The Circles of Lester, Evans, Parry, and Their Generalizations

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Abstract: Beginning with the famous Lester circle containing the circumcenter, nine-point center and the two Fermat points of a triangle, we survey a number of interesting circles in triangle geometry.

CONTENTS

1. Some common triangle centers

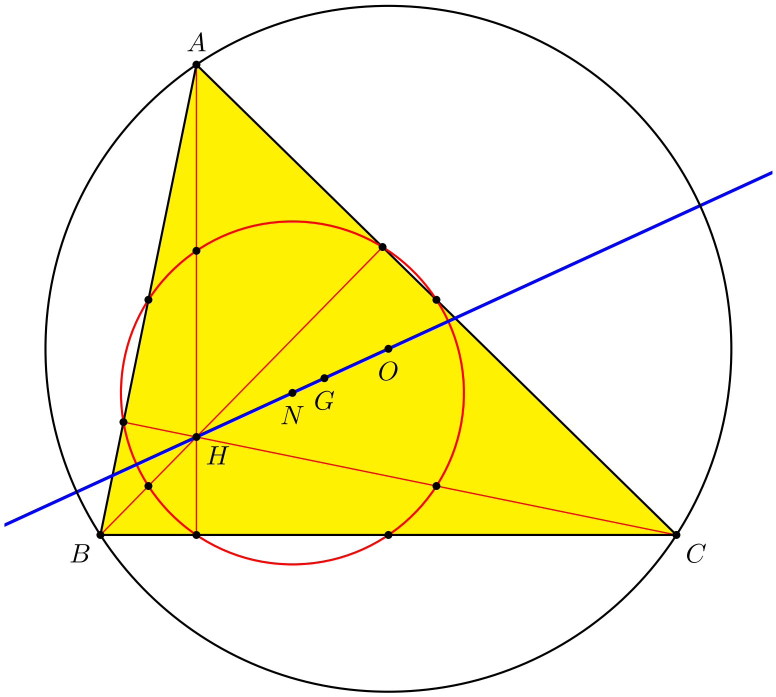


Figure 1. The Euler line and the nine-point circle

$$HN : NG : GO = 3 : 1 : 2.$$

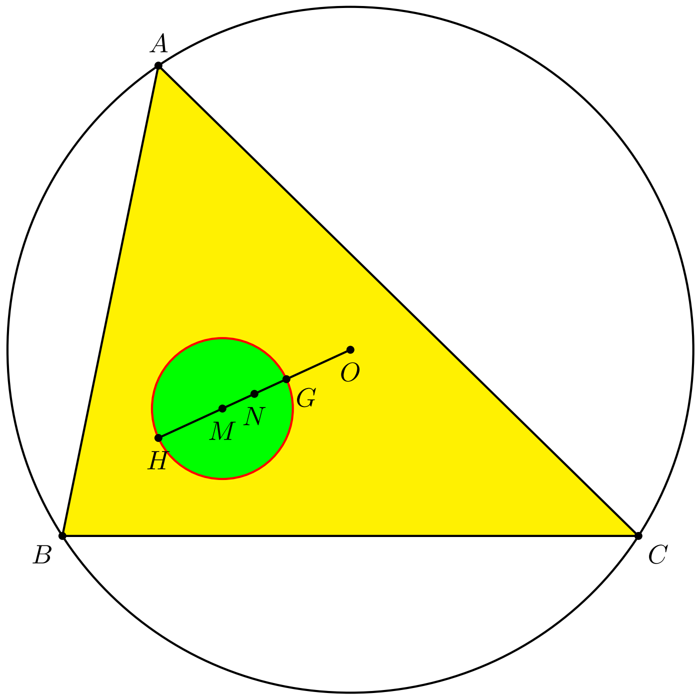


Figure 2. The orthocentroidal circle

O and N are inverse in the orthocentroidal circle.

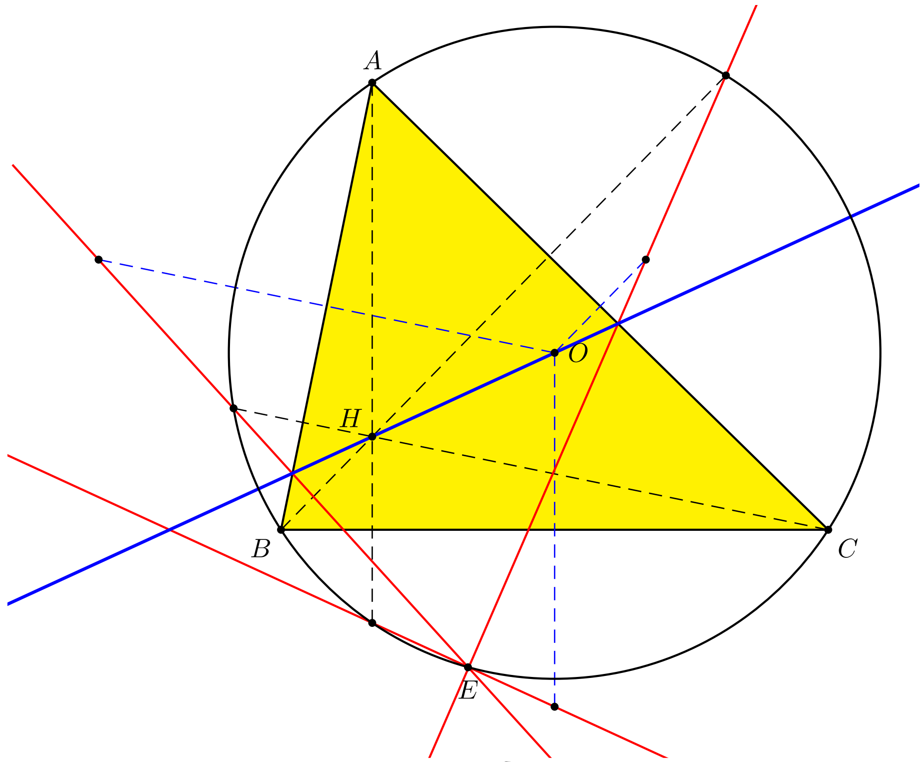


Figure 3. The Euler reflection point E

The reflections of the Euler line in the three sidelines intersect at a point on the circumcircle:

$$E = \left(\frac{a^2}{b^2 - c^2} : \frac{b^2}{c^2 - a^2} : \frac{c^2}{a^2 - b^2} \right).$$

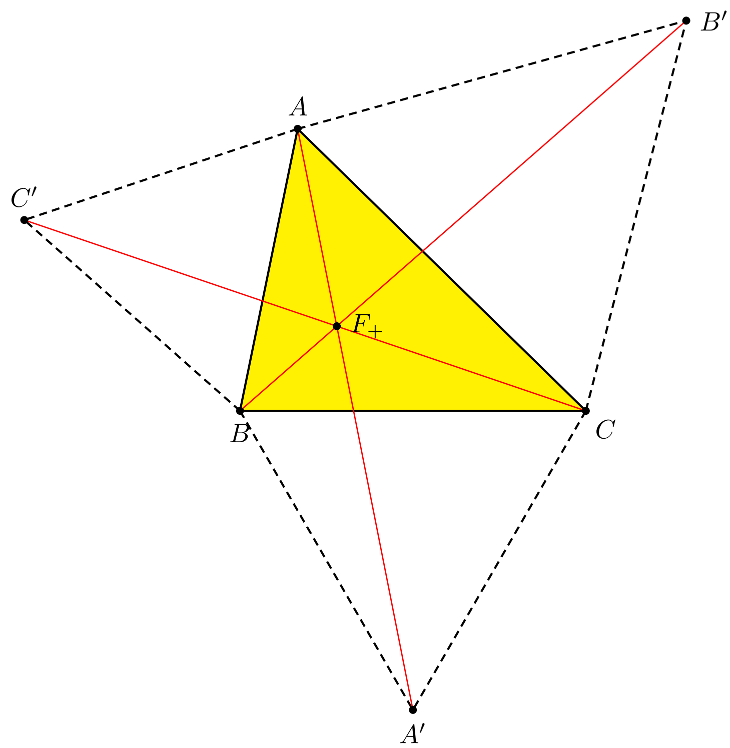


Figure 4. The Fermat point F_+

Construct equilateral triangles $A'BC$, $AB'C$, ABC' externally on the sides of triangle ABC.
AA', BB', and CC' concur at the **Fermat point**

$$F_+ = K\left(\frac{\pi}{3}\right) = \left(\frac{1}{\sqrt{3}S_A + S} : \frac{1}{\sqrt{3}S_B + S} : \frac{1}{\sqrt{3}S_C - S}\right).$$

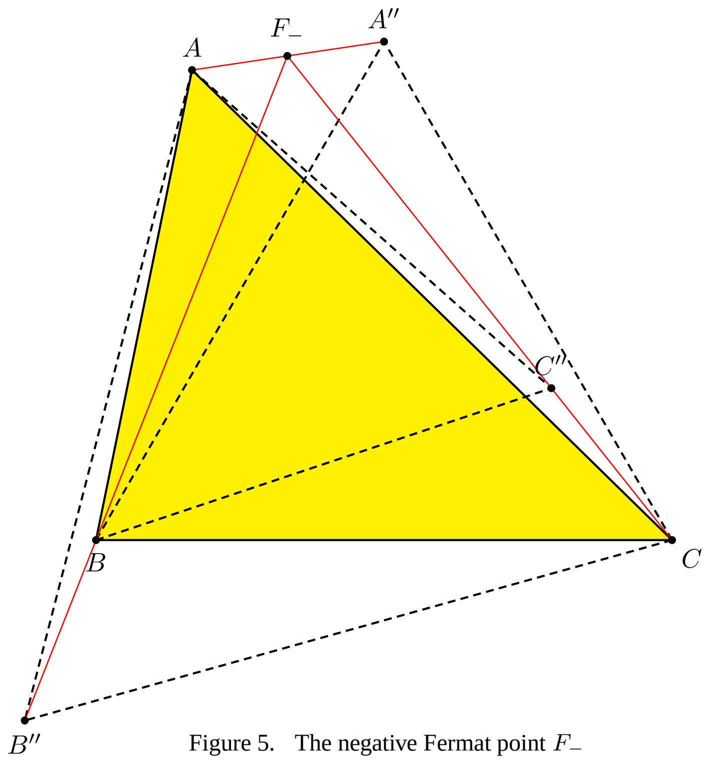


Figure 5. The negative Fermat point F_-

If the equilateral triangles $A''BC$, $AB''C$, ABC'' are constructed internally, AA'', BB'', and CC'' concur at the **negative Fermat point**

$$F_- = K\left(-\frac{\pi}{3}\right) = \left(\frac{1}{\sqrt{3}S_A - S} : \frac{1}{\sqrt{3}S_B - S} : \frac{1}{\sqrt{3}S_C + S}\right).$$

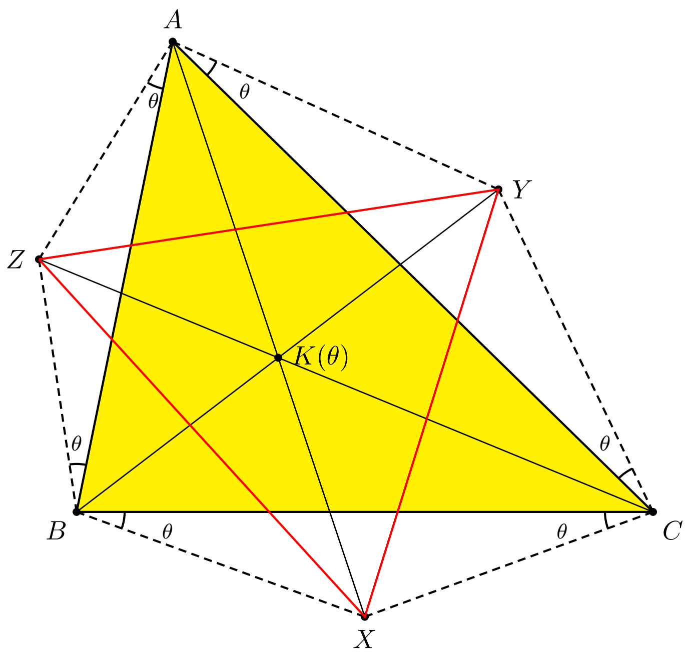


Figure 6. Kiepert triangle $\mathcal{K}(\theta)$ and Kiepert perspector $K(\theta)$

$$\text{Kiepert triangle } \mathcal{K}(\theta) := XYZ,$$

$$\text{Kiepert perspector } K(\theta) = \left(\frac{1}{S_A + S_\theta} : \frac{1}{S_B + S_\theta} : \frac{1}{S_C + S_\theta}\right).$$

