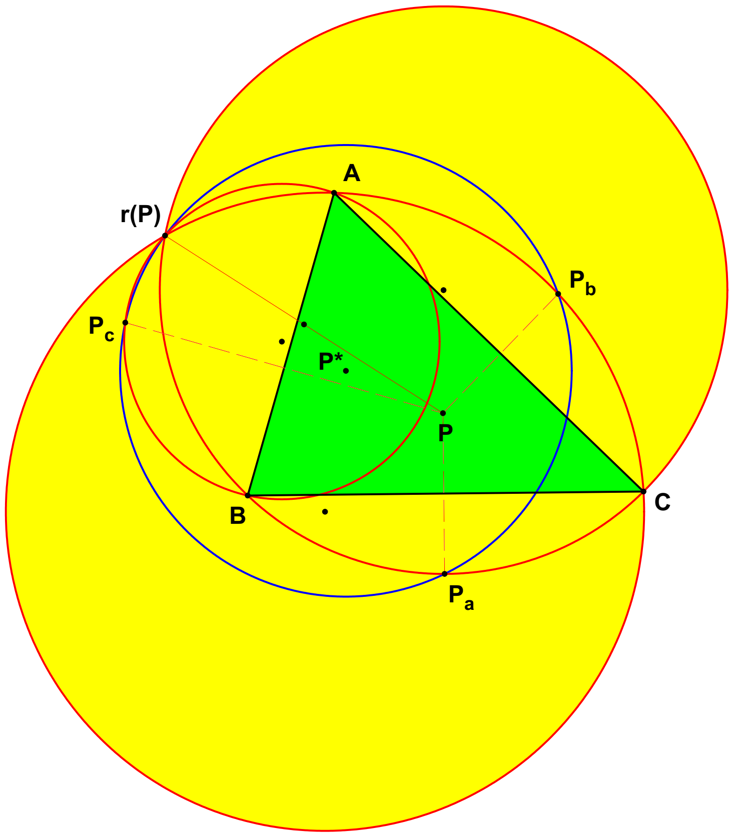

Figure 37.

The reflection conjugate $r(P)$ also lies on the rectangular circum-hyperbola $\mathcal{H}(P)$. Indeed, P and $r(P)$ are antipodal on this hyperbola. The center of the hyperbola $\mathcal{H}(P)$ is therefore the midpoint of P and its reflection conjugate.

6.2. Let P_a, P_b, P_c be the reflections of P in the sidelines of triangle AC. The circles AP_bP_c, BP_cP_a, CP_aP_b have a common point T on the circumcircle. If P is not on the circumcircle, and Q is the fourth intersection of the circumcircle and the rectangular circum-hyperbola $\mathcal{H}(P)$, then the line PQ intersects the circumcircle again at T.

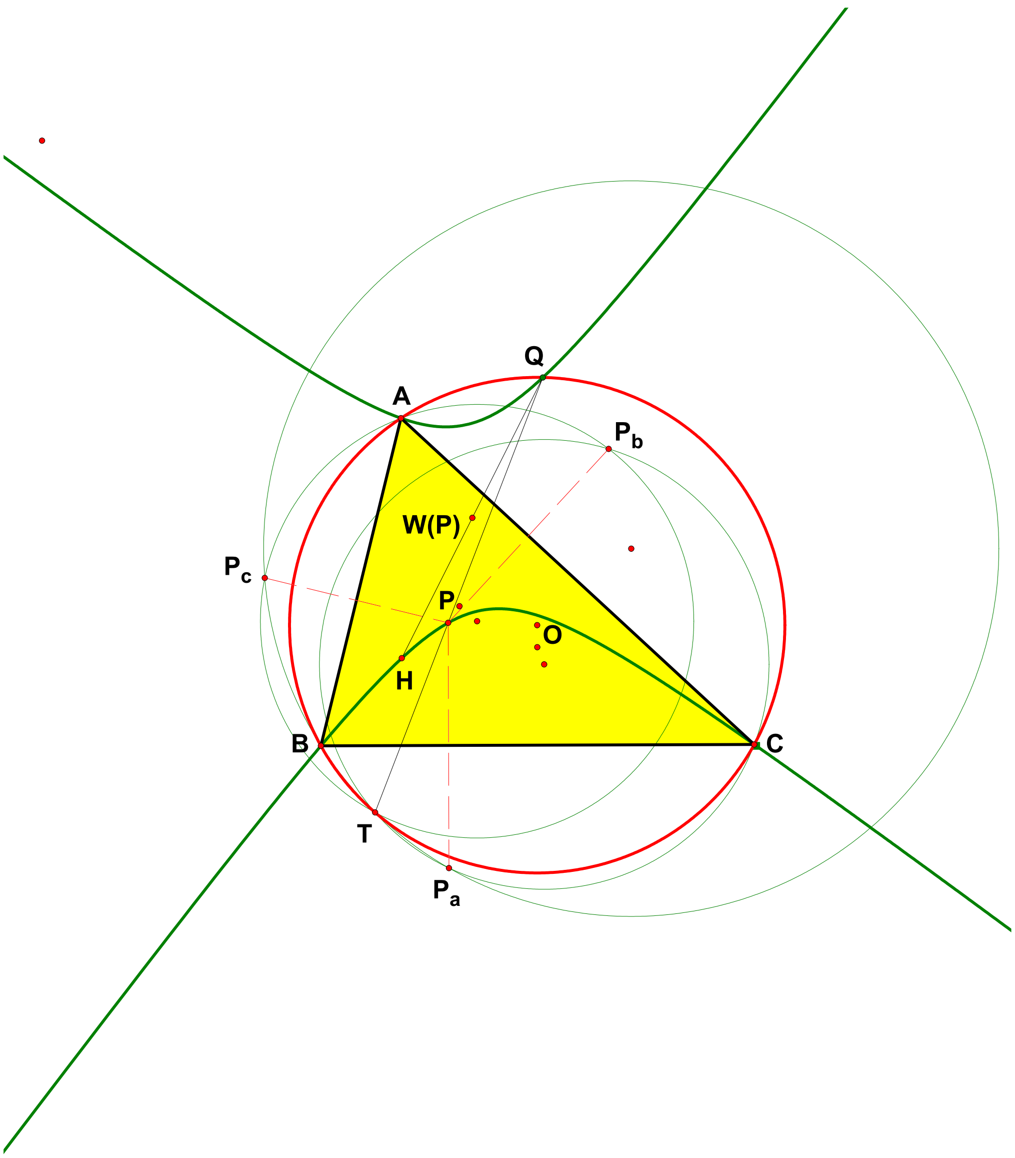

Figure 38.

6.3. *Reflections of H in cevian lines.* Given a point P, let X, Y, Z be the reflections of the orthocenter H in the lines AP, BP, CP respectively. The circles APX, BPY, CPZ have a second common point other than P. This is the second intersection of $\mathcal{H}(P)$ with the circumcircle.

