On the Original Malfatti Problem

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ON THE ORIGINAL MALFATTI PROBLEM

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1. Introduction. In 1803, Malfatti (1737–1807), of the University of Ferrara, proposed the following problem [1]:

Given a right triangular prism of any sort of material, such as marble, how shall three circular cylinders of the same height as the prism and of the greatest possible volume be related to one another in the prism and leave over the least possible amount of material?

This reduces to the plane problem of cutting three circles from a given triangle so that the sum of their areas is maximized.

Malfatti, and many others who considered the problem, assumed that the solution would be the three circles which are tangent to each other, while each circle is tangent to two sides of the triangle, as in Figure 1a. These circles have become known as the Malfatti circles. The construction of the Malfatti circles, and the derivation of their sizes, have been the subject of many elegant papers. A brief history of these is given by Eves [2], and a more extensive history is given by Lob and Richmond [3]. The solution by Schellbach is given by Dörrie [4].

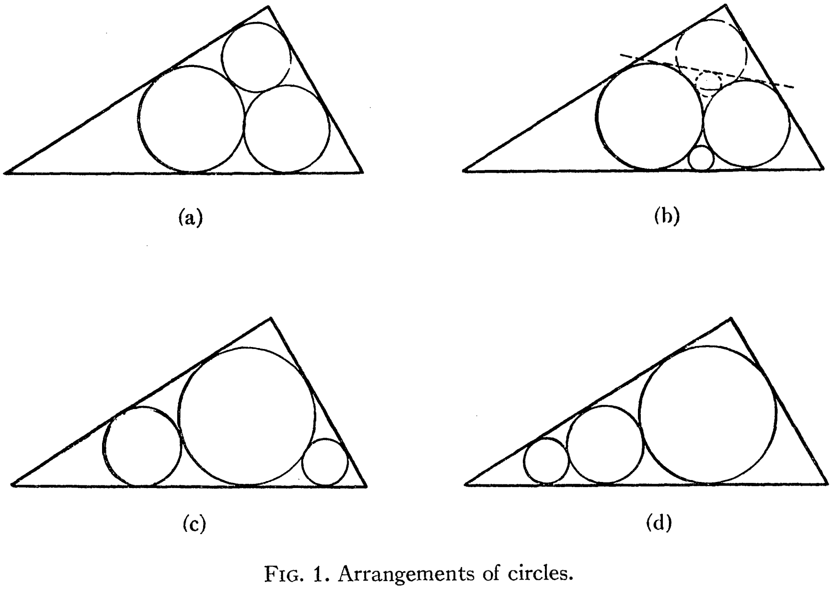

FIG. 1. Arrangements of circles.

It was not until 1929 that Lob and Richmond [3] noted that the Malfatti circles were not always the solution of the original Malfatti problem. In a brief note at the end of their paper, they remarked that for an equilateral triangle, the inscribed circle, with two little circles squeezed into the angles, contain a greater area than Malfatti's three circles. Eves [2] indicates that for very tall triangles, three circles placed one above the other can have a combined area greater than that of the Malfatti circles. It is the purpose of this note to show that the Malfatti circles are *never* the solution of the original Malfatti problem.

2. Cirles tangent to a line. A maximum area is not reached unless each circle is restrained from growing by making at least three contacts, either with the sides of the triangle or with other circles. These contacts must be distributed along the circumference so that they do not all lie within a semicircle; otherwise, the circle would not be in static equilibrium and could be enlarged by some adjustment. The Malfatti circles meet this condition and give a local maximum. However, other arrangements also meet this condition. The Malfatti arrangement is the only one in which each side touches only two circles, as shown in Figure 1a. In the other arrangements, the three circles touch the same side of the triangle. There are three such arrangements, shown in Figures 1b, 1c, and 1d. The middle circle may be the largest, the smallest, or the median circle. The case in which the smallest circle is in the middle, as shown in Figure 1b, can be improved by placing the smallest circle in the opposite angle where it can be larger, as shown in Figure 1a. The dotted line in Figure 1b is the other tangent to the two larger circles. By symmetry, the dotted circle is the same size as the smallest circle. However, by removing the constraint of the dotted line, the dotted circle can grow until it becomes the dashed circle when it touches one of the sides of the triangle. Each of the other cases may be best, depending upon the angles of the given triangle.