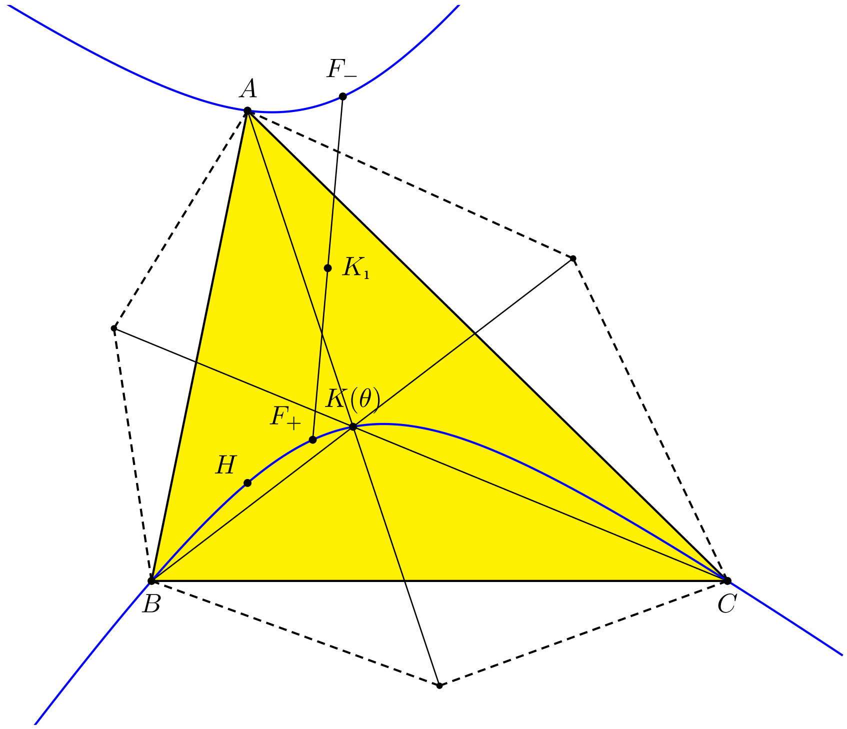


Figure 7. The Kiepert hyperbola

The locus of the Kiepert perspector is a rectangular hyperbola whose center is the midpoint of the Fermat points.

$$(b^2 - c^2)yz + (c^2 - a^2)zx + (a^2 - b^2)xy = 0.$$

2. The first Lester circle

Theorem 1 (Lester)**.** *The Fermat points are concyclic with the circumcenter and the nine-point center.*

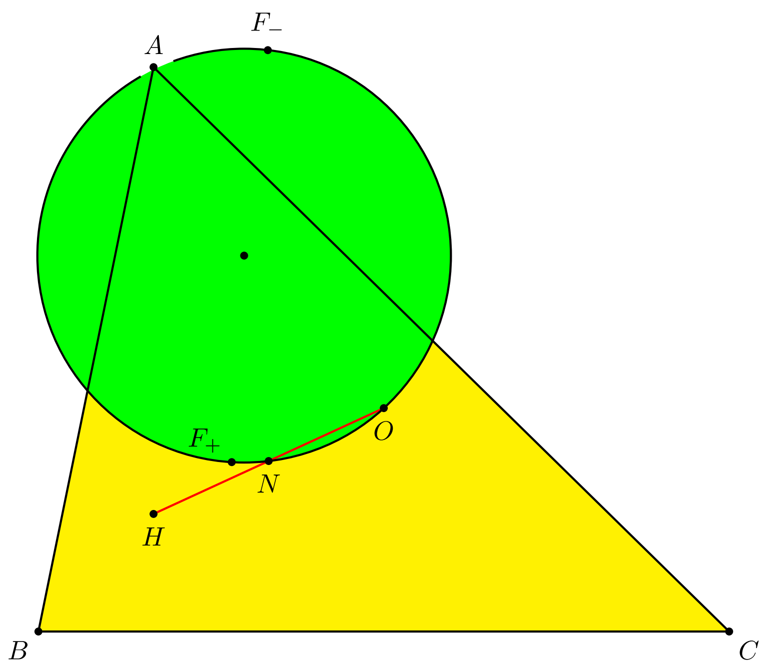


Figure 8. The first Lester circle through O, N and the Fermat points

Proof. (1) Let M be the intersection of F_+F_- and the Euler line. By the **intersection chords theorem**, it is enough to show that

$$MF_+ \cdot MF_- = MO \cdot MN.$$

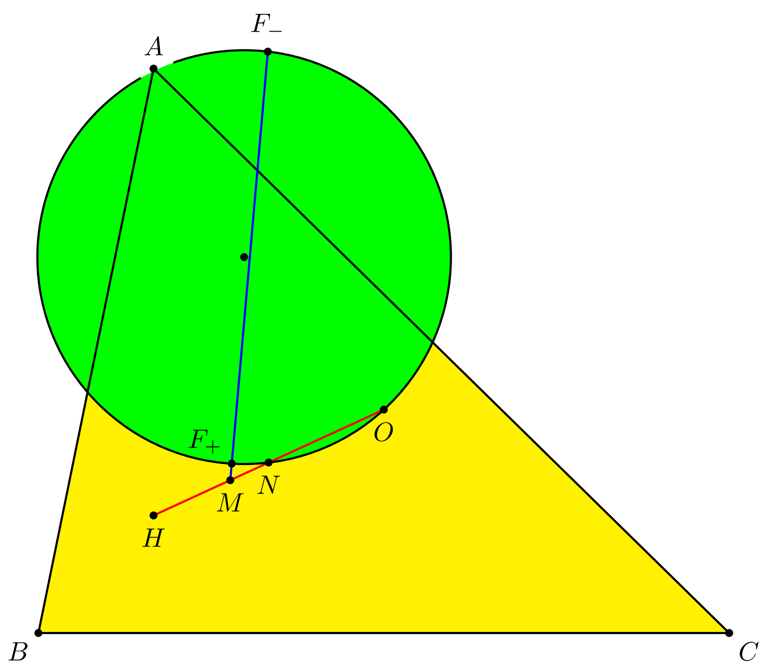


Figure 9. Intersection of Fermat line and Euler line

(2) Consider a Kiepert perspector $K(\theta)$ with homogeneous barycentric coordinates

$$K(\theta) = \left(\frac{1}{S_A + S_\theta} : \frac{1}{S_B + S_\theta} : \frac{1}{S_C + S_\theta}\right).$$

These homogeneous coordinates can be rewritten as

$$\begin{aligned}K(\theta) &= ((S_B + S_\theta)(S_C + S_\theta), (S_C + S_\theta)(S_A + S_\theta), (S_A + S_\theta)(S_B + S_\theta)) \\ &= (S_{BC} + S_{\theta\theta} + (S_B + S_C)S_\theta, \cdots, \cdots) \\ &=(S_{BC} + S_{\theta\theta}, S_{CA} + S_{\theta\theta}, S_{AB} + S_{\theta\theta}) + S_\theta(S_B + S_C, S_C + S_A, S_A + S_B).\end{aligned}$$

Similarly,

$$K(-\theta) =(S_{BC} + S_{\theta\theta}, S_{CA} + S_{\theta\theta}, S_{AB} + S_{\theta\theta}) - S_\theta(S_B + S_C, S_C + S_A, S_A + S_B).$$

From these, $K(\theta)$ and $K(-\theta)$ divide **harmonically**
the **symmedian point** $K = (S_B + S_C, S_C + S_A, S_A + S_B)$ and

$$\begin{aligned}Q(\theta) &=(S_{BC} + S_{\theta\theta}, S_{CA} + S_{\theta\theta}, S_{AB} + S_{\theta\theta}) \\ &=(S_{BC}, S_{CA}, S_{AB}) + S_{\theta\theta}(1, 1, 1)\end{aligned}$$

which is a point on the Euler line, dividing
the **orthocenter** $H = (S_{BC}, S_{CA}, S_{AB})$ and
the **centroid** $G = (1, 1, 1)$ in the ratio

$$GQ(\theta) : Q(\theta)H = 3S_{\theta\theta} : S^2 = 3\cot^2\theta : 1.$$

$$GQ(\theta) : Q(\theta)H = 3S_{\theta\theta} : S^2 = 3\cot^2\theta : 1.$$

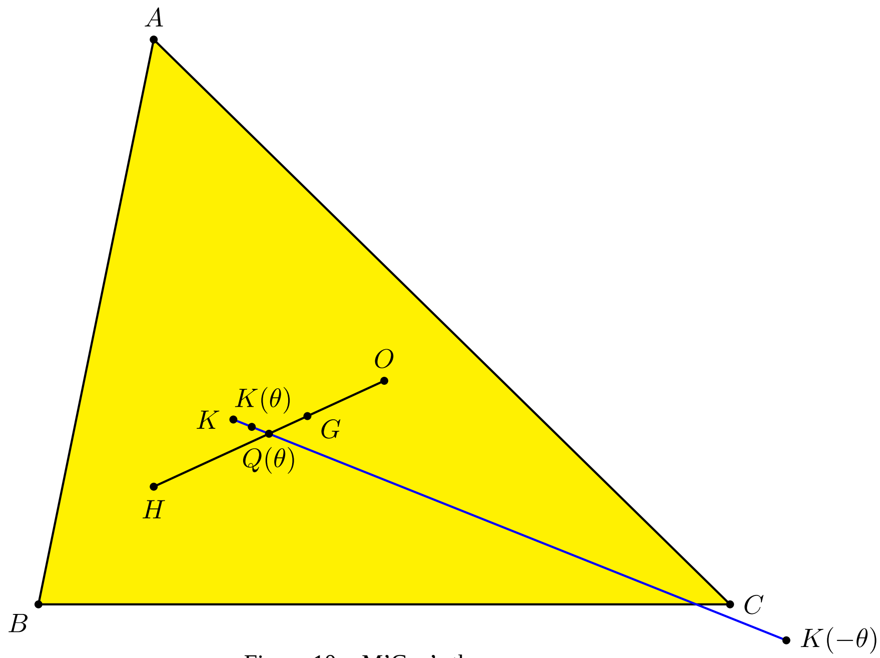


Figure 10. M'Cay's theorem

(3) For $\theta = \pm\frac{\pi}{3}$, this ratio is $1:1$.
$M = Q\left(\frac{\pi}{3}\right)$ is the midpoint of GH.
The Fermat line F_+F_- intersects the Euler line at the midpoint of H and G, which is the center of the **orthocentroidal circle** with HG as diameter.

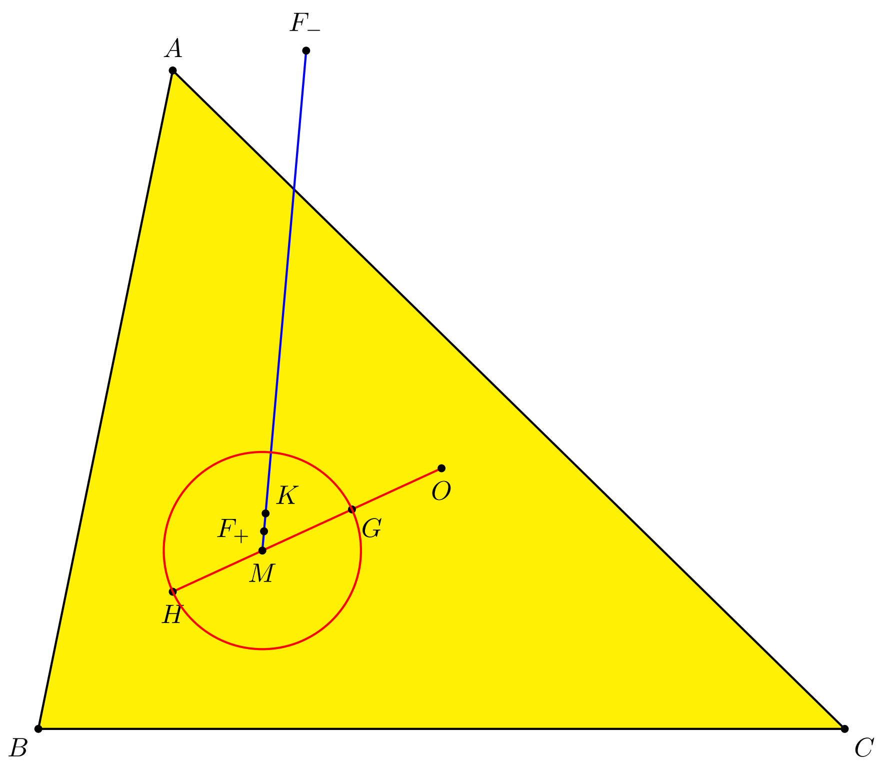


Figure 11. Fermat line and orthocentroidal circle

(4) If we put $OH = 6d$, then

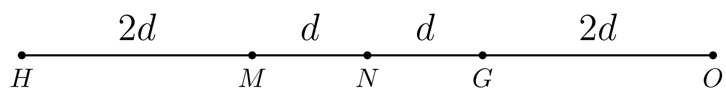


Figure 12. The Euler line

$$MO \cdot MN = 4d \cdot d = (2d)^2 = (MH)^2 = (MG)^2.$$

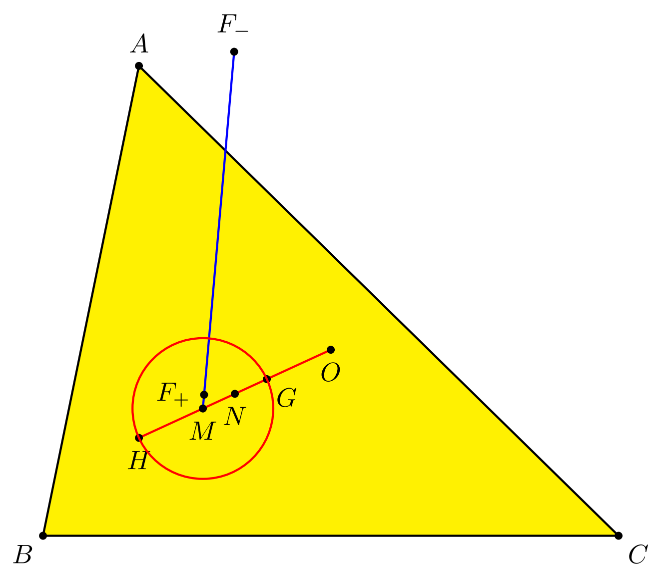


Figure 13. Fermat line and orthocentroidal circle

(5) Recall from (1) $MF_+ \cdot MF_- = MO \cdot MN = (MH)^2 = (MG)^2$.