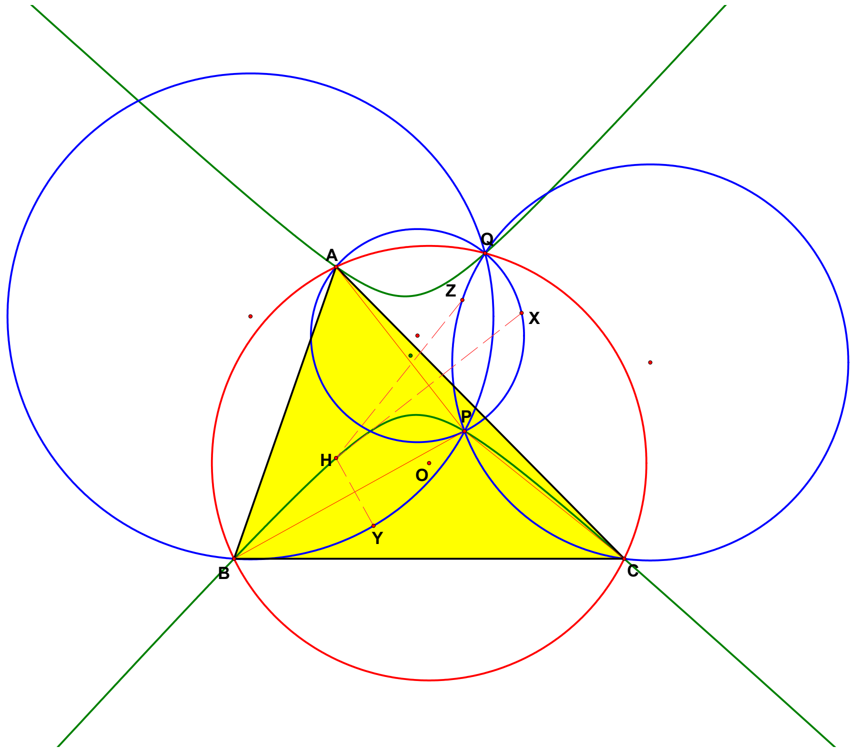

Figure 39.

6.4. *Perspector of orthic and reflection triangles* . The orthic triangle is the pedal triangle of H. Every reflection triangle is perspective with the orthic triangle. It is easy to see that the perspector is the isogonal conjugate of P in the orthic triangle.

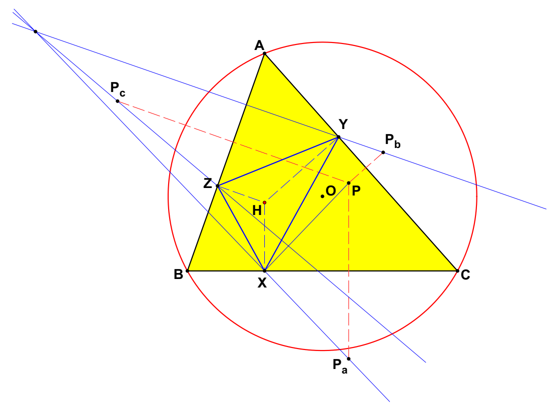

Figure 40.

6.4.1. *Reflection triangle of* K . An interesting example is $P = K$. The reflections are on the lines joining the corresponding vertices of the tangential and orthic triangles, which are homothetic. The homothetic center is a point on the Euler line. This point is the intersection of the Euler line and the tangent to the Jerabek hyperbola at K.

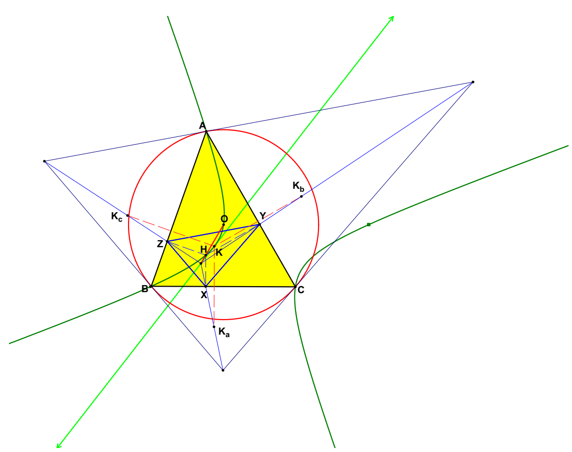

Figure 41.

6.5. *The reflection triangle.* Let $\begin{array}{c}A'\\B'\\C'\end{array}$ be the reflection of $\begin{array}{c}A\\B\\C\end{array}$ in its opposite side $\begin{array}{c}BC\\CA\\AB\end{array}$. We call $A'B'C'$ the reflection triangle of ABC.

Let O' be the circumcenter of the reflection triangle. [32] The midpoint of OO' is the isogonal conjugate of the nine-point center. [33]

Clearly, the circles $A'BC$, $AB'C$ and ABC' has H as a common point. On the other hand, the circles $AB'C'$, $A'BC'$ and $A'B'C$ also have a common point. [34] It lies on the line OO'.

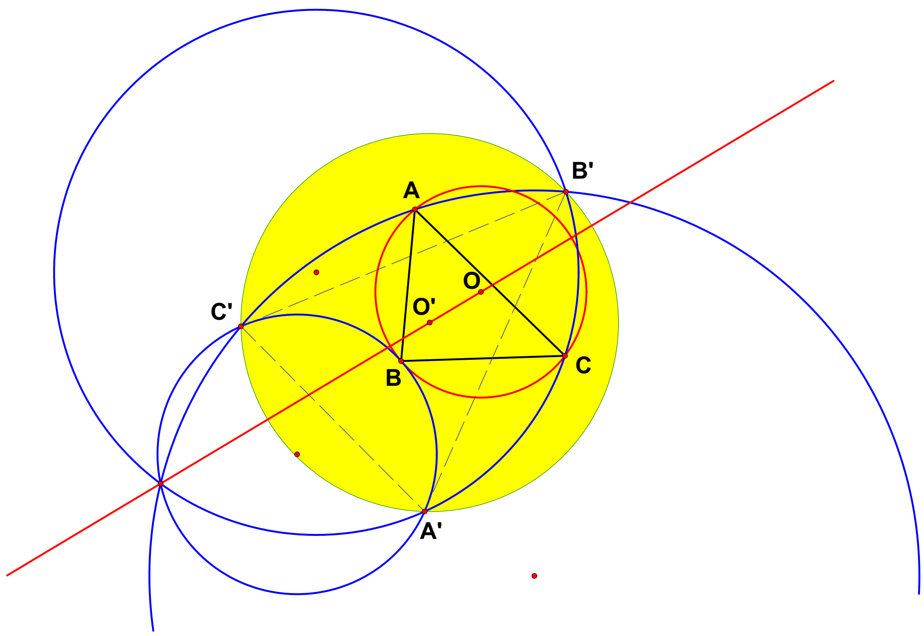

Figure 42.

[32] O' is X_{195} in ETC.
[33] The isogonal conjugate of the nine-point center is X_{54} in ETC.
[34] This common point is X_{1157} in ETC.