3. The Lob-Richmond-Goldberg construction (LRG). The following construction always yields a larger area than the Malfatti circles. First inscribe a circle in the given triangle. Then inscribe the second circle in the smallest angle and tangent to the first circle. The third circle may be inscribed in the same angle or in the next larger angle—whichever permits the larger circle. There are equivocal cases in which the two have the same area.

4. The radii of the Malfatti circles. The given triangles may have all possible shapes. Let us assume that they all have an inscribed circle of unit radius. Then, if the radii of the Malfatti circles are designated by r_1, r_2, r_3, it is shown by Lob and Richmond [**3**, p. 302] that

$$r_1 = (1+v)(1+w)/2(1+u), \qquad r_2 = (1+w)(1+u)/2(1+v),$$
$$r_3 = (1+u)(1+v)/2(1+w),$$

where $u=\tan A/4$, $v=\tan B/4$, $w=\tan C/4$, and A, B, C are the angles of the triangle.

If we maximize the sum of the squares of the radii, then this is equivalent to maximizing the area. For the Malfatti circles, let $M\equiv r_1^2+r_2^2+r_3^2$. Since $\tan\,(A/4+B/4+C/4)=\tan\,\pi/4=1$, we have $(u+v+w-uvw)/(1-vw-uw-uv)=1$, from which $w=\{(1-uv)-(u+v)\}/\{(1-uv)+(u+v)\}$ and $1+w=2(1-uv)/\{(1-uv)+(u+v)\}$. By means of this equation, the variable w can be eliminated, leaving M as a function of only u and v, namely:

$$M = (1 + u)^2(1 + v)^2(1 + u + v - uv)^2/16(1 - uv)^2$$
$$+ (1 - uv)^2\{(1 + u)^4 + (1 + v)^4\}/(1 + u)^2(1 + v)^2(1 + u + v - uv)^2.$$

5. The radii of the LRG circles. The first circle has unit radius. Let the smallest angle be called A, and the next larger angle (or equal) be called B. Then, the radius of the second circle is given by $r_2 = (1 - \sin A/2)/(1 + \sin A/2)$. If $\tan A/4 = u$, then, since $\sin A/2 = (2 \tan A/4)/(1 + \tan^2 A/4) = 2u/(1+u^2)$, we have

$$r_2 = \{(1 - u)/(1 + u)\}^2,$$
$$r_3 = \{(1 - v)/(1 + v)\}^2, \quad \text{if the third circle is in } B, \quad (\textit{Case } 1).$$

or

$$r_3 = \{(1 - u)/(1 + u)\}^4, \quad \text{if the third circle is in } A, \quad (\textit{Case } 2).$$

Hence, $\mathrm{LRG}(1) = r_1^2 + r_2^2 + r_3^2 = 1 + \{(1-u)/(1+u)\}^4 + \{(1-v)/(1+v)\}^4$, and $\mathrm{LRG}(2) = 1 + \{(1-u)/(1+u)\}^4 + \{(1-u)/(1+u)\}^8$.

6. The case of the equilateral triangle. The problem for the special case of the equilateral triangle has been extended by Procissi [5]. He inscribed a circle of radius y in one angle of an equilateral triangle of edge 2 and two circles of radius x in the other two angles, making the circles of radius x tangent to the circle of radius y. Then the relation between x and y is given by

$$y = \{2\sqrt{3} - x - \sqrt{8x(\sqrt{3} - x)}\}/3.$$

If S is the sum of the areas of the circles, then in Procissi's notation, $F(x) \equiv 9S/\pi = 9(2x^2 + y^2)$. The graph of the function $F(x)$ is shown in Figure 2. The curve has a horizontal tangent near $x = 0.27$. At this point, the function has the minimum value of 3.320.

The Malfatti circles are given by $x = y = (\sqrt{3} - 1)/2 = 0.366$ for which $F(x) = 3.618$. The LRG circles are given by $x = \sqrt{3}/9 = 0.192$, then $y\sqrt{3}/3 = 0.577$, and $F(x) = 3.667$. Each of these two values of $F(x)$ is a "corner maximum" since the slopes of the curve at these points are not zero.