Therefore, the Lester circle theorem is equivalent to each of the following.

(1) The Fermat points are inverse in the orthocentroidal circle.

(2) The circle F_+F_-G is tangent to the Euler line at G.

(3) The circle F_+F_-H is tangent to the Euler line at H.

$\square$

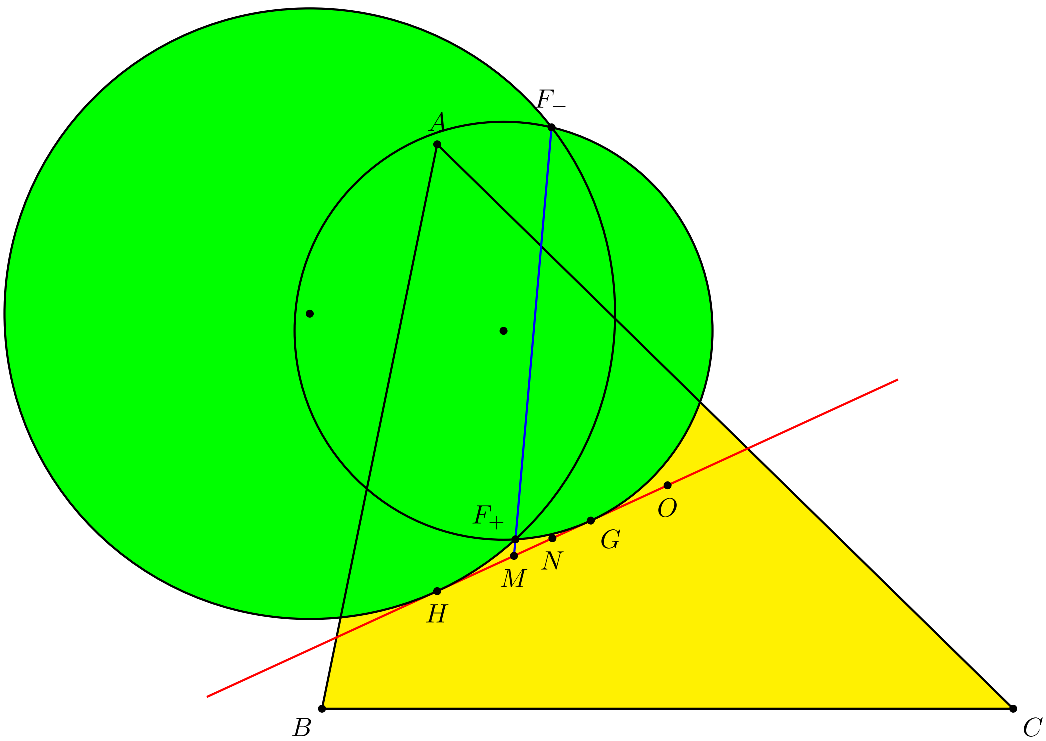


Figure 14. The circles F_+F_-G and F_+F_-H

Theorem 2. *The Fermat points are inverse in the orthocentroidal circle.*

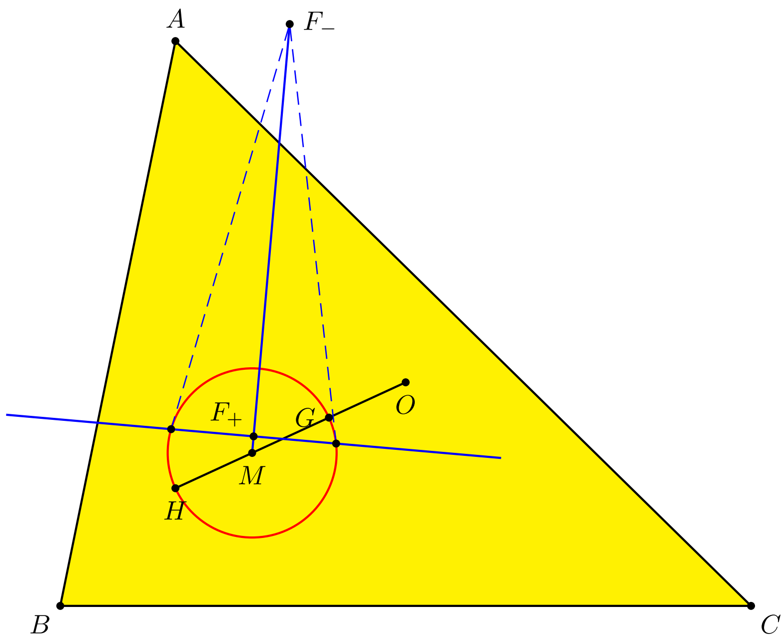


Figure 15. F_+ on the polar of F_- in the orthocentroidal circle

Proof. Let M be the matrix of the orthocentroidal circle.

$$\mathsf{M} = \begin{pmatrix} -4S_A & S_A+S_B & S_A+S_C \\ S_A+S_B & -4S_B & S_B+S_C \\ S_A+S_C & S_B+S_C & -4S_C \end{pmatrix}.$$

Write

$$F_+ = X + Y \quad \text{and} \quad F_- = X - Y,$$

with

$$X = \left(S_{BC} + \tfrac{1}{3}S^2 \quad S_{CA} + \tfrac{1}{3}S^2 \quad S_{AB} + \tfrac{1}{3}S^2\right),$$

$$Y = \frac{S}{\sqrt{3}}\left(S_B + S_C \quad S_C + S_A \quad S_A + S_B\right).$$

$$X\mathsf{M}X^t = Y\mathsf{M}Y^t = \frac{2}{3}(S_A(S_B - S_C)^2 + S_B(S_C - S_A)^2 + S_C(S_A - S_B)^2)S^2,$$

we have

$$F_+\mathsf{M}F_-^t = (X+Y)\mathsf{M}(X-Y)^t = X\mathsf{M}X^t - Y\mathsf{M}Y^t = 0.$$

This shows that the Fermat points are inverse in the orthocentroidal circle. □

Corollary 3. *Every circle through F_+ and F_- is orthogonal to the orthocentroidal circle.*

Theorem 4 (Gibert)**.** *Every circle with diameter a chord of the Kiepert hyperbola perpendicular to the Euler line passes through the Fermat points.*

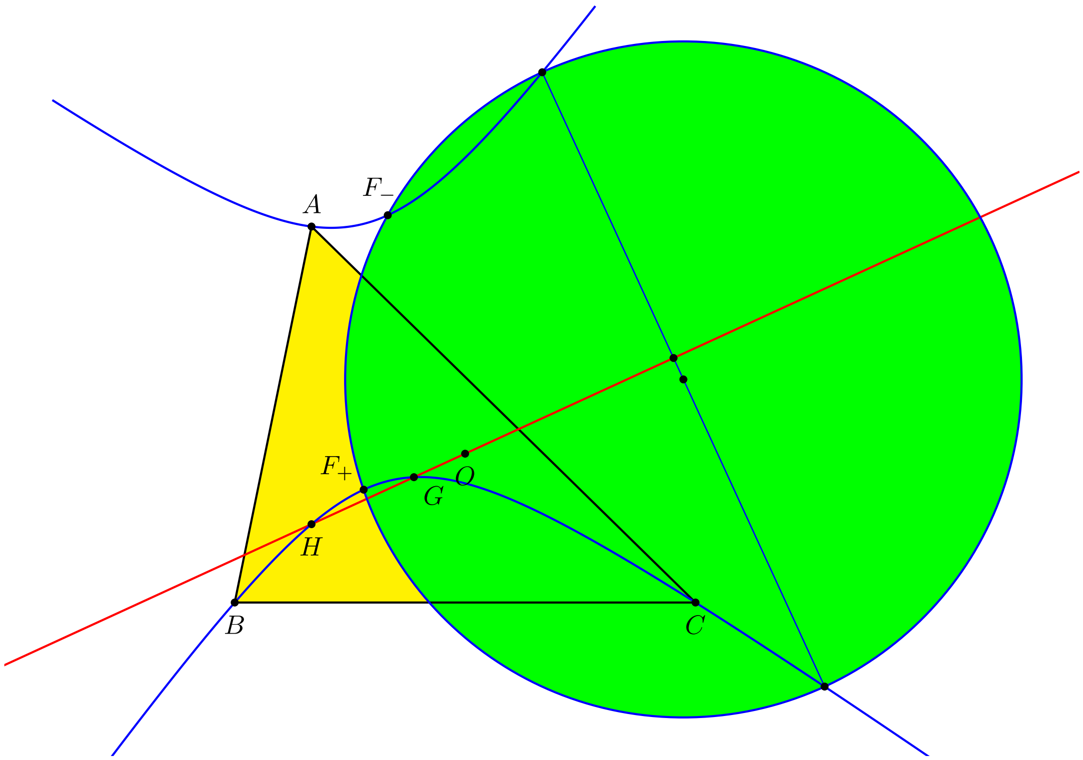


Figure 16. Gibert's generalization of Lester's circle

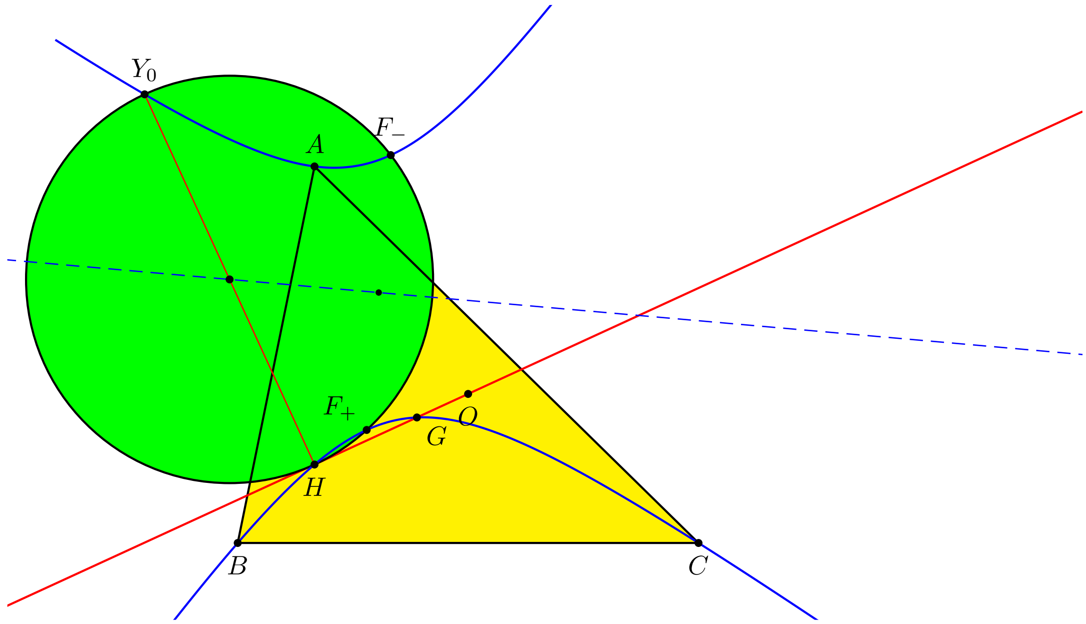


Figure 17. The circle F_+F_-H

Equation of line F_+F_-: $\quad L = 0$.
Perpendicular to Euler line at H: $L_0 = 0$,
intersecting Kiepert hyperbola at Y_0.
Circle F_+F_-H is one in the pencil of conics through F_+, F_-, H and Y_0:

$$(b^2 - c^2)yz + (c^2 - a^2)zx + (a^2 - b^2)xy - L \cdot L_0 = 0$$

by suitably adjusting the linear forms L, L_0 by constants.
It center lies on the perpendicular bisector of F_+F_-.