7. Some loci related to reflections

7.1. *Perspective reflection triangles.* The locus of P whose reflection triangle is perspective is the Neuberg cubic. This is also the locus of P for which the line PP^* is parallel to the Euler line. Furthermore, if P is on the Neuberg cubic, the perspector of the reflection triangle lies on the line PP^*.

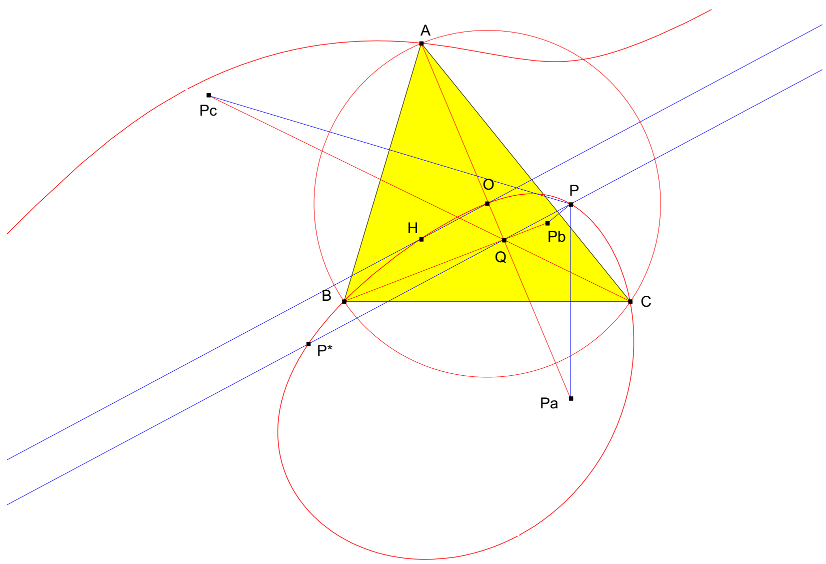

Figure 43.

7.2. *Reflections in perpendicular bisectors.* Given a point P with reflections X, Y, Z in the perpendicular bisectors of BC, CA, AB respectively, the triangle XYZ is perspective with ABC if and only if P lies on the circumcircle or the Euler line. If P is on the circumcircle, the lines AX, BY, CZ are parallel. The perspector is the isogonal conjugate of P. If P traverses the Euler line, the locus of the perspector is the Jerabek hyperbola. [35]

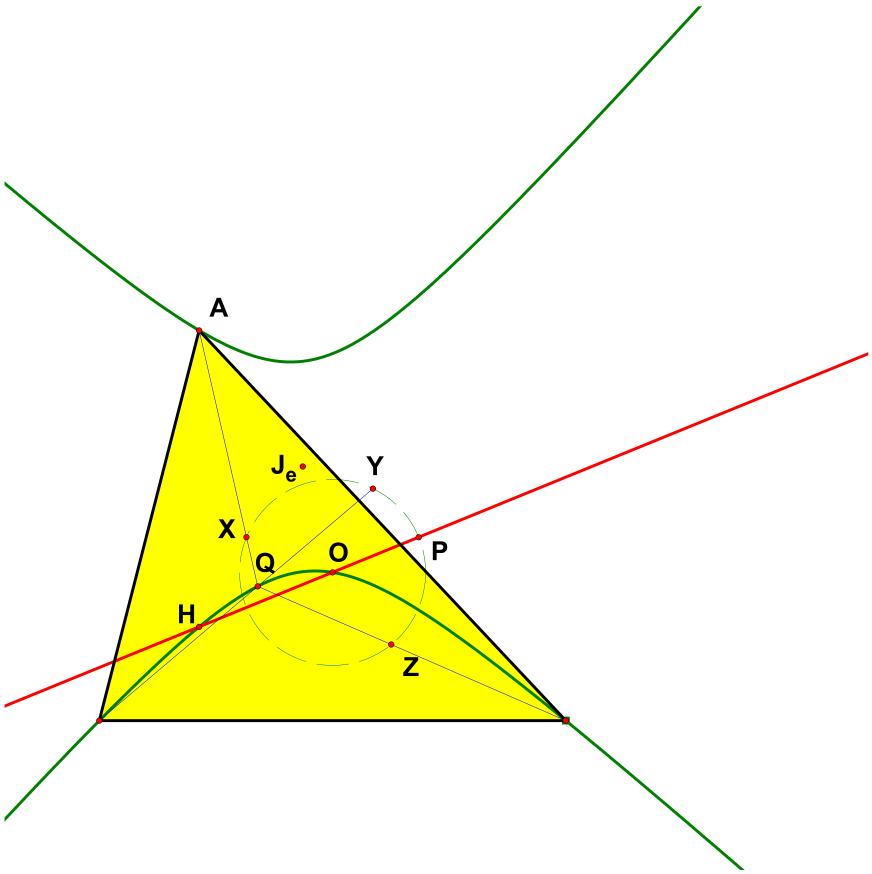

Figure 44.

[35] JPE, Hyacinthos, message 2204; PY, 2205, 12/26/00.

7.3. *Pedals on perpendicular bisectors.* In §4.2.4, we consider points whose pedals on the perpendicular bisectors are perspective. Such points lie on the Stammler hyperbola. The locus of the perspector is a singular cubic that can be constructed as the locus of the intersection $OP \cap HP^*$ for P on the circumcircle. [36]

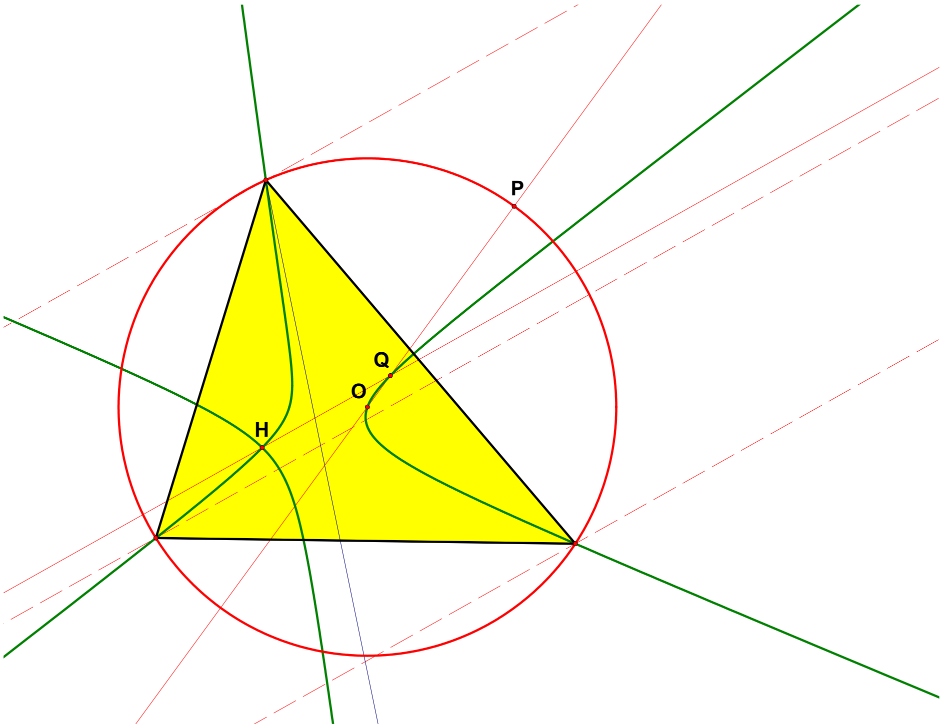

Figure 45.