For $x < 0.192$, the circle of radius y protrudes outside of the triangle, and for $x > 0.366$, the circles of radius x will overlap. Both of these cases are not admissible by the geometric conditions of the problem.

7. The general triangle. If a circle of radius x is inscribed in one angle of a given triangle and then circles of radii y and z are inscribed in the other angles, making the circles of radii y and z tangent to the circle of radius x, then y and z are expressible as functions of x. Therefore S is a function of x, say $F(x) = 9S/\pi$. This function is similar to the function for the equilateral triangle; namely, it will have a minimum for some value of x, and two "corner maxima," one of which corresponds to the Malfatti circles and the other to three circles tangent to one side. In general, these curves can be made in three ways, depending upon the choice of the angle in which the circle of radius x is inscribed. It will be

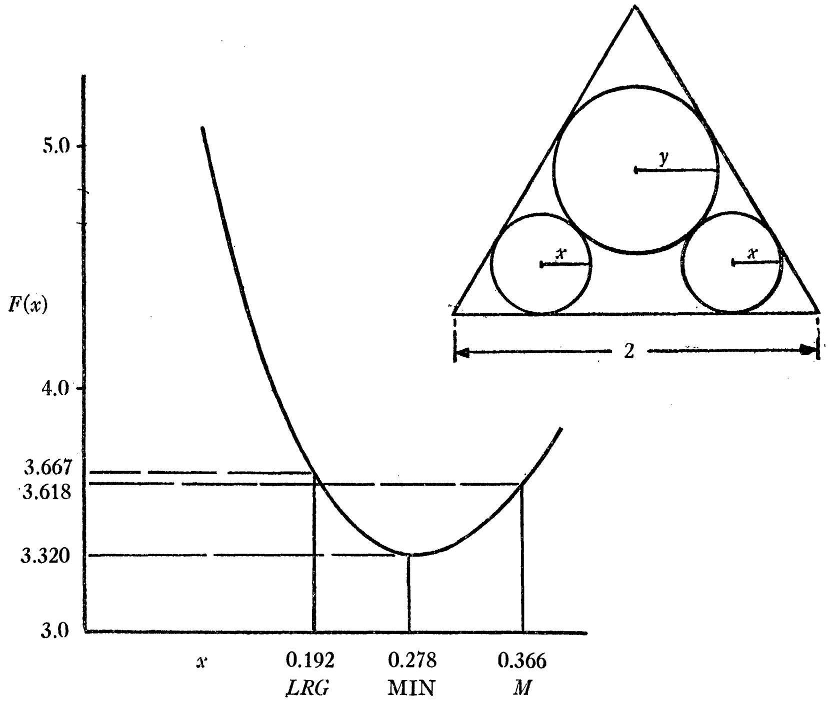

FIG. 2. Three circles in equilateral triangle.

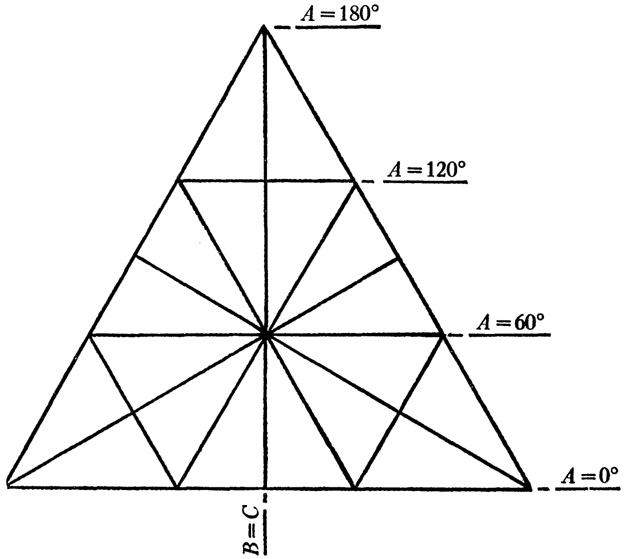

FIG. 3. Location of sections of surfaces.

shown that there is always a choice which makes the Malfatti sum the smaller of the two maxima.