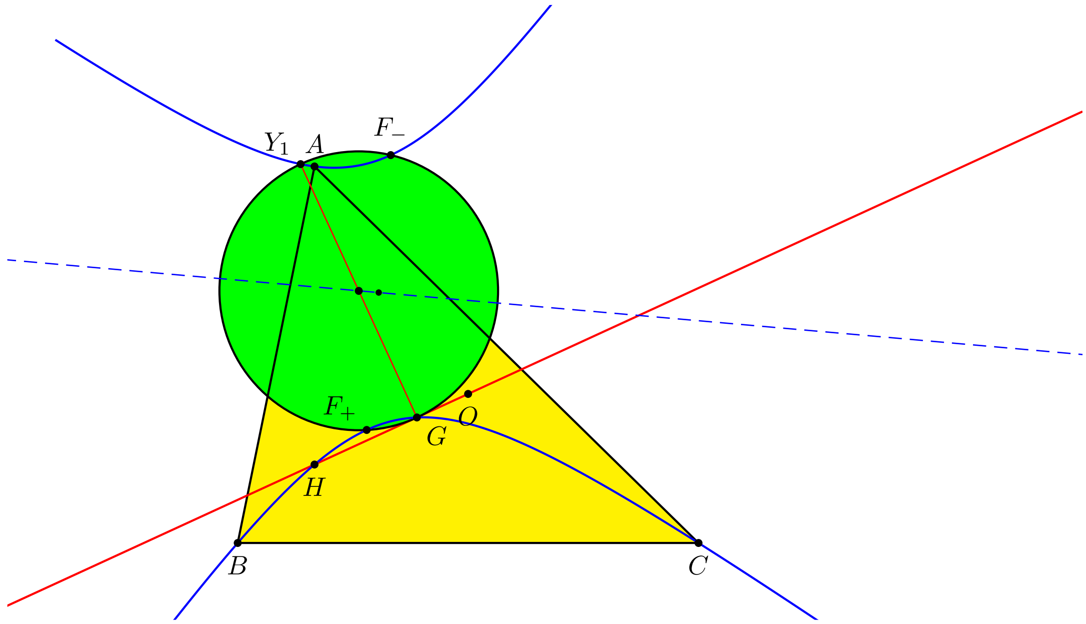


Figure 18. The circle F_+F_-G

Equation of line F_+F_-: $\quad L = 0$.
Perpendicular to Euler line at G: $L_1 = 0$,
intersecting Kiepert hyperbola at Y_1.
Circle F_+F_-G is one in the pencil of conics through F_+, F_-, G and Y_1:

$$(b^2 - c^2)yz + (c^2 - a^2)zx + (a^2 - b^2)xy - L \cdot L_1 = 0$$

by suitably adjusting the linear forms L, L_1 by constants.
It center lies on the perpendicular bisector of F_+F_-.

For arbitrary t, let $L_t = (1-t)L_0 + t \cdot L_1$.
The line $L_t = 0$ is perpendicular to the Euler line.
The equation

$$(b^2 - c^2)yz + (c^2 - a^2)zx + (a^2 - b^2)xy - L \cdot L_t = 0$$

represents a circle through the Fermat points.

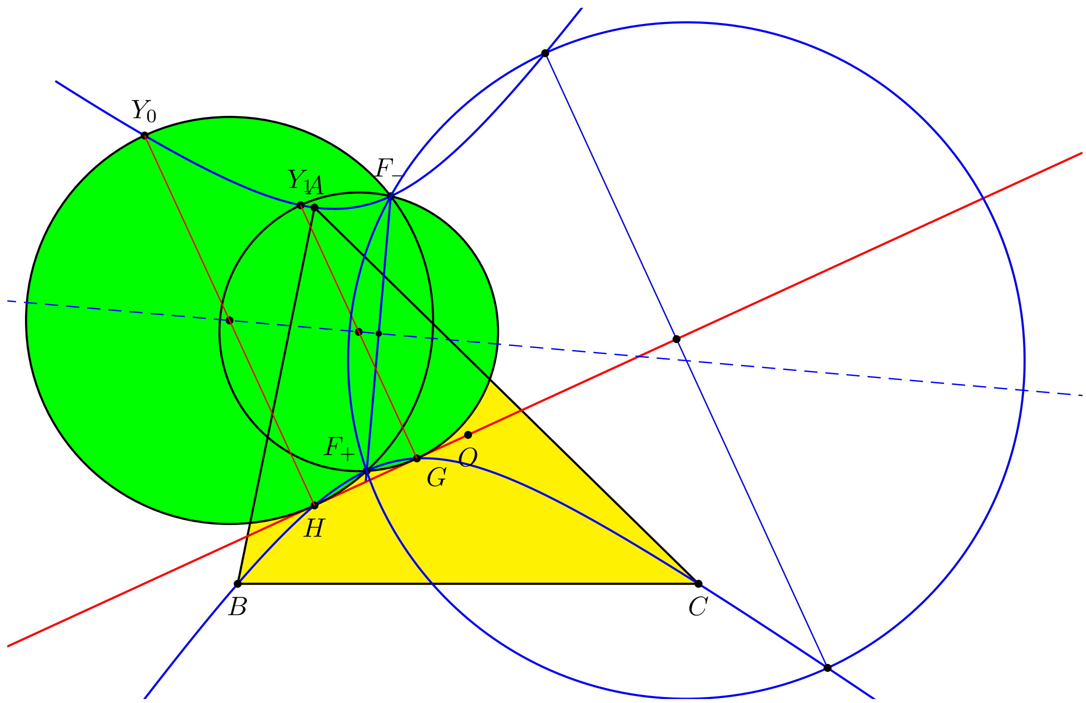


Figure 19. Gibert's generalization of Lester's circle

The line joining the midpoints of HY_0 and GY_1
contains the midpoint of the every chord cut out by $L_t = 0$.
This line is also the perpendicular bisector of F_+F_-.
Therefore the center of the circle is the midpoint of the chord.

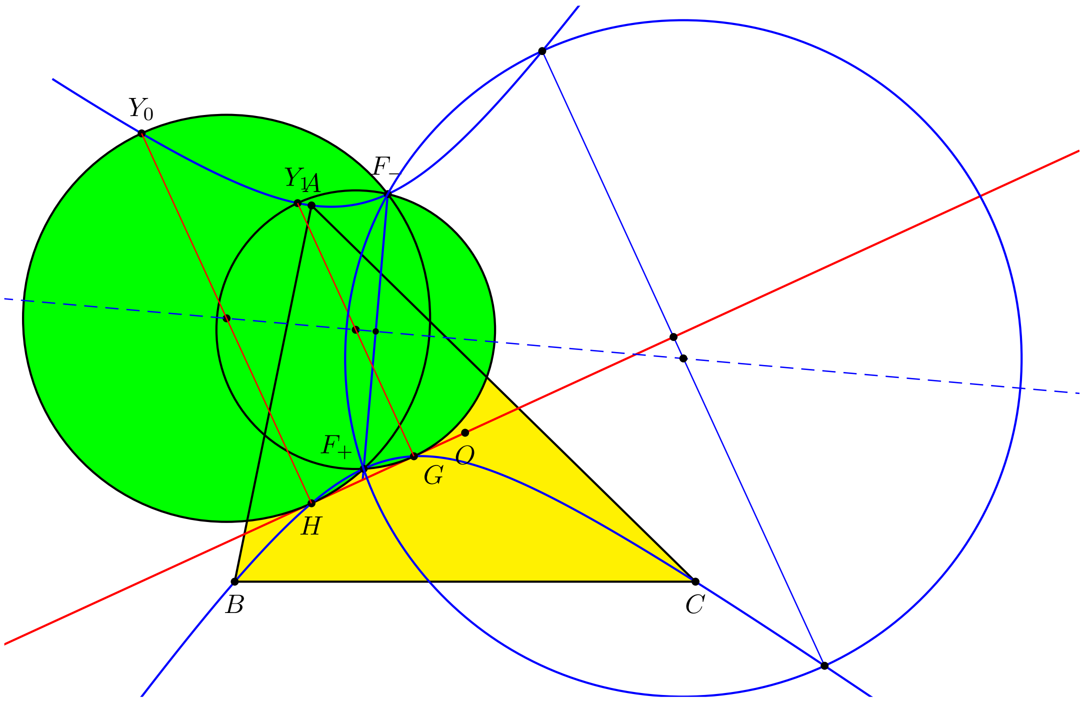


Figure 20. Gibert's generalization of Lester's circle

3. The symmedian and isodynamic points

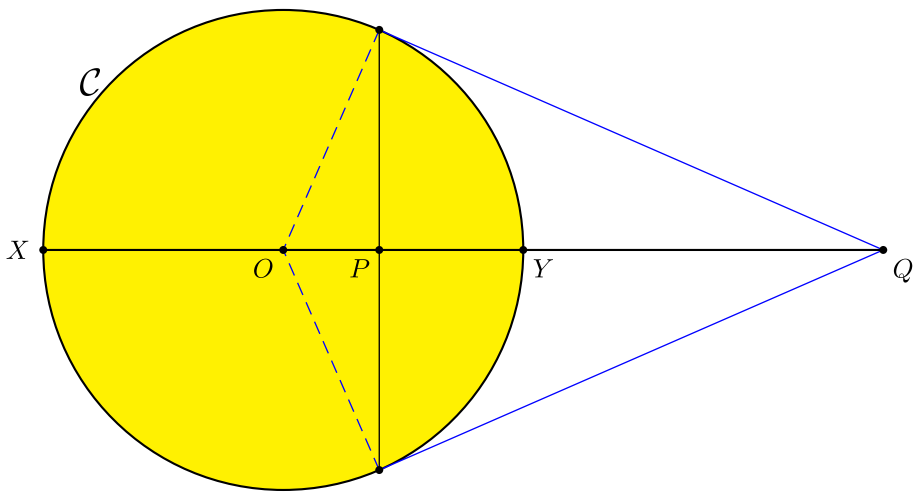


Figure 21

If P, Q divide X, Y harmonically,
then P and Q are inverse in the circle with diameter XY.

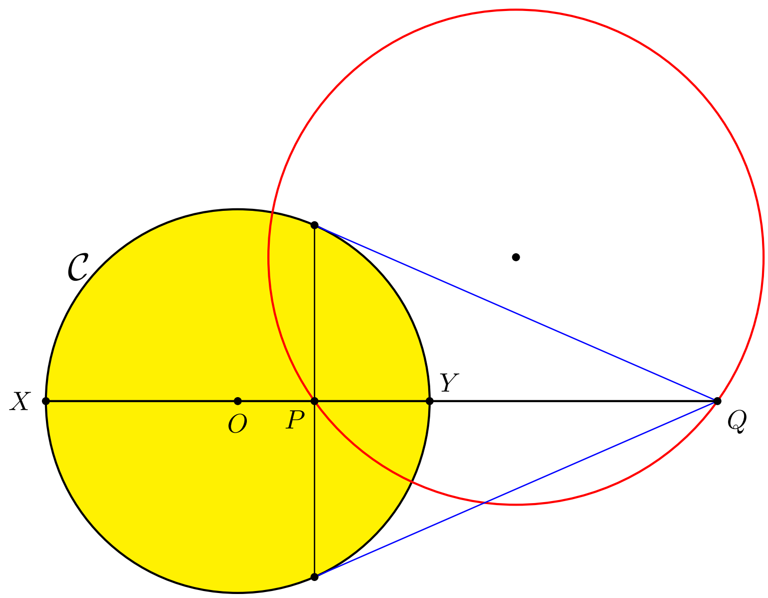


Figure 22

If P and Q are inverse in a circle $\mathcal{C}$,
then every circle through P and Q is orthogonal to $\mathcal{C}$.