[36]This is called the third Musselman cubic. in CTC.

7.4. *Reflections in altitudes.* Let X, Y, Z be the reflections of P in the altitudes of triangle ABC. The lines AX, BY, CZ are concurrent (at a point Q) if and only if P lies on a certain cubic curve which can be constructed as follows. We make use of an auxiliary triangle $A'B'C'$. Let DEF be the orthic triangle of ABC. The lines $\begin{matrix}EF\\FD\\DE\end{matrix}$ intersects $\begin{matrix}BC\\CA\\AB\end{matrix}$ at $\begin{matrix}D'\\E'\\F'\end{matrix}$ respectively. The points D', E', F' are collinear. The line $\mathcal{L}_H$ containing them is called the orthic axis of ABC. Construct parallels to $\mathcal{L}_H$ through $\begin{matrix}A\\B\\C\end{matrix}$ to intersect $\begin{matrix}EF\\FD\\DE\end{matrix}$ at $\begin{matrix}A'\\B'\\C'\end{matrix}$.[37]

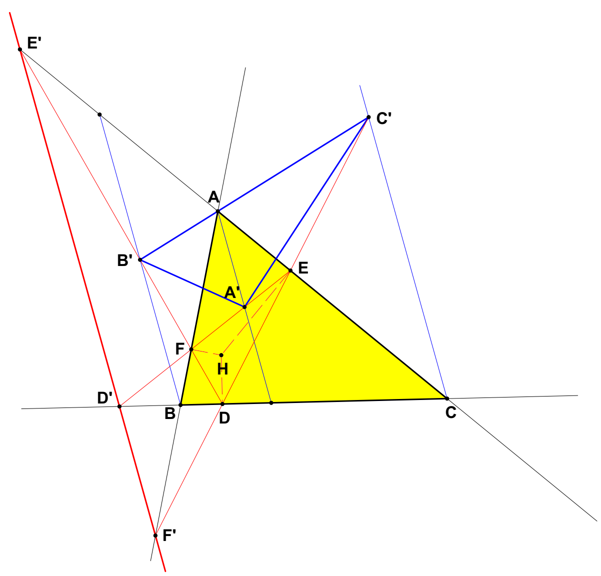

Figure 46.

Let ℓ be a line through the orthocenter H. There is a unique point P on ℓ for which XYZ is perspective. Construct the perpendiculars to ℓ from A', B', C' intersecting the corresponding sidelines of ABC at X', Y', Z'. The lines AX', BY', CZ' intersect at a point P'. The pedal of P' on ℓ is the unique point P on ℓ for which XYZ is perspective. See Figure 47.

[37] $A'B'C'$ is the anticevian triangle of the infinite point of the orthic axis.

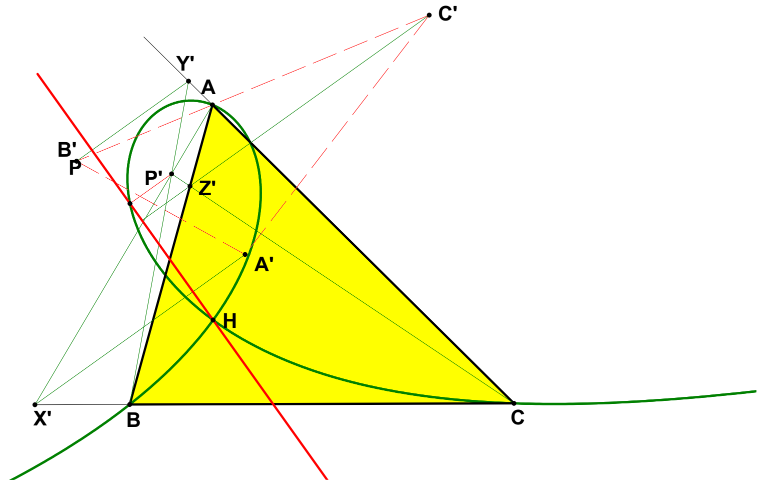

Figure 47.

8. Some more examples of loci

8.1. *Points collinear with isogonal and isotomic conjugates.* The locus of P which lies on the line joining its isogonal and isotomic conjugates [38] is the conic through the centroid, the incenter, and the vertices of the superior triangle. Its center is the Steiner point. For every P on the conic, the line containing P^* and P' is the tangent to the conic at P. This is a rectangular hyperbola since it passes through the excenters as well. The asymptotes are the lines joining the Steiner point to the intersections of the circumcircle with the Brocard axis OK.

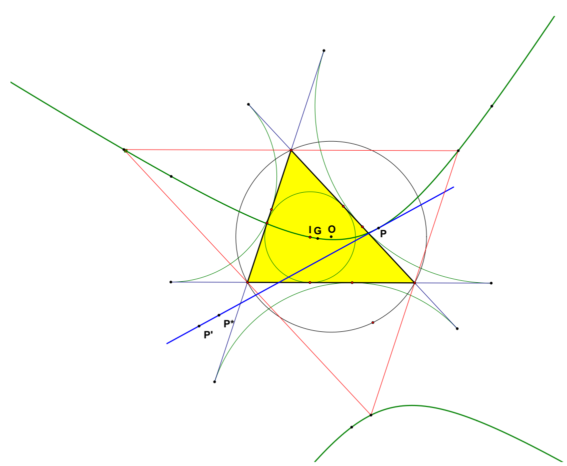

Figure 48.

[38]For the notion of isotomic conjugate, see 2.6.

8.2. *Inversive images of traces in circumcircle.* Let P be a point with traces X, Y, Z on the sidelines BC, CA, AB. The inverses of X, Y, Z form a perspective triangle if and only if P lies on the circumcircle or the Euler line. If P lies on the circumcircle, the locus of the perspector is the isogonal conjugate of the nine-point circle.

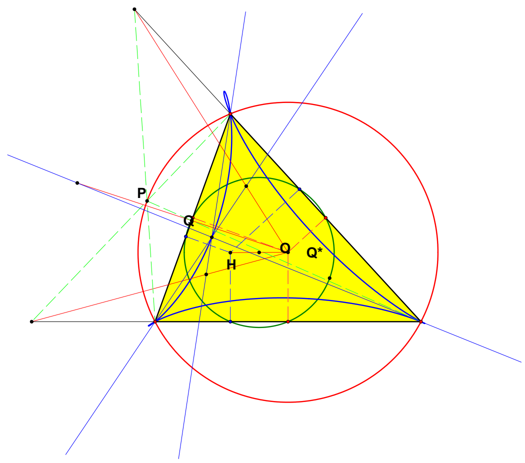

Figure 49.