8. Graphical demonstration of the nature of the surfaces. A rigorous algebraic proof of the greater values of the LRG sums over the M sums could become quite involved. It is proposed, therefore, to compute and describe the surfaces which represent the LRG and M sums as functions of the variables A, B, and C. Since $A+B+C=\pi$, we can indicate a triangle of angles A, B, and C as a point P in an equilateral triangle of height π (Figure 3). Then, the distances of the point from the sides of the triangle are A, B, and C. The continuous one-parameter family of isosceles triangles is represented by a median of the equilateral triangle. The values of LRG and M were computed for these isosceles triangles and are shown on the graph of Figure 4. Figure 5 shows the values for $A=0$, and $B+C=180°$. Figure 6 shows the values for $A=60°$ and $B+C=120°$. Figure 7 shows the values for $A=120°$ and $B+C=60°$. The curves on these graphs correspond to the sections of the surfaces cut by the planes indicated by the lines of Figure 3.

The M surface resembles a paraboloid of revolution. The LRG surface can be approximated by a segment of a paraboloid of revolution which has been deformed so that the circle at the top edge has been distorted into an equilateral triangle.

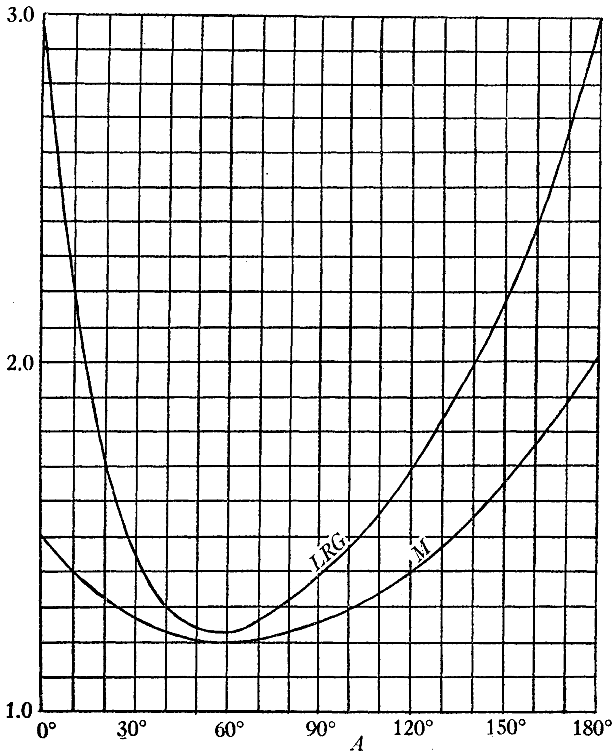

FIG. 4. Isosceles triangles, $B=C$.

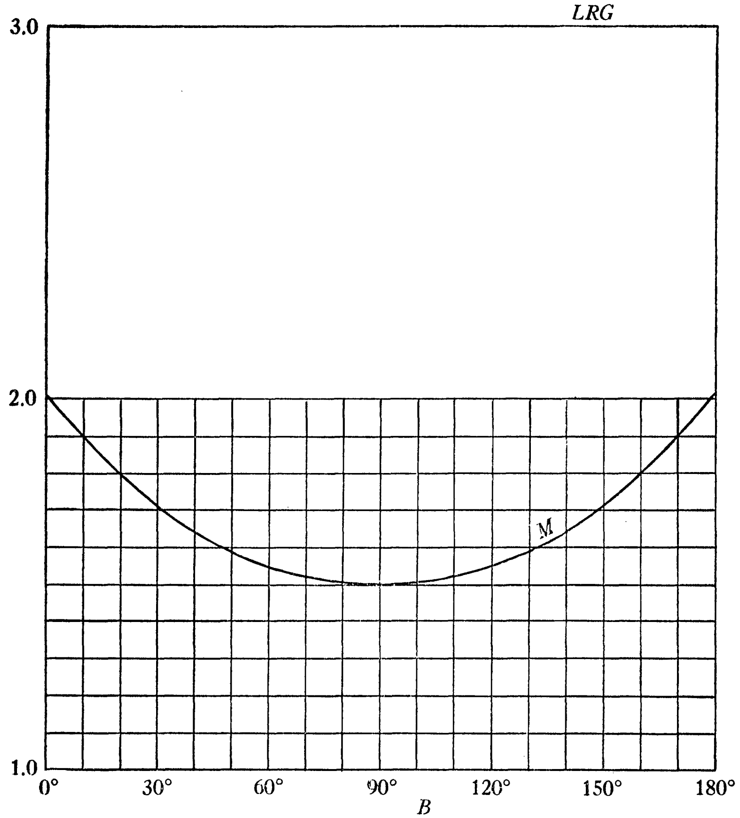

FIG. 5. $A=0°$.