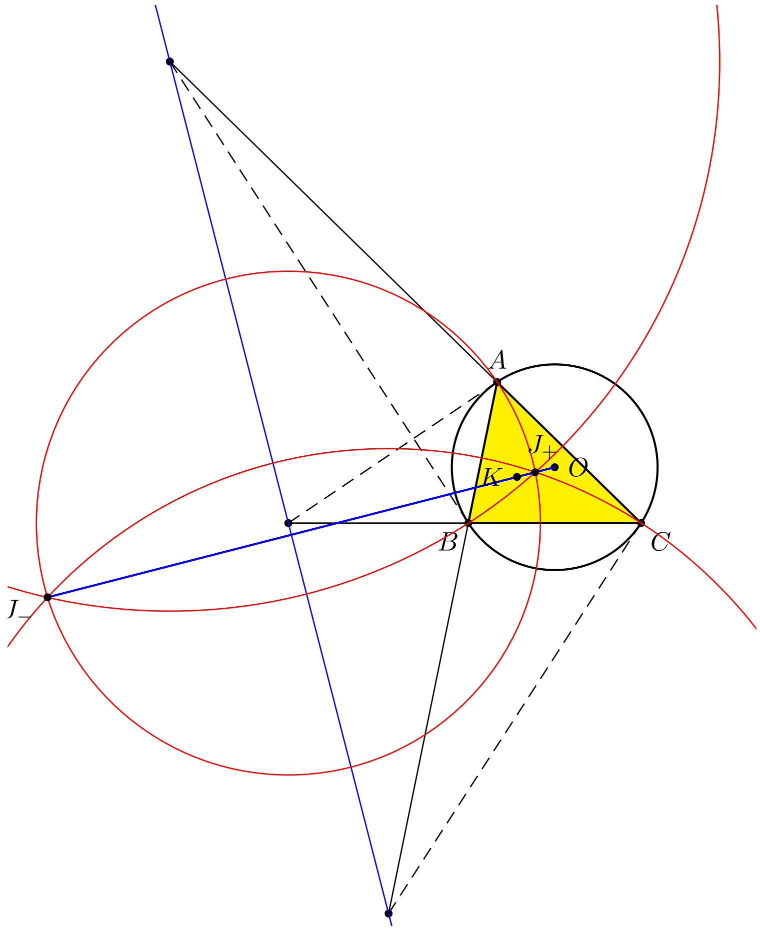
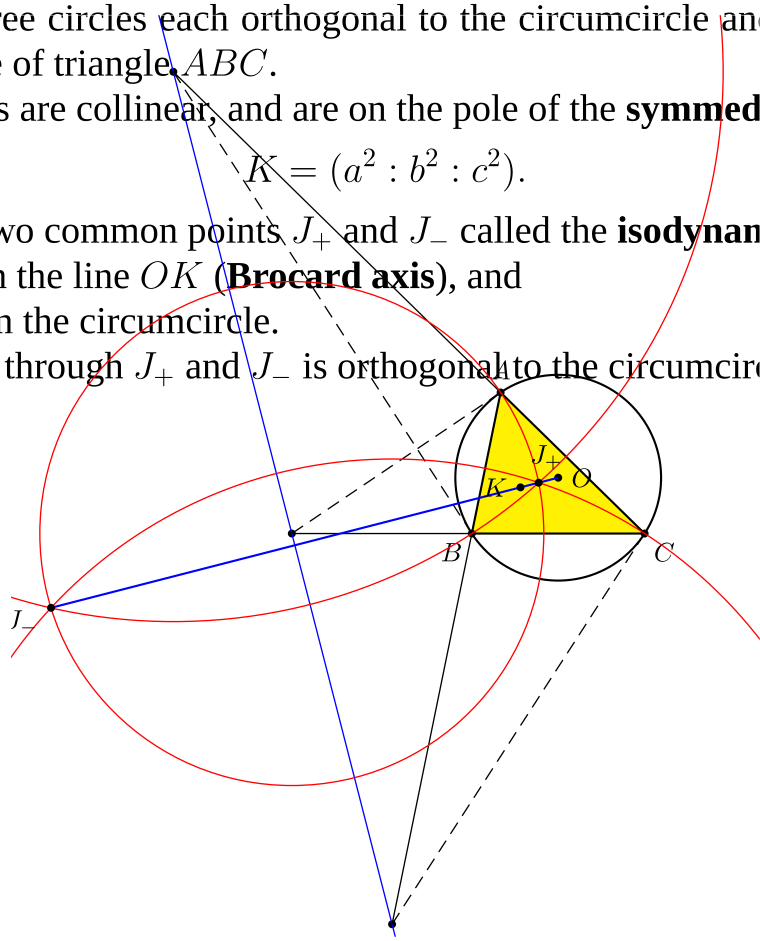


Figure 23.

Consider three circles each orthogonal to the circumcircle and with center on a sideline of triangle ABC.
Their centers are collinear, and are on the pole of the **symmedian point**

$$K = (a^2 : b^2 : c^2).$$

They have two common points J_+ and J_- called the **isodynamic points**, which are on the line OK (**Brocard axis**), and are inverse in the circumcircle.
Every circle through J_+ and J_- is orthogonal to the circumcircle.



The **isodynamic points** have coordinates

$$J_+ = (a^2(\sqrt{3}S_A + S), b^2(\sqrt{3}S_B + S), c^2(\sqrt{3}S_C + S)),$$
$$= \sqrt{3}(a^2S_A, b^2S_B, c^2S_C) + S(a^2, b^2, c^2);$$
$$J_- = \sqrt{3}(a^2S_A, b^2S_B, c^2S_C) - S(a^2, b^2, c^2).$$

They divide O and K **harmonically**.
Therefore, every circle through $J_\pm$ is orthogonally to the **Brocard circle** (with diameter OK).

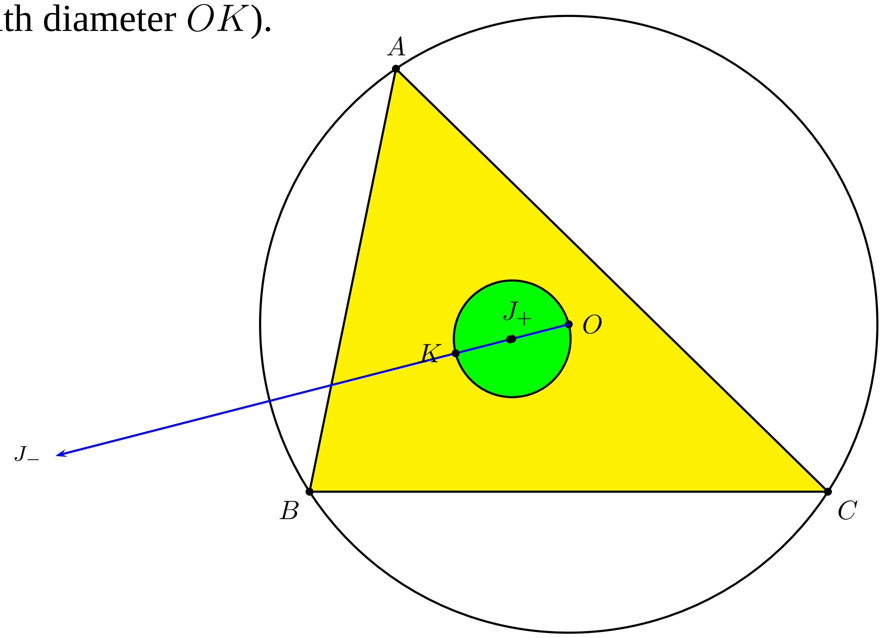


Figure 24. The Brocard circle and the isodynamic points

The isodynamic points are the only points whose **pedal triangles** are equilateral.

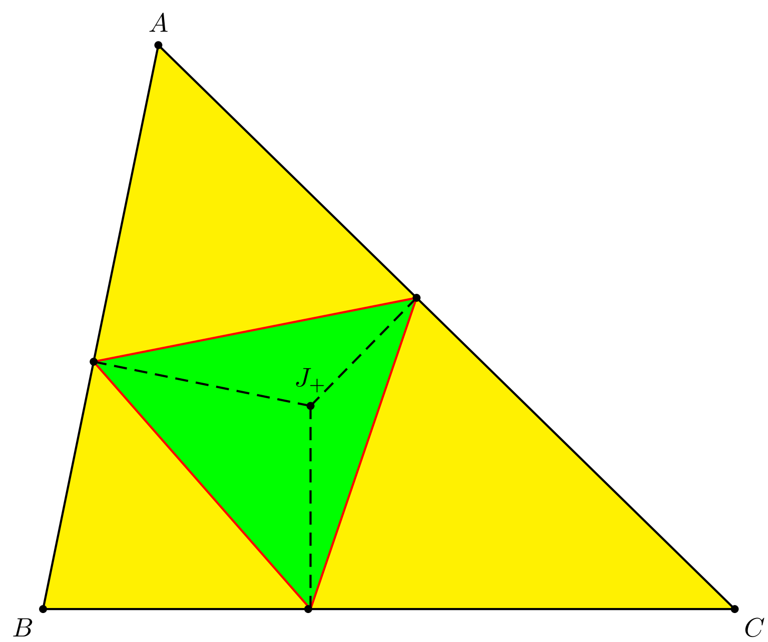


Figure 25. The pedal triangle of J_+ is equilateral

The isodynamic points are the **isogonal conjugates** of the Fermat points

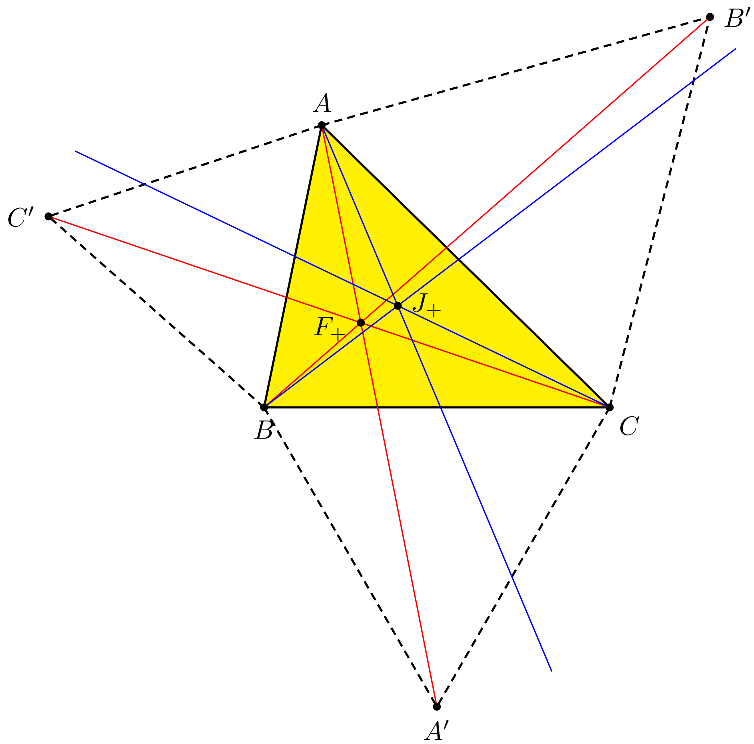


Figure 26. $J_+ =$ isogonal conjugate of F_+

4. The first Evans circle

The **excentral triangle** $I_aI_bI_c$ has circumradius $2R$ and circumcenter $I' :=$ reflection of I in O.