If P lies on the Euler line, the locus of the perspector is the conic $\mathcal{C}(J_e^*, K_i^*)$.[39]

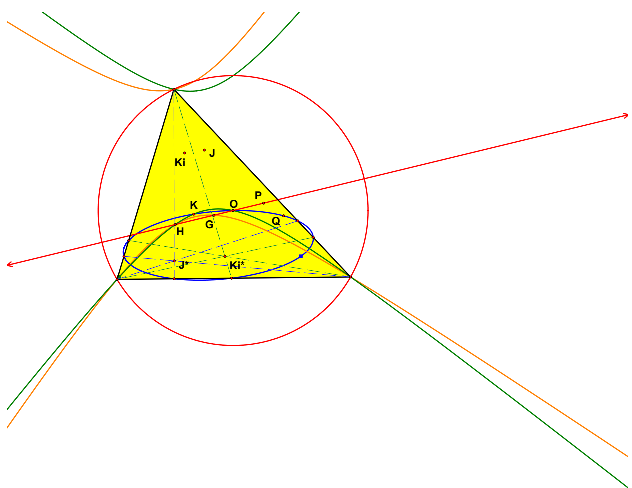

Figure 50.

[39]The isogonal conjugates of the Jerabek and Kiepert centers are the points X_{250} and X_{249} respectively in ETC.

8.3. In §4.2.1 we have seen that the Jerabek hyperbola is the locus of P the triangle formed by the reflections of the vertices of whose circumcevian triangle is perspective. The locus of the point of concurrency Q is a curve which is the isotomic conjugate of the conic $\mathcal{C}(J'_{\rm e}, K'_{\rm i})$ through the traces of of $J'_{\rm e}$ and $K'_{\rm i}$, the isotomic conjugates of $J_{\rm e}$ and $K_{\rm i}$. [40]

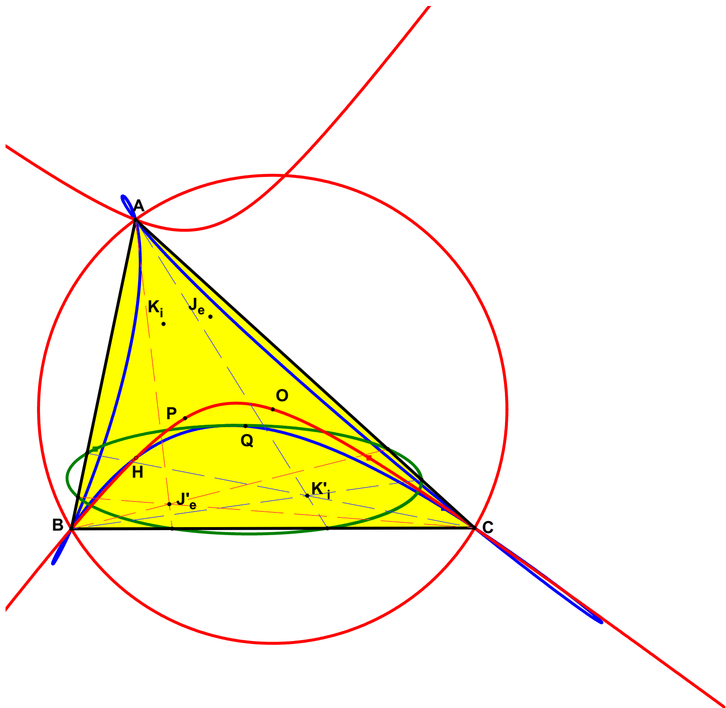

Figure 51.

[40]The isotomic conjugates of the Jerabek and Kiepert centers do not appear in the current (April, 2004) edition of ETC.

9. Select properties of triangle centers

9.1. *The symmedian point* .

9.1.1. The reflection triangle of K is perspective to the tangential triangle at a point on the Euler line. [41] See Figure 52. Note that this is also the homothetic center of the tangential and orthic triangles.

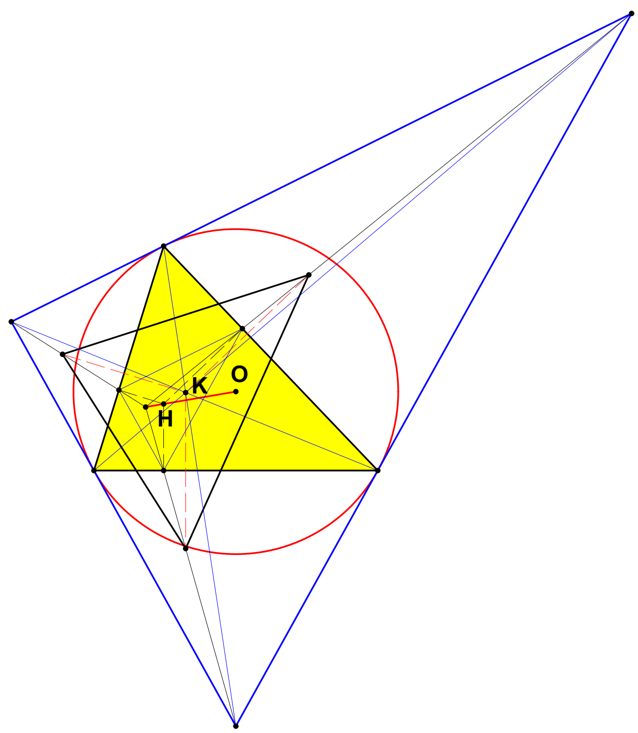

Figure 52.

This phenomenon is true in general. The reflection triangle and the anticevian triangle of a point P are always perspective. [42] See §6.4.

[41] This is the point X_{25} in ETC.

[42] The vertices of the anticevian triangle of P are $AP \cap P_aX$, $BP \cap P_bY$, $CP \cap P_cZ$, where XYZ is the orthic triangle, *i.e.*, X, Y, Z are the pedals of the orthocenter H on the sidelines of triangle ABC.

9.1.2. *The first Lemoine circle.* The symmedian point K is the only point P with the property that the 6 intercepts of parallels through P to the sides lie on a circle. See Figure 53. The center of the circle is the midpoint of OK. [43]

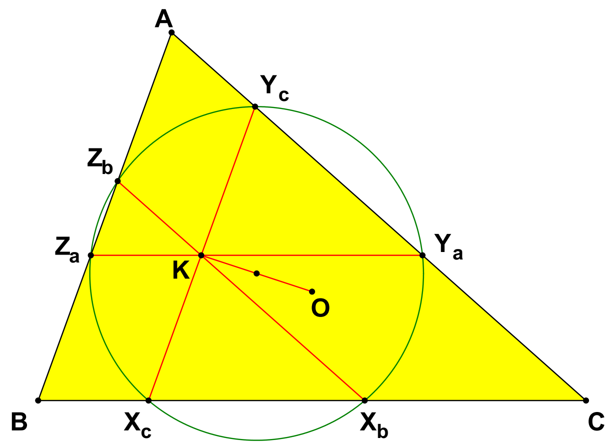

Figure 53.

9.1.3. *The second Lemoine circle.* Let XYZ be the orthic triangle. The 6 intercepts of parallels through K to YZ, ZX, XY on the sidelines are on a circle whose center is K.