For both surfaces, the lowest point occurs at $(A, B, C)=(60°, 60°, 60°)$. The axis of the surfaces is a vertical line through this point. A plane passing through this axis cuts the surfaces in two curves. One case is shown in Figure 4. Another case is shown in Figure 6. For other directions, an interpolated pair of curves is obtained. From the nature of the functions from which the surfaces are computed the curves are well behaved; they are continuous and have continuously increasing first derivatives. The surfaces made from these curves are, similarly, well behaved.

For each section through the axis, the curves have horizontal tangents at the axis. The ordinate of the lowest point of the M surface is $9(2-\sqrt{3})/2 = 1.206$. The ordinate of the lowest point of the LRG surface is slightly greater, namely, $11/9=1.222$. As we move away from the axis, the ordinates increase monotonically. The rate of increase on the LRG surface is always greater than the rate of increase on the M surface. Hence, over any point of the base triangle, the ordinate of the LRG surface is always greater than the ordinate of the M surface. This is evident from the numerical computation and graphing of the curves. A rigorous demonstration of this fact would be desirable, but it has not yet been developed.

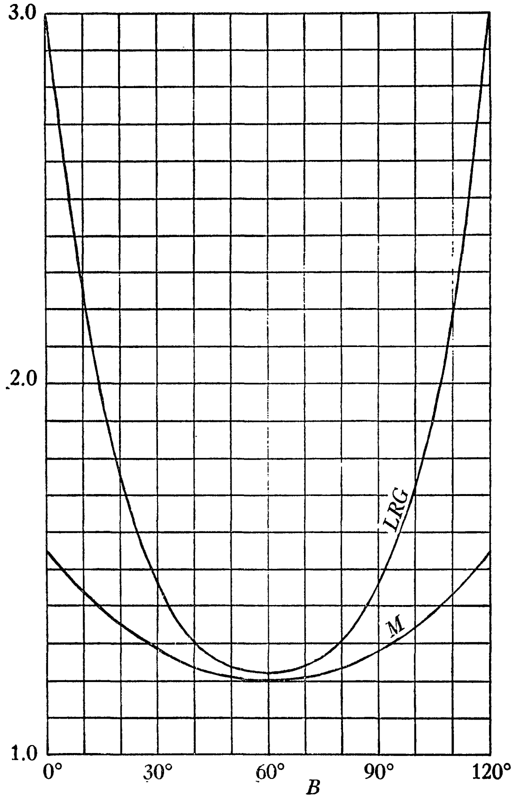

FIG. 6. $A=60°$.

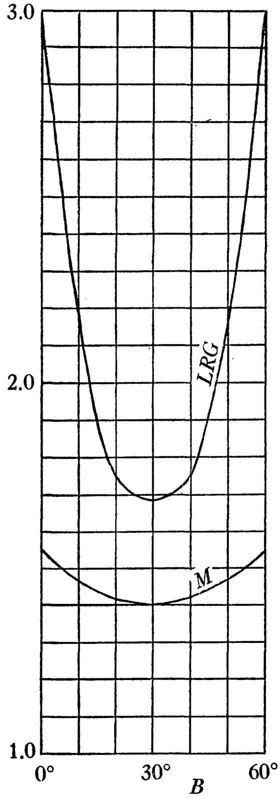

FIG. 7. $A=120°$.

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GENERALIZATIONS OF THE A.M. AND G.M. INEQUALITY

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Throughout this note Greek letters $\alpha, \beta, \cdots$ denote real numbers, Roman letters $a, b, \cdots$ denote positive real numbers, and capital Roman letters