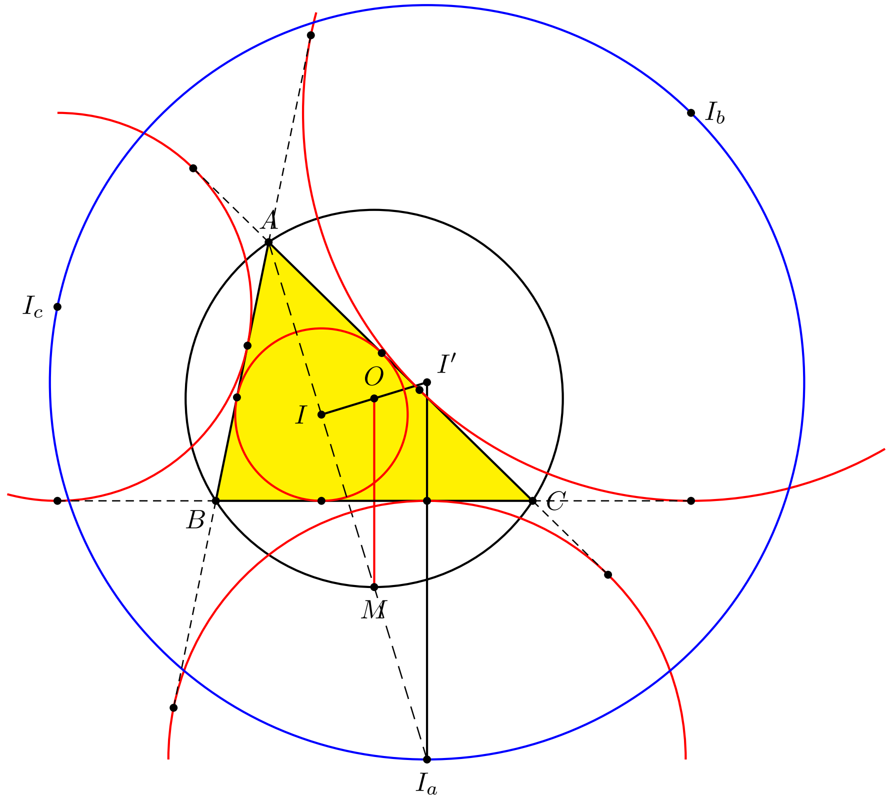


Figure 27. The excentral triangle and its circumcircle

The triangle of reflections

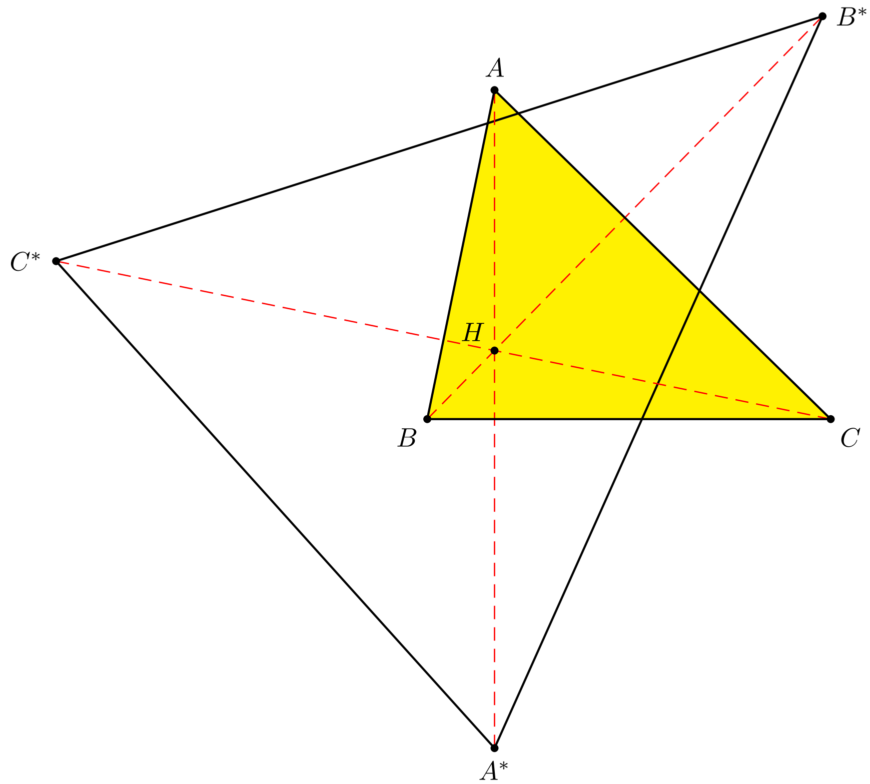


Figure 28. The triangle of reflections

The Evans perspector W of the excentral triangle and the triangle of reflections

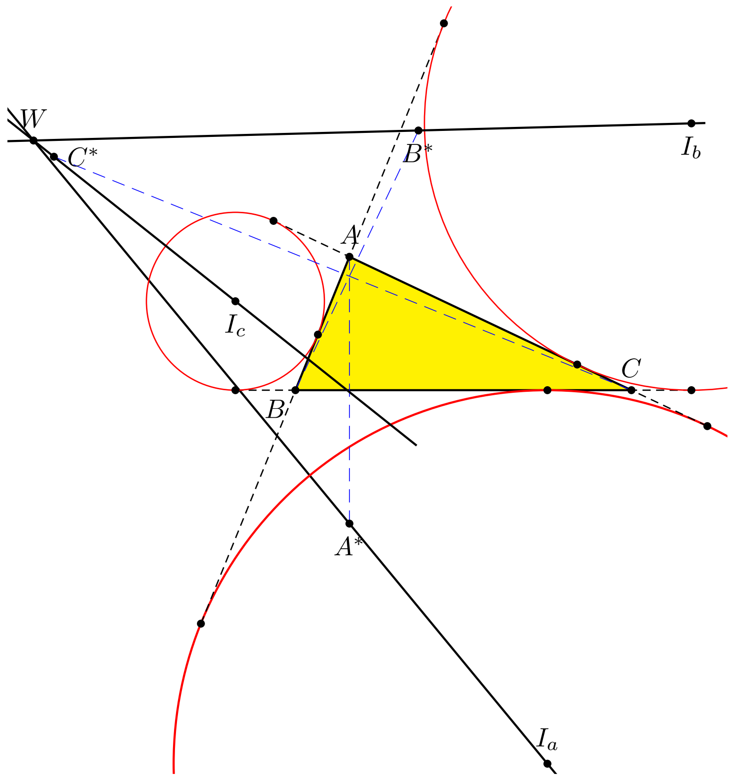


Figure 29. The Evans perspsector W

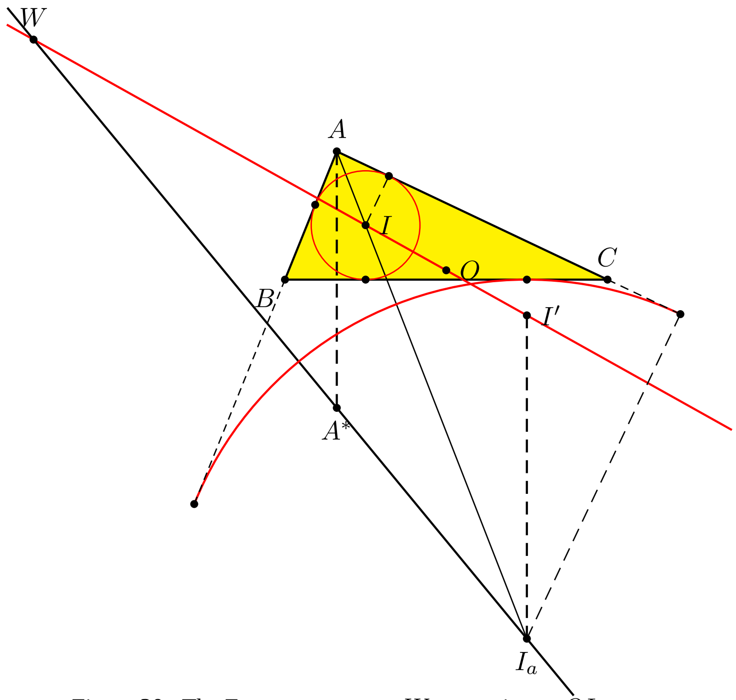


Figure 30. The Evans perspsector W as a point on OI

Let I_aA^* intersect OI at W. A routine calculation shows that

$$I'W : WI = R : -2r.$$

Similarly, I_bB^* and I_cC^* intersect OI at points given by the same ratio. Therefore the lines I_aA^*, I_bB^* and I_cC^* concur at W on OI.

Theorem 5. *The Evans perspector W and the incenter I are inverse in the circumcircle of the excentral triangle.*

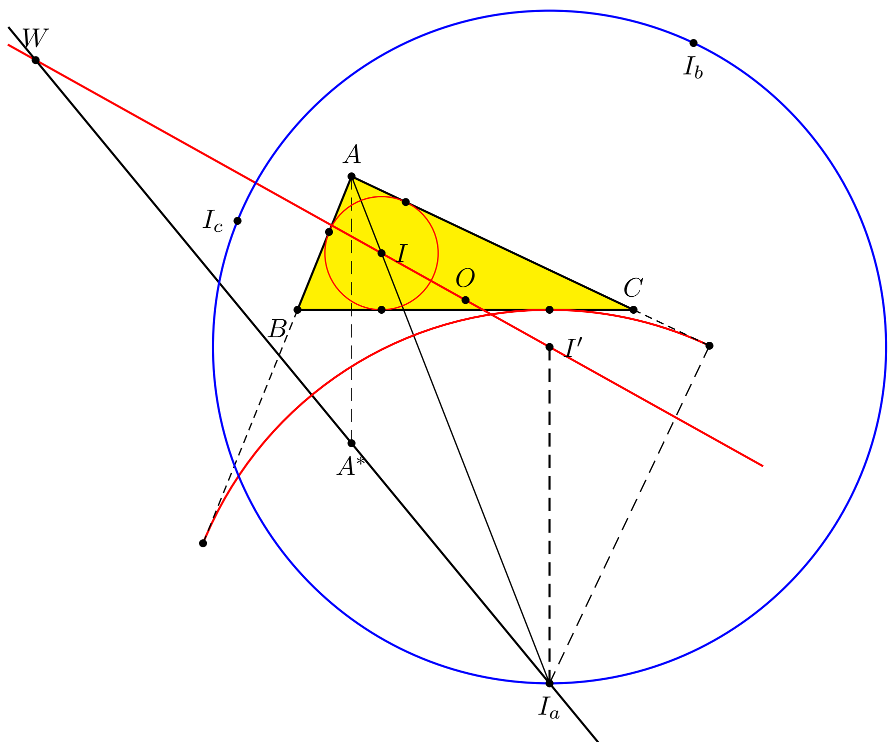


Figure 31. W and I are inverse in the circumcircle of the excentral triangle

Proof. $I'W \cdot I'I = \frac{R}{R-2r} \cdot I'I^2 = \frac{R^2}{R(R-2r)} \cdot (2 \cdot OI)^2 = (2R)^2$. □

Evans also found that the excentral triangle is perspective with each of the Kiepert triangles $\mathcal{K}\left(\frac{\pi}{3}\right)$ and $\mathcal{K}\left(-\frac{\pi}{3}\right)$.
He denoted these perspectors by V_+ and V_- and conjectured that V_+, V_-, I and W are concyclic.

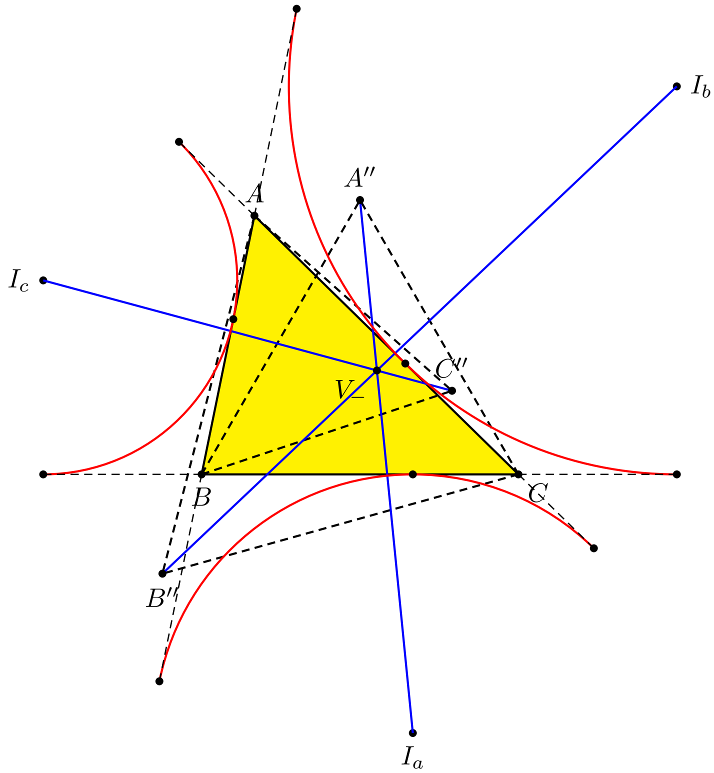


Figure 32. Evans' perspector V_- of $\mathcal{K}\left(-\frac{\pi}{3}\right)$ and excentral triangle

Proposition 6. *Let XBC and $X'I_bI_c$ be* **oppositely oriented similar isosceles triangles** *with bases BC and I_bI_c respectively. The lines I_aX and I_aX' are isogonal with respect to angle I_a the excentral triangle.*