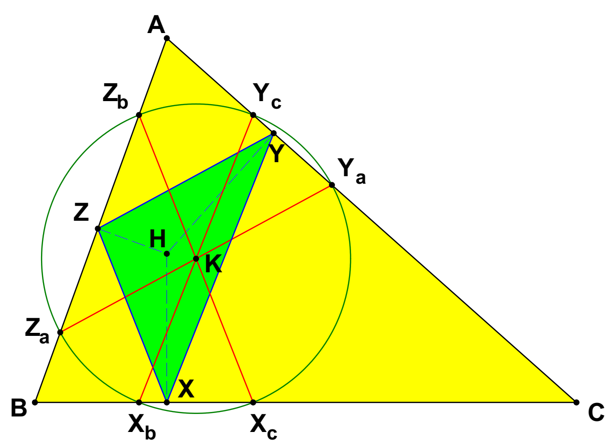

Figure 54.

[43]The midpoint of OK is the point X_{182} in ETC.

9.1.4. *The symmedian point and squares erected on the sides.* The symmedian point is also the perspector of the triangle $A'B'C'$ bounded by the outer sides of the squares erected externally on the sides of triangle ABC. See Figure 55. For other interesting properties of K, see §**??**.

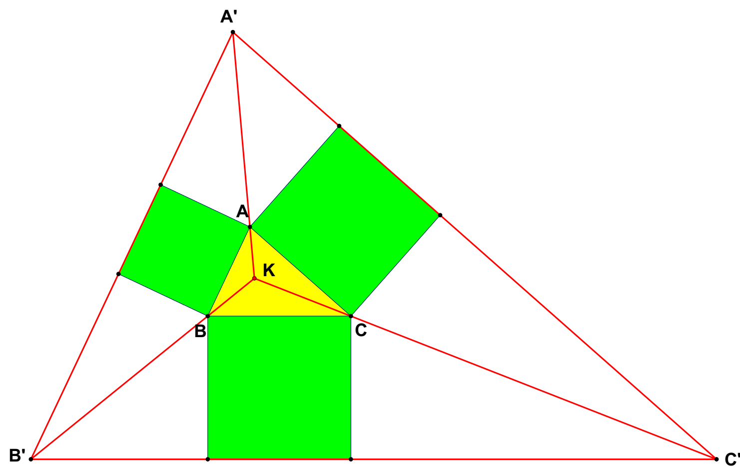

Figure 55.

9.2. *The deLongchamps point* L. This is the reflection of H in O.

9.2.1. A tetrahedron is said to be isosceles if its four faces are congruent triangles. In an isosceles tetrahedron, the pedal [44] of each vertex on its opposite face is the deLongchamps point of the triangle formed by the remaining three vertices.

9.2.2. It is the radical center of the three circles $A(a)$, $B(b)$, $C(c)$.

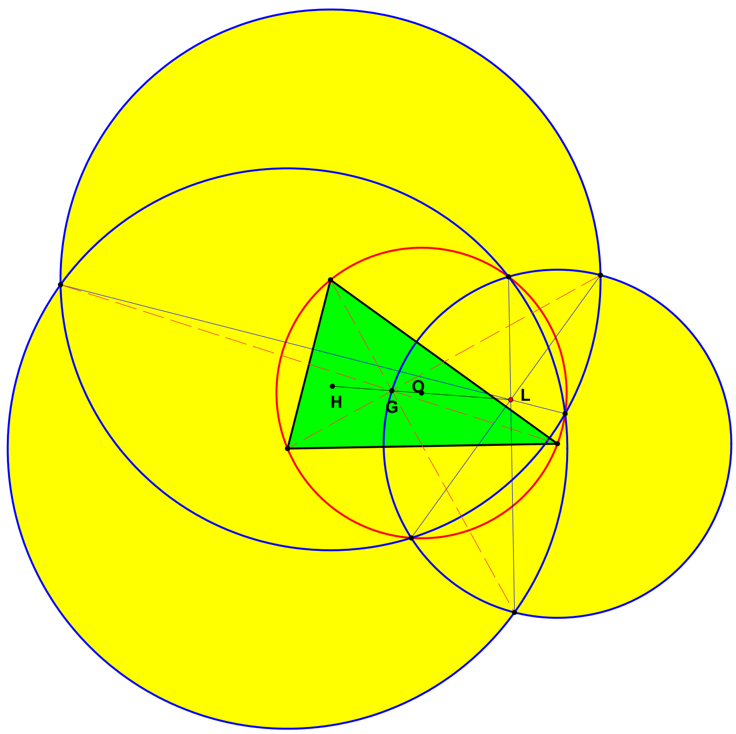

Figure 56.

[44] In this paper, we use the word *pedal* in the sense of *orthogonal projection*.

9.2.3. Let $\mathcal{E}(A)$ be the ellipse with foci B, C, and passing through A. Similarly, consider the ellipses $\mathcal{E}(B)$ and $\mathcal{E}(C)$. Each pair of these ellipses has a common chord. The three common chords intersect at the deLongchamps point L.

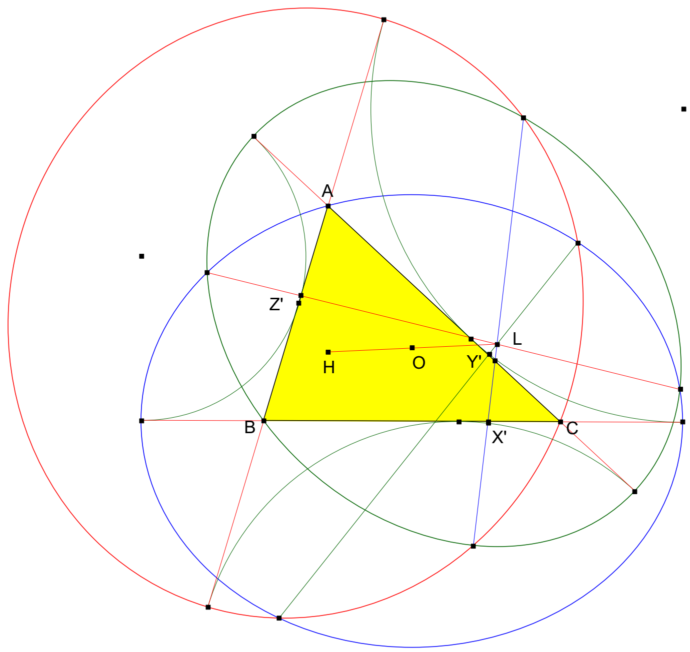

Figure 57.

9.3. *The Schiffler point* S_c. The Schiffler point S_c is the common point of the Euler lines of the triangles IBC, ICA, IAB, and ABC. [45] Here, I is the incenter of triangle ABC.

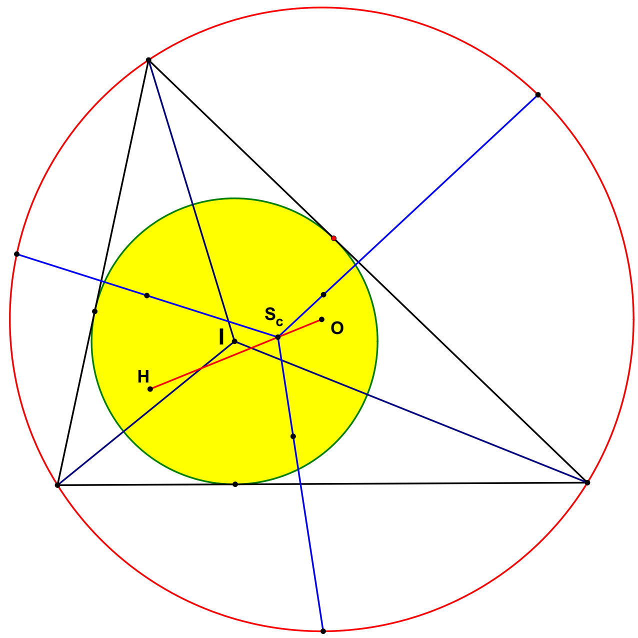

Figure 58.

[45]The Schiffler point is X_{21} in ETC. For interesting properties of the Schiffler point, see [5].

9.3.1. If $\begin{matrix} X \\ Y \\ Z \end{matrix}$ is the intersection of $\begin{matrix} OI_a \\ OI_b \\ OI_c \end{matrix}$ and $\begin{matrix} BC \\ CA \\ AB \end{matrix}$, the triangle XYZ is perspective with ABC at the Schiffler point $S_{\rm c}$.

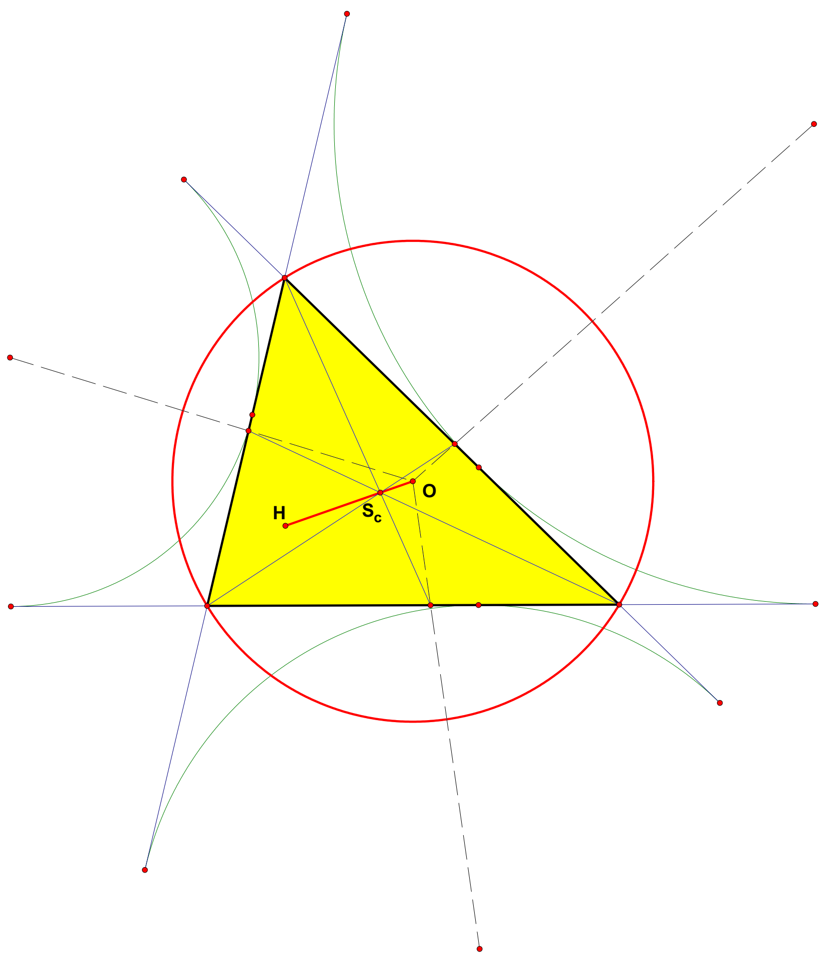

Figure 59.

9.3.2. *The centroid of the cevian triangle of the Euler infinity point.* Construct lines through the vertices parallel to the Euler line to intersect the opposite sidelines at X, Y, Z respectively. The centroid of triangle XYZ is a point on the Euler line. [46] The Euler line is the only line through O with this property. [47]

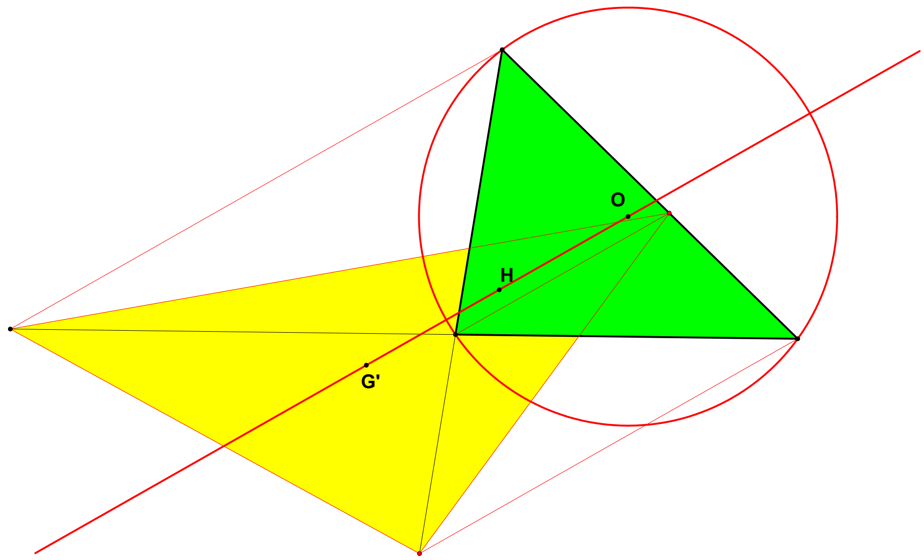

Figure 60.

[46] This point does not appear in the current (April, 2004) edition of ETC.

[47] More generally, let P be a given point. Select a line ℓ through P (not parallel to any of the sidelines) and construct the parallels to ℓ through the vertices A, B, C to intersect their opposite sides at X, Y, Z. Construct the centroid of triangle ABC. There is a unique line ℓ which contains this centroid. What is this line? What minor modification should be made on the point P to guarantee uniqueness?