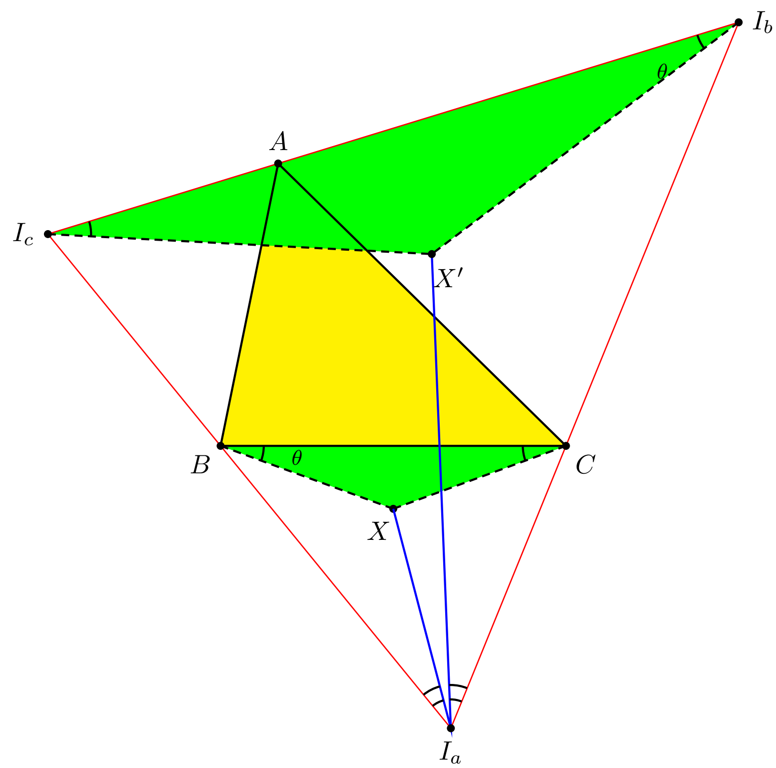


Figure 33. Isogonal lines joining I_a to apices of similar isosceles on BC and I_bI_c

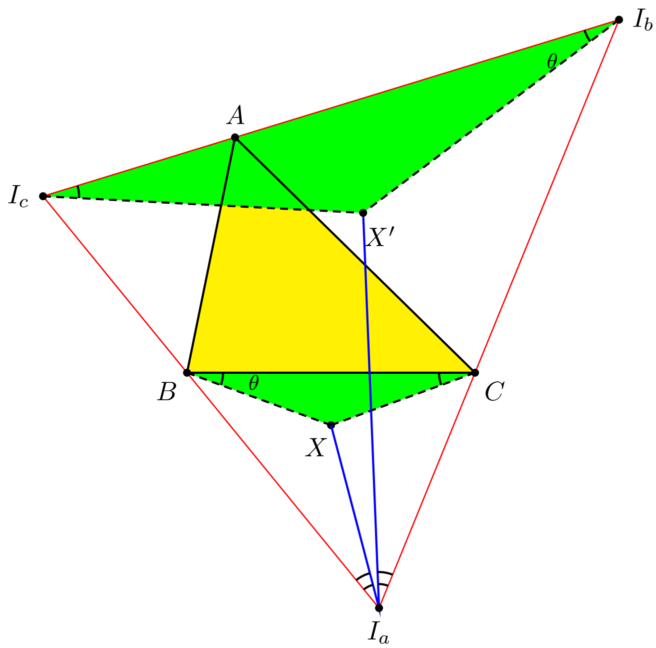


Figure 34. Isogonal lines joining I_a to apices of similar isosceles on BC and I_bI_c

Proof. Triangles XBI_a and $X'I_bI_a$ are similar since
$\angle XBI_a = \angle X'I_bI_a = \frac{\pi}{2} - \frac{B}{2} - \theta$, and
as BC and I_bI_c are antiparallel,

$$XB : X'I_b = BC : I_bI_c = I_aB : I_aI_b.$$

It follows that $\angle BI_aX = \angle I_bI_aX'$ and
the lines I_aX, I_aX' are isogonal in the excentral triangle. $\square$

Theorem 7. *Let* XYZ *be the Kiepert triangle* $\mathcal{K}(\theta)$ *of* ABC.
The lines I_aX, I_bY, I_cZ *concur at a point*
$V(\theta)$ *which is the isogonal conjugate of* $K_{\rm e}(-\theta)$ *in the excentral triangle.*

Proof. (i) I_aX', I_bY', I_cZ' concur at the Kiepert perspector $K_{\rm e}(-\theta)$ of the excentral triangle.

(ii) Since I_aX and I_aX' are isogonal with respect to I_a, and similarly for the pairs I_bY, I_bY' and I_cZ and I_cZ', the lines I_aX, I_bY, I_cZ concur at the isogonal conjugate of $K_{\rm e}(-\theta)$ in the excentral triangle. □

Corollary 8. $V_\pm$ *are the isodynamic points of the excentral triangle.*

Therefore, every circle through V_+ and V_- is orthogonal to the circumcircle of the excentral triangle.

If such a circle contains the incenter I, it also contains the inverse of I in the circumcircle of the excentral triangle.

This latter is the Evans perspector W.

Theorem 9 (Evans)**.** *The points $V_\pm$ are concyclic with I and W.*

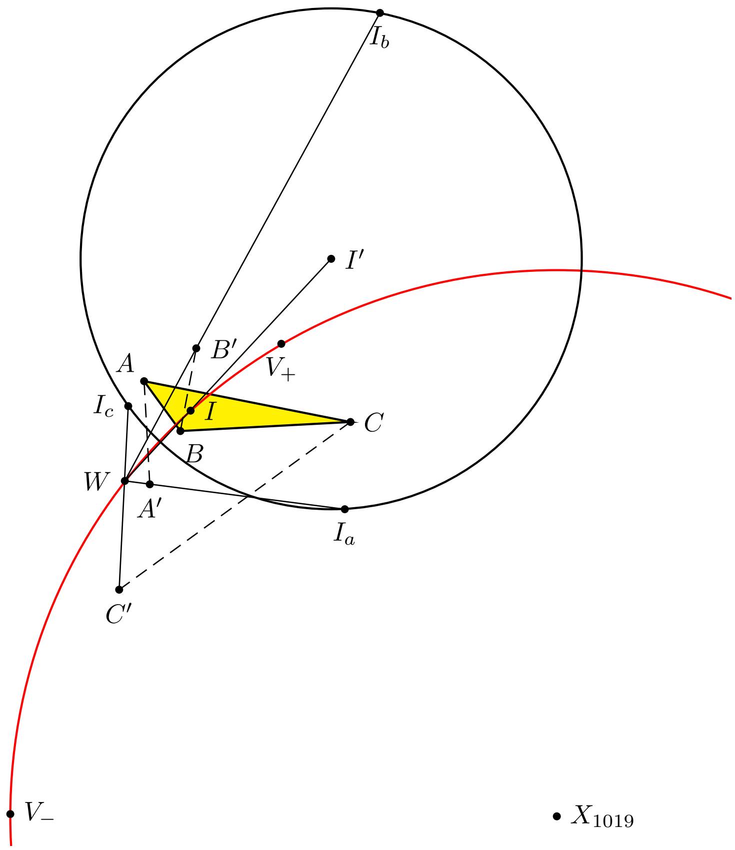


Figure 35. The first Evans circle

Proposition 10. *The center of the first Evans circle is the point*

$$X_{1019} = \left(\frac{a(b-c)}{b+c} : \frac{b(c-a)}{c+a} : \frac{c(a-b)}{a+b} \right).$$

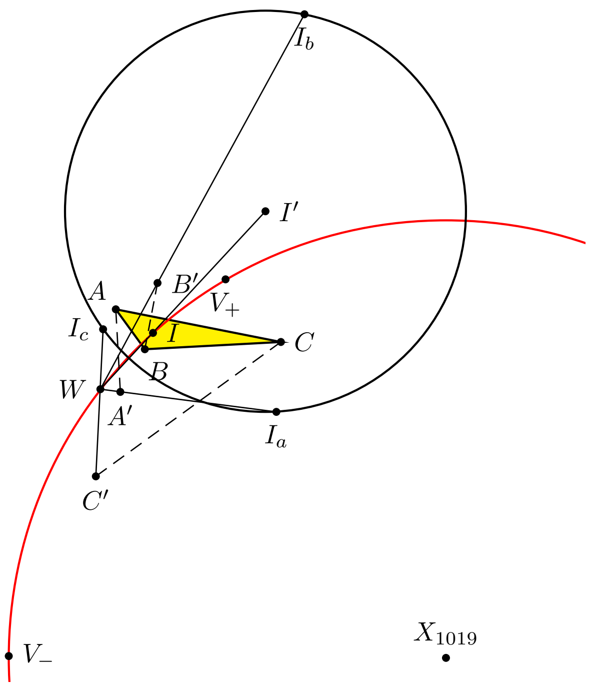


Figure 36. The first Evans circle

5. The Parry circle and the Parry point

The **Parry circle** is the one passing through the **isodynamic points** $J_{\pm}$ and the **centroid** G.

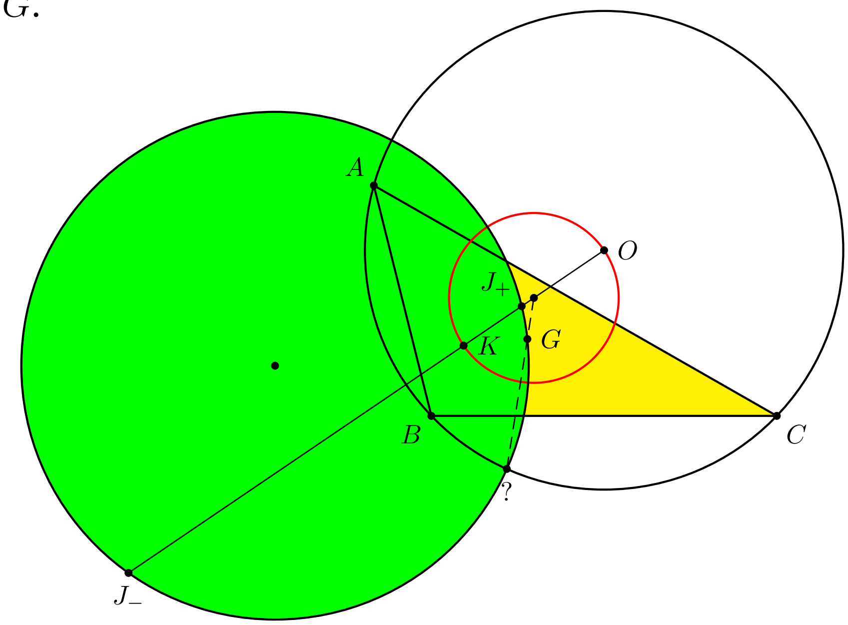


Figure 37. The Parry circle through the centroid and isodynamic points

Since $J_{\pm}$ are inverse in the Brocard circle,
the Parry circle is orthogonal to the Brocard circle,
and also contains the inverse of G in the Brocard circle.

The same is true with the Brocard circle replaced by the circumcircle.

Theorem 11. *The inverse of the centroid G in the Brocard circle is the* **Euler reflection point** E.

Proof. The equation of the Brocard circle is

$$(a^2+b^2+c^2)(a^2yz+b^2zx+c^2xy)-(x+y+z)(b^2c^2x+c^2a^2y+a^2b^2z)=0.$$

The polar of the centroid is the line

$$(b^2-c^2)^2x+(c^2-a^2)^2y+(a^2-b^2)^2z=0.$$

This clearly contains the Euler reflection point

$$E=\left(\frac{a^2}{b^2-c^2}:\frac{b^2}{c^2-a^2}:\frac{c^2}{a^2-b^2}\right),$$

which also lies on the line

$$\sum(b^2-c^2)(S_{AA}-S_{BC})x=0$$

joining G to the midpoint of OK. □

The lines GE and F_+F_- are parallel.

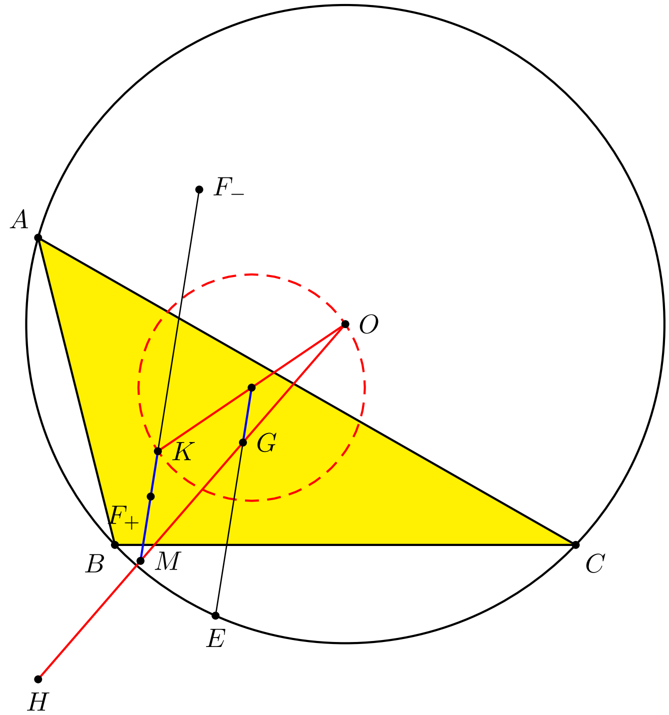


Figure 38. GE and F_+F_- are parallel

Since the Parry circle is orthogonal to the circumcircle,
the polar O is the radical axis of the circles.
This line passes through the symmedian point K.
The **Parry point** P is the second intersection of
the Parry circle and the circumcircle.
It lies on a number of interesting circles.