9.4. *The reflection of I in O.* TIf X', Y', Z' are the points of tangency of the excircles with the respective sides, the circles $AY'Z'$, $BZ'X'$, $CX'Y'$ has a common point, which is the reflection of I in O. [48]

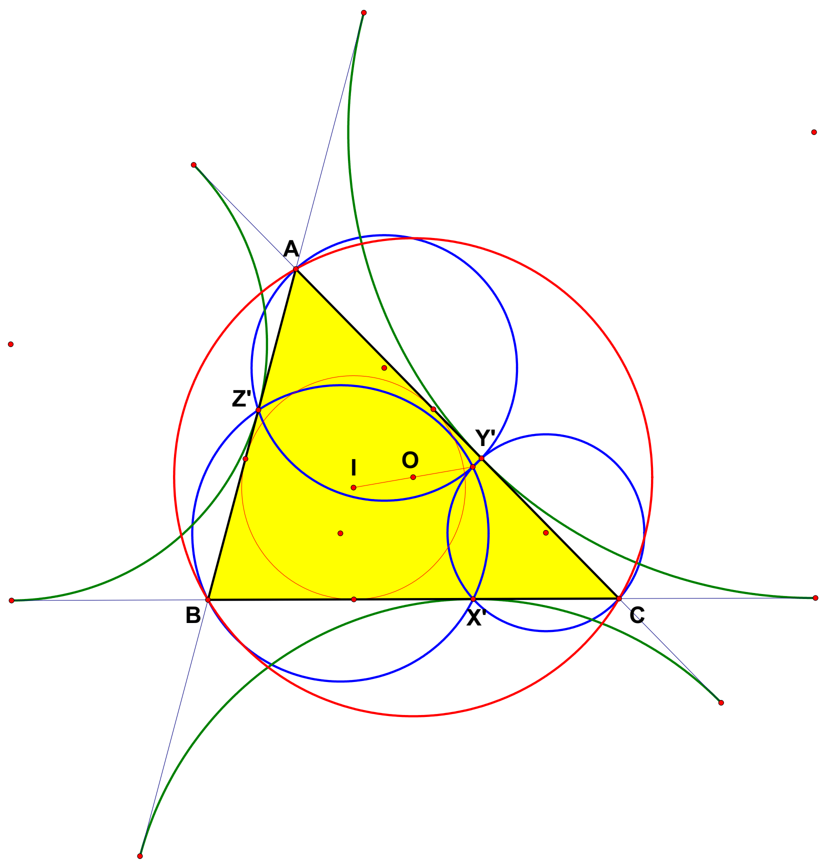

Figure 61.

The reflection of I in O is also the circumcenter of the excentral triangle.

[48]The reflection of the incenter in the circumcenter is X_{40} in ETC.

9.5. *Reflection conjugate of I*. The reflection conjugate of I is the reflection of I in the Feuerbach point F_e. [49] It is the isogonal conjugate of the inversive image of I in the circumcircle. It is also the perspector of the reflections of the excenters in the respective sides.

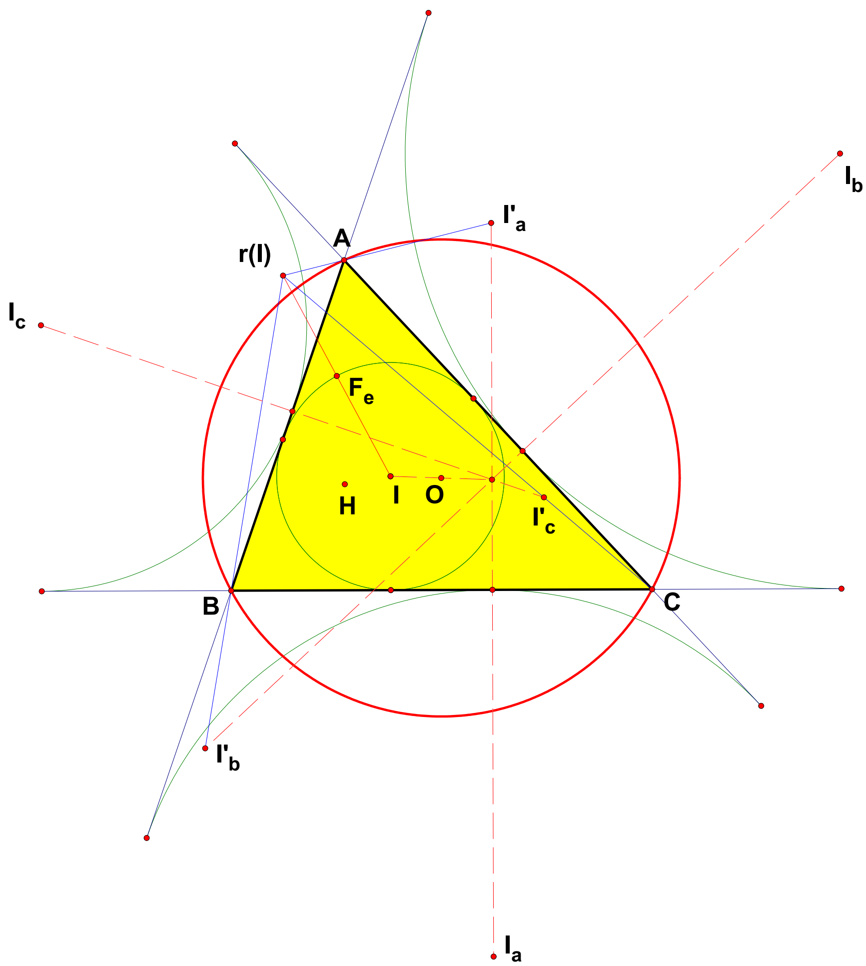

Figure 62.

[49]The reflection conjugate $r(I)$ is X_{80} in ETC.

10. Constructions for conics

We list a number of basic ruler-and-compass constructions for conics defined by 5 points, no three of which are concurrent, and no four are concyclic.

10.1. *The tangent at a point on* $\mathcal{C}$.
(1) $P := AC \cap BD$;
(2) $Q := AD \cap CE$;
(3) $T := PQ \cap BE$.
AT is the tangent at A.

10.2. *The second intersection of* $\mathcal{C}$ *and a line* ℓ *through* A.
(1) $P := AC \cap BE$;
(2) $Q := \ell \cap BD$;
(3) $R := PQ \cap CD$;
(4) $A' := \ell \cap ER$.
A' is the second intersection of $\mathcal{C}$ and ℓ.

10.3. *The center of* $\mathcal{C}$.
(1) B' := the second intersection of $\mathcal{C}$ with the parallel through B to AC;
(2) ℓ_b := the line joining the midpoints of BB' and AC;
(3) C' := the second intersection of $\mathcal{C}$ with the parallel through C to AB;
(4) ℓ_c := the line joining the midpoints of CC' and AB;
(5) $O := \ell_b \cap \ell_c$ is the center of the conic $\mathcal{C}$.

10.4. *Principal axes of* $\mathcal{C}$. (1) $K(O)$:= any circle through the center O of the conic $\mathcal{C}$.

(2) Let M be the midpoint of AB. Construct (i) OM and (ii) the parallel through O to AB each to intersect the circle at a point. Join these two points to form a line ℓ.

(3) Repeat (2) for another chord AC, to form a line ℓ'.

(4) $P := \ell \cap \ell'$.

(5) Let KP intersect the circle $K(O)$ at X and Y.

Then the lines OX and OY are the principal axes of the conic $\mathcal{C}$.

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