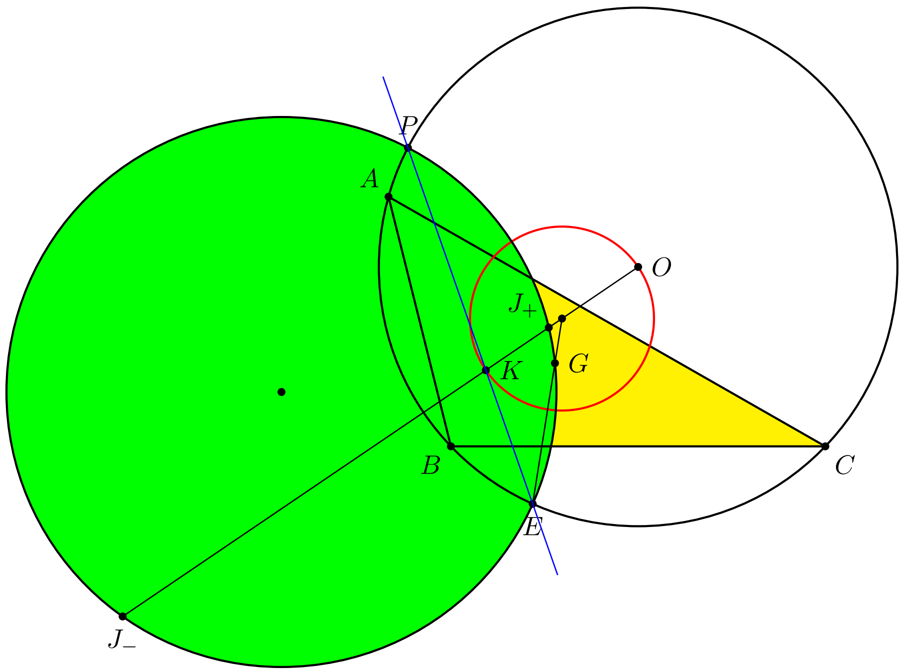


Figure 39. The Parry circle and Parry point

(1) **The circle** F_+F_-G contains the Parry point P.

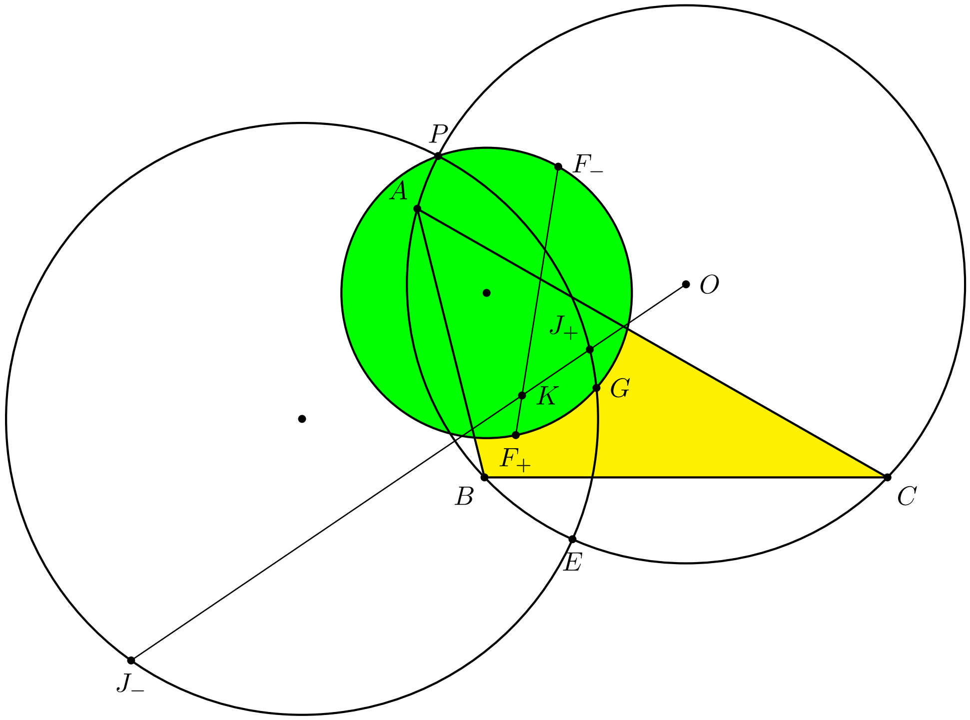


Figure 40. The circle through F_+F_-G contains the Parry point

(2) **The circle** OGK contains the Parry point P.

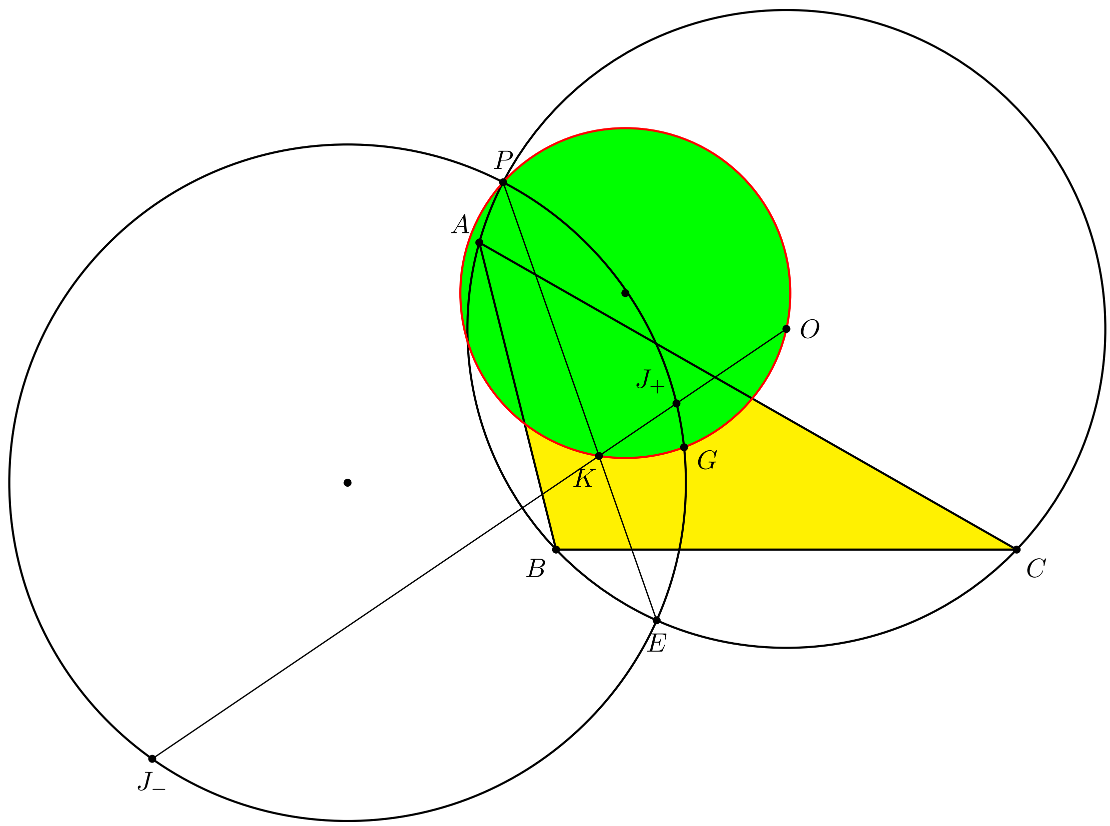


Figure 41. The circle OGK contains the Parry point P

Proposition 12. *The circle F_+F_-G intersects the circumcircle at the Parry point and*
the reflection of E in the **Euler line**.

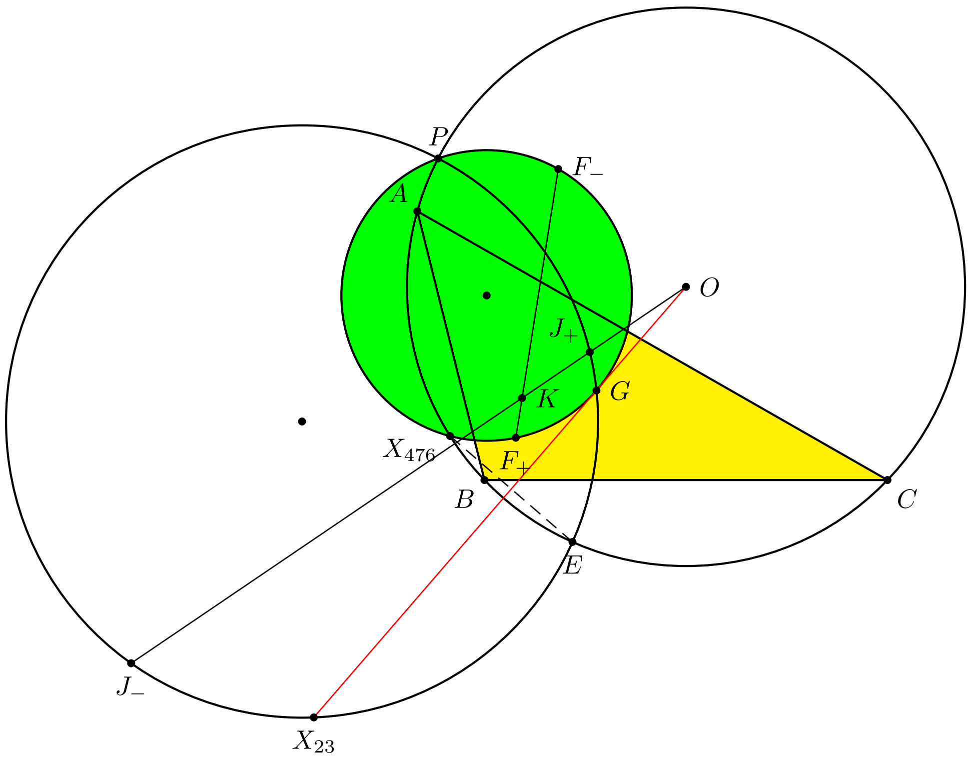


Figure 42. Intersections of F_+F_-G and the circumcircle

Proposition 13. *The circle OKG intersects the circumcircle at the Parry point P and the reflection of E in the* **Brocard axis**.

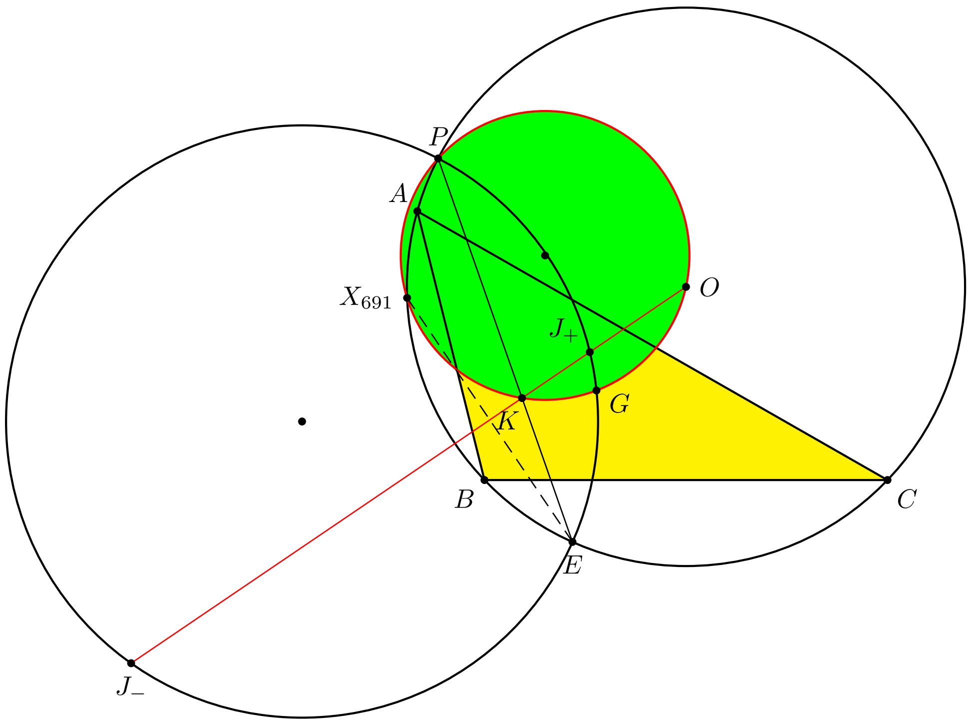


Figure 43. Intersections of OKG with the circumcircle